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η-Ricci Solitons on η-Einstein $(LCS)_n$-Manifolds

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Abstract

The object of the present paper is to study η-Ricci solitons on η-Einstein $(LCS)_n$-manifolds. It is shown that if ξ is a recurrent torse forming η-Ricci soliton on an η-Einstein $(LCS)_n$-manifold then ξ is (i) concurrent and (ii) Killing vector field.

Key words: η-Ricci soliton, η-Einstein manifold, $(LCS)_n$-manifold.

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1 Introduction

In 2003 Shaikh [33] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [23] and also by Mihai and Rosca [24]. Then Shaikh and Baishya ([36, 37]) investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. The $(LCS)_n$-manifolds are also studied by Atceken et al. ([3, 4, 19]), Hui [18], Narain and Yadav [28], Prakasha [32], Shaikh and his co-authors ([34, 35], [38]–[42]) and many others.

In 1982, Hamilton [14] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([30, 31]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generelization of Einstein metric such that [15]

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor and $\pounds_V$ is the Lie derivative along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. In [44] Sharma studied the Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Bagewadi et al. ([1, 2, 5, 22]), Bejan and Crasmareanu [6], Blaga [7], Hui et al. ([9, 20, 21]), Chen and Deshmukh [10], Deshmukh et. al [13], Nagaraja and Premalatta [27], Tripathi [45] and many others. In this connection it may be mentioned that Hinterleitner and Kiosak ([16, 17]) studied special Ricci Solitons.

In [12] Cho and Kimura studied on Ricci solitons of real hypersurfaces in a non-flat complex space form and they defined η-Ricci soliton, which satisfies the equation

$$\pounds_\xi g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where λ and μ are real constants.

Motivated by the above studies the object of the present paper is to study η-Ricci solitons on η-Einstein $(LCS)_n$-manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of η-Ricci solitons on η-Einstein $(LCS)_n$-manifolds. It is proved that if ξ is a recurrent torse forming η-Ricci soliton on an η-Einstein $(LCS)_n$-manifold $(M, g, \xi, \lambda, \mu, a, b)$ then ξ is (i) concurrent and (ii) Killing vector field. Also it is shown that if the torse forming η-Ricci soliton on η-Einstein manifold is regular, then any parallel symmetric (0,2) tensor field is a constant multiple of the metric.

2 Preliminaries

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0,2) such that for each point $p \in M$, the tensor $g_p \colon T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, \cdots, +)$, where T_pM denotes the tangent vector space of M at p and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_pM$ is said to be timelike (resp., non-spacelike, null, spacelike) if it satisfies $g_p(v, v) < 0$ (resp, ≤ 0, $= 0$, > 0) [29].

We now recall the definitions of concircular, torse-forming, recurrent, concurrent and parallel vector fields, see [8, 11, 26, 25, 46].

Definition 2.1. In a pseudo Riemannian manifold (M, g), a vector field P defined by

$$g(X, P) = A(X),$$

for any $X \in \Gamma(TM)$, the section of all smooth tangent vector fields on M, is said to be a *concircular vector field* if

$$(\nabla_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\}$$

where α is a non-zero scalar and ω is a closed 1-form and ∇ denotes the operator of covariant differentiation with respect to the metric g.

Definition 2.2. A vector field ξ is called *torse forming* if it satisfies

$$\nabla_X \xi = fX + \gamma(X)\xi \tag{2.1}$$

for a smooth function $f \in C^\infty(M)$ and γ is an 1-form, for all vector field X on M. A torse forming vector field ξ is called *recurrent* if $f = 0$.

Definition 2.3. A vector field v is called *concurrent* vector field if it satisfies

$$\nabla_X v = 0 \tag{2.2}$$

for any vector field X on M.

Definition 2.4. A tensor h of second order is said to be a *parallel tensor* if $\nabla h = 0$.

Let M be an n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2.3}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{2.4}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \quad \alpha \neq 0 \tag{2.5}$$

that is,

$$\nabla_X \xi = \alpha[X + \eta(X)\xi]$$

for all vector fields X, Y, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X), \tag{2.6}$$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi, \tag{2.7}$$

then from (2.5) and (2.7) we have

$$\phi X = X + \eta(X)\xi, \tag{2.8}$$

from which it follows that ϕ is a symmetric (1,1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and an (1,1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly, $(LCS)_n$-manifold), [34]. Especially, if we take $\alpha = 1$, then we can obtain the LP-Sasakian structure of Matsumoto [23]. In a $(LCS)_n$-manifold ($n > 2$), the following relations hold ([34, 36, 37, 38]):

$$\eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.9}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.10}$$

$$S(X, \xi) = (n-1)(\alpha^2 - \rho)\eta(X), \tag{2.11}$$

$$R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{2.12}$$

$$R(\xi, Y)Z = (\alpha^2 - \rho)[g(Y, Z)\xi - \eta(Z)Y], \tag{2.13}$$

$$(\nabla_X \phi)Y = \alpha\{g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X\}, \tag{2.14}$$

$$(X\rho) = d\rho(X) = \beta\eta(X), \tag{2.15}$$

$$R(X, Y)Z = \phi R(X, Y)Z + (\alpha^2 - \rho)\big\{g(Y, Z)\eta(X) - g(X, Z)\eta(Y)\big\}\xi, \tag{2.16}$$

for any vector fields X, Y, Z on M and $\beta = -(\xi\rho)$ is a scalar function, where R is the curvature tensor and S is the Ricci tensor of the manifold.

Definition 2.5. A $(LCS)_n$-manifold (M^n, g) is said to be η-Einstein if its Ricci tensor S of type (0,2) is of the form

$$S = ag + b\eta \otimes \eta, \tag{2.17}$$

where a and b are smooth functions on M.

Proposition 2.1. *In an η-Einstein $(LCS)_n$-manifold, the following relations hold:*

$$S(\phi X, Y) = S(X, \phi Y) = ag(\phi X, Y), \tag{2.18}$$

$$S(X, \xi) = (a-b)\eta(X), \quad S(\xi, \xi) = -(a-b), \tag{2.19}$$

$$S(\phi X, \phi Y) = S(X, \phi^2 Y) = S(X, Y) + (a-b)\eta(X)\eta(Y). \tag{2.20}$$

Proof. By virtue of (2.8) and (2.9), we have from (2.17) that (2.18). In view of (2.4) and (2.9), we get from (2.17 that (2.19). Replacing Y by ϕY in (2.18) and using (2.10) and (2.19), we get the relation (2.20). □

3 η-Ricci solitons on η-Einstein $(LCS)_n$-manifolds

Definition 3.1. The metric tensor g on η-Einstein $(LCS)_n$-manifold is said to be *η-Ricci soliton* if it satifies the relation (1.2).

In this section, we study η-Ricci solitons on η-Einstein $(LCS)_n$-manifolds $(M, g, \xi, \lambda, \mu, a, b)$ and prove the following:

Theorem 3.1. *If $(M, g, \xi, \lambda, \mu, a, b)$ is an η-Ricci soliton on an η-Einstein $(LCS)_n$-manifold, then*
(i) $a - b + \lambda - \mu = 0$,
(ii) ξ *is a geodesic vector field,*
(iii) $(\nabla_\xi \phi)\xi = 0$ *and* $\nabla_\xi \eta = 0$,
(iv) $\nabla_\xi S = 0$ *and* $\nabla_\xi Q = 0$.

Proof. Let $(M, g, \xi, \lambda, \mu, a, b)$ be an η-Ricci soliton on η-Einstein $(LCS)_n$-manifold. In view of (2.17) we have from (1.2) that

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2[(a + \lambda)g(X, Y) + (b + \mu)\eta(X)\eta(Y)] = 0. \tag{3.1}$$

Putting $X = Y = \xi$ in (3.1) and using (2.9) we obtain $g(\nabla_\xi \xi, \xi) = a - b + \lambda - \mu$, but $g(\nabla_X \xi, \xi) = 0$ for any vector field X on M, since ξ has a constant norm. Hence we get (i).

Consequently (3.1) becomes

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2(a + \lambda)[g(X, Y) + \eta(X)\eta(Y)] = 0. \tag{3.2}$$

Setting $Y = \xi$ in (3.2) we get $g(\nabla_\xi \xi, X) = 0$ for any vector field X on M and hence we have $\nabla_\xi \xi = 0$, i.e., ξ is a geodesic vector field. Thus we get (ii).

(iii) is obvious from (ii).

More precisely, the general expression for ∇S and ∇Q are

$$(\nabla_X S)(Y, Z) = b[\eta(Y)g(Z, \nabla_X \xi) + \eta(Z)g(Y, \nabla_X \xi)]$$

and

$$(\nabla_X Q)Y = b[\eta(Y)\nabla_X \xi + g(Y, \nabla_X \xi)\xi].$$

Putting $X = Y = Z = \xi$ in above we get (iv). □

Theorem 3.2. *If ξ is a torse forming η-Ricci soliton on an η-Einstein $(LCS)_n$-manifold $(M, g, \xi, \lambda, \mu, a, b)$ then $f = -(a + \lambda)$, η is closed and*

$$b = a - (n - 1)(a + \lambda)^2 \quad \textit{and} \quad \mu = \lambda + (n - 1)(a + \lambda)^2.$$

Proof. Let ξ be a torse forming η-Ricci soliton on an η-Einstein $(LCS)_n$-manifold $(M, g, \xi, \lambda, \mu, a, b)$. Then we have from (2.1) that $g(\nabla_X \xi, \xi) = f\eta(X) - \gamma(X)$ and hence we get $\gamma = f\eta$. Consequently (2.1) becomes

$$\nabla_X \xi = f[X + \eta(X)\xi]. \tag{3.3}$$

Using (3.3) in (3.2), we get

$$(f + a + \lambda)[g(X,Y) + \eta(X)\eta(Y)] = 0 \tag{3.4}$$

for all vector fields X and Y and hence it follows that $f = -(a+\lambda)$. Thus we get from (3.3) that

$$\nabla_X \xi = -(a+\lambda)[X + \eta(X)\xi], \tag{3.5}$$

which means that $\nabla_X \xi$ is collinear to $\phi^2 X$ for all X and hence we get $d\eta = 0$, i.e., η is closed.

It is known that

$$R(X,Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi. \tag{3.6}$$

In view of (3.5), (3.6) yields

$$R(X,Y)\xi = (a+\lambda)^2[\eta(Y)X - \eta(X)Y]. \tag{3.7}$$

From (3.7), we get

$$S(X,\xi) = (n-1)(a+\lambda)^2\eta(X). \tag{3.8}$$

From (2.19) and (3.8), we get $b = a-(n-1)(a+\lambda)^2$ and $\mu = \lambda+(n-1)(a+\lambda)^2$. Thus we get the theorem. □

Corollary 3.1. *If ξ is a recurrent torse forming η-Ricci soliton on an η-Einstein $(LCS)_n$-manifold $(M, g, \xi, \lambda, \mu, a, b)$ then ξ is (i) concurrent and (ii) Killing vector field.*

Proof. Since ξ is recurrent, therefore $f = 0$ and hence $a + \lambda = 0$. So, by virtue of (3.5) we get $\nabla_X \xi = 0$ for all X on M, which means that ξ is concurrent vector field, i.e., (i). Also in that case

$$(\pounds_\xi g)(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0$$

for all X, Y that means ξ is Killing vector field, i.e., (ii). □

Corollary 3.2. *If ξ is a torse forming Ricci soliton on an η-Einstein $(LCS)_n$-manifold $(M, g, \xi, \lambda, a, b)$ then the Ricci soliton is shrinking, steady and expanding according as $a > b$, $a = b$ and $a < b$ respectively.*

Proof. In particular, if $\mu = 0$ then from Theorem 3.2, we get

$$\lambda + (n-1)(a+\lambda)^2 = 0$$

and hence we obtain $b = a + \lambda$, i.e., $\lambda = b - a$. Hence the proof is complete. □

Let ξ be a torse forming η-Ricci soliton on an η-Einstein $(LCS)_n$-manifold $(M, g, \xi, \lambda, a, b)$ and let h be a (0,2) symmetric tensor field on $(M, g, \xi, \lambda, a, b)$ such that $\nabla h = 0$. Then applying the Ricci identity [43]

$$\nabla^2 h(X,Y;Z,W) - \nabla^2 h(X,Y;W,Z) = 0, \tag{3.9}$$

we obtain

$$h\big(R(X,Y)Z,W\big)+h\big(Z,R(X,Y)W\big)=0 \tag{3.10}$$

for arbitrary vector fields X, Y, Z and W on (M,g,ξ,λ,a,b). Putting $X=Z$ $=W=\xi$ in (3.10) and since h is symmetric, we get

$$h\big(\xi,R(\xi,Y)\xi\big)=0, \tag{3.11}$$

Using (3.7) in (3.11) we get

$$(a+\lambda)^2\big[h(Y,\xi)+\eta(Y)h(\xi,\xi)\big]=0. \tag{3.12}$$

Definition 3.2. The η-Ricci soliton on an η-Einstein manifold $(LCS)_n$-manifold is regular if $(a+\lambda)\neq 0$.

From regularity we have from (3.12) that

$$h(Y,\xi)+\eta(Y)h(\xi,\xi)=0. \tag{3.13}$$

Differentiating (3.13) covariantly and using (3.5) we get

$$h(X,Y)=-h(\xi,\xi)g(X,Y).$$

So by following the same method of [9], we can state the following:

Theorem 3.3. *If the torse forming η-Ricci soliton on η-Einstein manifold (M,g,ξ,λ,a,b) is regular, then any parallel symmetric* (0,2) *tensor field is a constant multiple of the metric.*

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