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Stability and Boundedness of Solutions of Some Third-order Nonlinear Vector Delay Differential Equation

Larbi FATMI [a], Moussadek REMILI [b*]

Department of Mathematics, University of Oran 1 Ahmed Ben Bella,
31000 Oran, Algeria
[a] *e-mail: fatmilrb@yahoo.fr*
[b] *e-mail: remilimous@gmail.com*

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Abstract

This paper investigates the stability of the zero solution and uniformly boundedness and uniformly ultimately boundedness of all solutions of a certain vector differential equation of the third order with delay. Using the Lyapunov–Krasovskiĭ functional approach, we obtain a new result on the topic and give an example for the related illustrations.

Key words: Lyapunov functional, third-order vector delay differential equation, boundedness, stability.

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1 Introduction

In recent years much attention have been drawn to the stability and ultimate boundedness of solutions of ordinary scalar and vector nonlinear differential equations of third-order. However, it should be clarified that the number of results related to the ultimate boundedness of certain third order nonlinear vector differential equations is very few in comparison to that on the certain scalar nonlinear differential equations of third order. An effective method for studying the stability and boundedness of nonlinear differential equations is the second Lyapunov's method; see, for example, [1, 2, 8, 9, 10, 15, 16, 22]. The major advantage of this method is that information about stability, boundedness can be obtained without any prior knowledge of solutions.

[*] Corresponding author.

In 1966, 1983 and 1993, respectively, Ezeilo & Tejumola [9], Afuwape [1] and Meng [15] investigated the ultimately boundedness and existence of periodic solutions of the nonlinear vector differential equation of the form

$$X'''(t) + AX''(t) + BX'(t) + H(X(t)) = P(t, X(t), X'(t), X''(t)).$$

Tunç ([24]–[27]) investigated the asymptotic stability, boundedness, ultimate boundedness and periodicity for certain third-order nonlinear vector differential equations. Moreover, In 2014, Tunç [23] investigated the qualitative behaviors of solutions, stability, boundedness, uniform boundedness and existence of periodic solutions, for some kind of the following vector differential equation

$$X'''(t) + X''(t) + G(X'(t)) + H(X(t)) = P(t, X(t), X'(t), X''(t)).$$

Recently, that is in 2015 Omeike [16] by defining a complete lyapunov functional, discussed conditions for uniform stability of the trivial solution and uniform ultimate boundedness of solutions of equation

$$X'''(t) + AX''(t) + BX'(t) + H\Big(X\big(t-r\big)\Big) = P(t), \tag{1.1}$$

when $P(t) = 0$ and $P(t) \neq 0$, respectively. in which $X \in \mathbb{R}^n, P\colon \mathbb{R} \to \mathbb{R}^n$, A and B are real $n \times n$ constant matrices, $0 \le r(t) \le \gamma$, γ is a positive constant.
The present work is concerned with the differential more general third order nonlinear vector differential equation with delay of the form

$$\Big(G\big(X(t)\big)X'(t)\Big)'' + AX''(t) + BX'(t) + H\Big(X\big(t - r(t)\big)\Big) = P(t), \tag{1.2}$$

in which $X \in \mathbb{R}^n$, $H\colon \mathbb{R}^n \to \mathbb{R}^n$ is a continuous differentiable function with $H(0) = 0$, $P\colon \mathbb{R} \to \mathbb{R}^n$, $G\colon \mathbb{R}^n \to \mathbb{R}^{n\times n}$ with G is twice differentiable, A and B are real $n \times n$ constant matrices, $0 \le r(t) \le \gamma$, γ is a positive constant, and $r'(t) \le \xi$, $0 < \xi < 1$, and the dots indicate differentiation with respect to t. Clearly the equation (1.1) discussed in [16] is a special case of equation (1.2) when $G(X) = I$. A complete Lyapunov–Krasovskiĭ functional was defined and investigated to study uniform ultimate boundedness and asymptotic behaviour of solutions.

2 Preliminaries

The following notations (see [16]) will be useful in subsequent sections. For $x \in \mathbb{R}^n$, $|x|$ is the norm of x. For a given $r > 0$, $t_1 \in \mathbb{R}$,

$$C(t_1) = \{\phi\colon [t_1 - r, t_1] \to \mathbb{R}^n/\phi \ \text{ is continuous}\}.$$

In particular, $C = C(0)$ denotes the space of continuous functions mapping the interval $[-r, 0]$ into $\mathbb{R}^n$ and for $\phi \in C$, $\|\phi\| = \sup_{-r\le\theta\le 0} |\phi(0)|$. C_H will denote the set of ϕ such that $\|\phi\| \le H$. For any continuous function $x(u)$ defined on $-h \le u < A$ for $A > 0$, and any fixed t such that $0 \le t < A$, the symbol x_t will denote the restriction of $x(u)$ to the interval $[t-r,t]$, that is, x_t is an element of C defined by

$$x_t(\theta) = x(t+\theta), -r \leq \theta \leq 0.$$

The following results will be basic to the proofs of Theorems.

Lemma 2.1. [1, 2, 8, 9, 10, 22] *Let D be a real symmetric positive definite $n \times n$ matrix, then for any X in $\mathbb{R}^n$, we have*

$$\delta_d \|X\|^2 \leq \langle DX, X\rangle \leq \Delta_d \|X\|^2,$$

where δ_d, Δ_d are the least and the greatest eigenvalues of D, respectively.

Lemma 2.2. [1, 2, 8, 9, 10, 22] *Let Q, D be any two real $n \times n$ commuting symmetric matrices, then,*

(i) *The eigenvalues $\lambda_i(QD)\,(i = 1, 2, \ldots, n)$ of the product matrix QD are all real and satisfy*

$$\min_{1 \leq j,k \leq n} \lambda_j(Q)\,\lambda_k(D) \leq \lambda_i(QD) \leq \max_{1 \leq j,k \leq n} \lambda_j(Q)\,\lambda_k(D).$$

(ii) *The eigenvalues $\lambda_i(Q+D)\,(i = 1, 2, \ldots, n)$ of the sum of matrices Q and D are all real and satisfy.*

$$\left\{\min_{1 \leq j \leq n} \lambda_j(Q) + \min_{1 \leq k \leq n} \lambda_k(D)\right\} \leq \lambda_i(Q+D) \leq \left\{\max_{1 \leq j \leq n} \lambda_j(Q) + \max_{1 \leq k \leq n} \lambda_k(D)\right\}.$$

Lemma 2.3. [1, 2, 8, 9, 10, 22] *Let $H(X)$ be a continuous vector function and that $H(0) = 0$ then,*

$$\frac{d}{dt}\left(\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma\right) = \langle H(X), X'\rangle.$$

Lemma 2.4. *Let $H(X)$ be a continuous vector function and that $H(0) = 0$ then,*

$$\delta_h \|X\|^2 \leq \int_0^1 \langle H(\sigma X), X\rangle\, d\sigma \leq \Delta_h \|X\|^2.$$

where δ_h, Δ_h are the least and the greatest eigenvalues of $J_h(X)$ (Jacobian matrix of H), respectively.

Definition 2.5. We definite the spectral radius $\rho(A)$ of a matrix A by

$$\rho(A) = \max\{\lambda / \ \lambda \text{ is eigenvalue of } A\}.$$

Lemma 2.6. *For any $A \in \mathbb{R}^{n\times n}$, we have the norm $\|A\| = \sqrt{\rho(A^TA)}$ if A is symmetric then*

$$\|A\| = \rho(A).$$

We shall note all the equivalents norms by the same notation $\|X\|$ for $X \in \mathbb{R}^n$ and $\|A\|$ for a matrix $A \in \mathbb{R}^{n\times n}$.

3 Stability

Consider the functional differential equation

$$x' = f(t, x_t), \quad x_t(\theta) = x(t+\theta), \quad -r \le \theta \le 0,\ t \ge 0, \tag{3.1}$$

where $f\colon I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t,0) = 0$, $C_H := \{\phi \in (C[-r,0],\ \mathbb{R}^n) : \|\phi\| \le H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t,\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Definition 3.1. [5] An element $\psi \in C$ is in the *ω-limit* set of ϕ, say $\Omega(\phi)$, if $x(t,0,\phi)$ is defined on $[0,+\infty)$ and there is a sequence $\{t_n\}, t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n+\theta, 0, \phi)$ for $-r \le \theta \le 0$.

Definition 3.2. [5] A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (3.1), $x(t,0,\phi)$, is defined on $[0,\infty)$ and $x_t(\phi) \in Q$ for $t \in [0,\infty)$.

Lemma 3.3. [6] *If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (3.1) with $x_0(\phi) = \phi$ is defined on $[0,\infty)$ and $\|x_t(\phi)\| \le H_1 < H$ for $t \in [0,\infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*

$$\operatorname{dist}(x_t(\phi), \Omega(\phi)) \to 0 \quad as \quad t \to \infty.$$

Lemma 3.4. [6] *Let $V(t,\phi)\colon I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(t,0) = 0$, and such that:*

(i) $W_1(|\phi(0)|) \le V(t,\phi) \le W_2(|\phi(0)|) + W_3(\|\phi\|_2)$
where $\|\phi\|_2 = (\int_{t-r}^{t} \|\phi(s)\|^2 ds)^{\frac{1}{2}}$.

(ii) $\dot{V}_{(3.1)}(t,\phi) \le -W_4(|\phi(0)|)$, *where, W_i $(i = 1,2,3,4)$ are wedges.*

Then the zero solution of (3.1) is uniformly asymptotically stable.

We shall state here some assumptions which will be used on the functions that appeared in equation (1.1), and suppose that there are positive constants $\delta_a, \delta_b, \delta_g, \delta_{g^{-1}}, \delta_h, \Delta_a, \Delta_b, \Delta_g, \Delta_{g^{-1}}$ and Δ_h, such that the matrices A, B, G^{-1} and $J_h(X)$ (Jacobian matrix of $H(X)$) are symmetric and positive definite, and furthermore the eigenvalues $\lambda_i(A)$, $\lambda_i(B)$, $\lambda_i(G^{-1})$ and $\lambda_i(J_h(X))$ $(i = 1,2,\dots,n)$ of A, B and $J_h(X)$, respectively satisfy,

$$\begin{aligned}
&0 < \delta_a \le \lambda_i(A) \le \Delta_a, && 0 < \delta_b \le \lambda_i(B) \le \Delta_b,\\
&0 < \delta_g \le \lambda_i(G) \le \Delta_g, && 0 < \delta_{g^{-1}} \le \lambda_i\left(G^{-1}\right) \le \Delta_{g^{-1}},\\
&0 < \delta_h \le \lambda_i(J_h(X)) \le \Delta_h, && \Delta_{g^{-1}} = \delta_g^{-1};\ \ \delta_{g^{-1}} = \Delta_g^{-1}.
\end{aligned}$$

Before stating the major theorems, we introduce the following notations

$$\begin{aligned}
\theta(t) &= \frac{d}{dt}G^{-1}(X(t)) = -G^{-1}\big(X(t)\big)\Big[\frac{d}{dt}G\big(X(t)\big)\Big]G^{-1}\big(X(t)\big),\\
\mu(t) &= \int_0^t \|\theta(s)\|\, ds.
\end{aligned}$$

For the case $P \equiv 0$, the following result is introduced.

Theorem 3.5. *In addition to the basic assumptions imposed on the matrices A, B, G^{-1} and $J_h(X)$ witch commute pairwise, let us assume that the following conditions hold:*

(H_0) *There is a constant such that* $\dfrac{1}{\delta_a} < \beta < \dfrac{\Delta_g^{-2}\delta_b}{\Delta_h \delta_g^{-1}}$,

(H_1) $\displaystyle\int_{-\infty}^{+\infty} \left\| \frac{d}{ds} G\big(X(s)\big) \right\| ds < +\infty.$

Then the zero solution of (1.2) is uniformly asymptotically stable, provided that

$$\gamma < \min\{\alpha_1, \alpha_2\}, \tag{3.2}$$

where

$$\alpha_1 = \frac{2(\beta\delta_a - 1)}{\beta\Delta_h\delta_g^{-1}}, \qquad \alpha_2 = \frac{2\left(\Delta_g^{-2}\delta_b - \beta\Delta_h\delta_g^{-1}\right)(1-\xi)}{\Delta_h\delta_g^{-1}\left[2\left(\beta + \Delta_g^{-1}\right) + (1-\xi)\,\delta_g^{-1}\right]}.$$

Proof. We write the equation (1.2) as the following equivalent system

$$\begin{cases} X' = G^{-1}(X)\,Y, \\ Y' = Z, \\ Z' = -AG^{-1}(X)\,Z - \left[A\left(G^{-1}\right)' + BG^{-1}\right] Y - H(X) \\ \qquad + \int_{t-r(t)}^{t} J_h(X)\,G^{-1}(X)\,Y ds. \end{cases} \tag{3.3}$$

The proof depends on some fundamental properties of a continuously differentiable Lyapunov–Krasovskiĭ functional $W_1 = W_1(X, Y, Z)$ defined by

$$W_1 = V_1 \exp\big(-\frac{1}{\rho_0}\mu(t)\big), \tag{3.4}$$

where

$$V_1(X_t, Y_t, Z_t) = 2\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma \tag{3.5}$$

$$+\,\omega_0 \int_{-r(t)}^{0}\int_{t+s}^{t} \langle Y(\theta), Y(\theta)\rangle\, d\theta ds + \left\langle AG^{-2}Y, Y\right\rangle$$

$$+\,\beta\left\langle Y, BG^{-1}Y\right\rangle + \beta\langle Z, Z\rangle + \left\langle G^{-1}Y, Z\right\rangle + 2\beta\langle Y, H(X)\rangle, \tag{3.6}$$

ω_0, ρ_0 are some positive constants which will be specified later in the proof. Since

$$\omega_0 \int_{-r(t)}^{0}\int_{t+s}^{t} \langle Y(\theta), Y(\theta)\rangle\, d\theta ds$$

is non-negative, we have

$$
\begin{aligned}
V_1(X_t,Y_t,Z_t) \geq\ & 2\int_0^1 \langle H(\sigma X), X\rangle\, d\sigma + \left\langle AG^{-2}Y, Y\right\rangle + \beta\left\langle Y, BG^{-1}Y\right\rangle \\
& +\beta\langle Z, Z\rangle + \left\langle G^{-1}Y, Z\right\rangle + 2\beta\langle Y, H(X)\rangle \\
\geq\ & 2\int_0^1\int_0^1 \sigma\langle J_h(\sigma\tau X)X, X\rangle\, d\tau d\sigma \\
& -\beta\left\langle \left(BG^{-1}\right)^{-\frac{1}{2}} H(X), \left(BG^{-1}\right)^{-\frac{1}{2}} H(X)\right\rangle \\
& +\beta\left\|\left(BG^{-1}\right)^{\frac{1}{2}} Y + \left(BG^{-1}\right)^{-\frac{1}{2}} H(X)\right\|^2 + \beta\left\|Z + \frac{1}{2}\beta^{-1}G^{-1}Y\right\|^2 \\
& +\left\langle \left[A - \frac{1}{4}\beta^{-1}I\right] G^{-2}Y, Y\right\rangle.
\end{aligned}
$$

Using Lemma 2.1, Lemma 2.2 and Lemma 2.4, and the fact that

$$
\beta\left\|\left(BG^{-1}\right)^{\frac{1}{2}} Y + \left(BG^{-1}\right)^{-\frac{1}{2}} H(X)\right\|^2 \geq 0,
$$

it follows that

$$
\begin{aligned}
V_1(X_t,Y_t,Z_t) \geq\ & 2\int_0^1\int_0^1 \sigma\left\langle\left[I - \beta GB^{-1}J_h(\sigma X)\right] J_h(\sigma\tau X)X, X\right\rangle d\tau d\sigma \\
& +\beta\left\|Z + \frac{1}{2}\beta^{-1}G^{-1}Y\right\|^2 + \left\langle\left[A - \frac{1}{4}\beta^{-1}I\right] G^{-2}Y, Y\right\rangle.
\end{aligned}
$$

Next, in view of the assumption of Theorem 3.5 and Lemma 2.2, respectively, it follows that

$$
\begin{aligned}
V_1(X_t,Y_t,Z_t) \geq\ & \left(1 - \beta\delta_b^{-1}\Delta_g\Delta_h\right)\delta_h\|X\|^2 + \beta\left\|Z + \frac{1}{2}\beta^{-1}G^{-1}Y\right\|^2 \\
& +\left(\delta_a - \frac{1}{4}\beta^{-1}\right)\delta_{g^{-1}}^2\|Y\|^2.
\end{aligned} \tag{3.7}
$$

Clearly from the terms contained in (3.7), there exists a constant $k > 0$ small enough such that

$$
V_1(X_t,Y_t,Z_t) \geq k\left(\|X\|^2 + \|Y\|^2 + \|Z\|^2\right). \tag{3.8}
$$

Assumption (3.5) implies the following

$$
\begin{aligned}
\mu(t) &= \int_0^t \|\theta(s)\|\, ds \\
&= \int_0^t \left\|G^{-1}(X(s))\left[\frac{d}{ds}G(X(s))\right]G^{-1}(X(s))\right\| ds \\
&\leq \int_0^t \left\|G^{-1}(X(s))\right\|^2 \left\|\frac{d}{ds}G(X(s))\right\| ds.
\end{aligned}
$$

Hence from Lemma 2.6 together with, condition (H_1), we get

$$\mu(t) \leq (\Delta_{g^{-1}})^2 \int_0^t \left\| \frac{d}{ds} G\big(X(s)\big) \right\| ds < \infty.$$

Therefore, we can find a continuous function $W_1(|\Phi(0)|)$ with $0 \leq W_1(|(\Phi(0)|)$ $\leq W((\|\Phi\|)$. The existence of a continuous function $W_2(|\phi(0)|) + W_3(\|\phi\|_2)$ which satisfies the inequality $W(\Phi) \leq W_2(|\phi(0)|) + W_3(\|\phi\|_2)$, is easily verified. Thus, subject to the above discussion, it can be shown that condition (i) of Lemma (3.4) holds.

Now let $(X, Y, Z) = (X_t, Y_t, Z_t)$ be any solution of differential system (3.3). Differentiating the function $V_1 = V_1(X_t, Y_t, Z_t))$ with respect to t along system (3.3) and using Lemma 2.3

$$\begin{aligned}
\frac{d}{dt} V_1(X_t, Y_t, Z_t) &= -2 \left\langle \left[G^{-1}BG^{-1} - \beta J_h\left(X\right) G^{-1} - \omega_0 r\left(t\right) I\right] Y, Y \right\rangle \\
&\quad + \beta \left\langle Y, B\theta(t)Y \right\rangle - 2 \left\langle [\beta A - I]\, \theta(t)Y, Z \right\rangle - 2 \left\langle [\beta A - I]\, G^{-1}Z, Z \right\rangle \\
+ 2\beta \int_{t-r(t)}^t & \left\langle J_h\left(X\right) G^{-1} Y\left(s\right), Z\left(t\right) \right\rangle ds + 2 \int_{t-r}^t \left\langle J_h\left(X\right) G^{-1} Y\left(s\right), G^{-1} Y\left(t\right) \right\rangle ds \\
&\quad - \omega_0 \left(1 - r'\left(t\right)\right) \int_{t-r(t)}^t \left\langle Y\left(\eta\right), Y\left(\eta\right) \right\rangle d\eta. \qquad (3.9)
\end{aligned}$$

Then, by (3.8), Lemma 2.1 and Lemma 2.2, and the identity $2\left|\langle U, V\rangle\right| \leq \|U\|^2 + \|V\|^2$, we obtain,

$$\begin{aligned}
&2\beta \int_{t-r}^t \left\langle J_h\left(X\right) G^{-1} Y\left(s\right), Z\left(t\right) \right\rangle ds \\
&\leq 2\beta \int_{t-r(t)}^t \|Z(t)\| \left\|J_h\left(X\right) G^{-1} Y\left(s\right)\right\| ds \\
&\leq \beta \Delta_h \delta_g^{-1} \int_{t-r(t)}^t (\|Z(t)\|^2 + \|Y\left(s\right)\|^2) ds \\
&\leq \beta \Delta_h \delta_g^{-1} \gamma \|Z(t)\|^2 + \beta \Delta_h \delta_g^{-1} \int_{t-r(t)}^t \|Y\left(s\right)\|^2 ds,
\end{aligned}$$

and

$$\begin{aligned}
&2 \int_{t-r}^t \left\langle J_h\left(X\right) G^{-1} Y\left(s\right), G^{-1} Y\left(t\right) \right\rangle ds \\
&\leq 2 \int_{t-r(t)}^t \left\|G^{-1} Y(t)\right\| \left\|J_h\left(X\right) G^{-1} Y\left(s\right)\right\| ds \\
&\leq \Delta_h \delta_g^{-2} \int_{t-r(t)}^t (\|Y(t)\|^2 + \|Y\left(s\right)\|^2) ds \\
&\leq \Delta_h \delta_g^{-2} \gamma \|Y\|^2 + \Delta_h \delta_g^{-2} \int_{t-r(t)}^t \|Y\left(s\right)\| ds,
\end{aligned}$$

from which we deduce that

$$\begin{aligned}\frac{d}{dt}V_1(X_t,Y_t,Z_t) &\le -2\left[\Delta_g^{-2}\delta_b-\beta\Delta_h\delta_g^{-1}-\omega_0\gamma-\frac{1}{2}\Delta_h\delta_g^{-2}\gamma\right]\|Y\|^2\\ &-2\left[\beta\delta_a-1-\frac{1}{2}\beta\Delta_h\delta_g^{-1}\gamma\right]\Delta_g^{-1}\|Z\|^2+\left(\frac{\beta^2\Delta_a^2+3}{k}+\frac{\beta\Delta_b}{k}\right)\|\theta(t)\|V_1\\ &+\left(-\omega_0(1-\xi)+\Delta_h\delta_g^{-2}+\beta\Delta_h\delta_g^{-1}\right)\int_{t-r(t)}^{t}\langle Y(\eta),Y(\eta)\rangle\,d\eta. \quad (3.10)\end{aligned}$$

Choosing $\omega_0=\dfrac{\left[\beta+\delta_g^{-1}\right]\Delta_h\delta_g^{-1}}{1-\xi}$, we get

$$\frac{d}{dt}V_1(X_t,Y_t,Z_t)\le -M_1\|Z\|^2-M_2\|Y\|^2+\frac{1}{\rho_0}\|\theta\|V_1, \quad (3.11)$$

where,

$$M_1=\left[2(\beta\delta_a-1)-\beta\Delta_h\delta_g^{-1}\gamma\right]>0,$$

and

$$M_2=\left[2\Delta_g^{-2}\delta_b-2\beta\Delta_h\delta_g^{-1}-2\omega_0\gamma-\Delta_h\delta_g^{-2}\gamma\right]>0$$

From (3.4), we have

$$\frac{d}{dt}W_1(X_t,Y_t,Z_t)=\left(\frac{d}{dt}V_1-\frac{1}{\rho_0}\|\theta\|V_1\right)e^{-\frac{1}{\rho_0}\mu(t)}.$$

Using (3.11) and (3.8) by choosing $\rho_0=\frac{k}{2k_1}$, such that

$$k_1=\max\left\{\beta\Delta_b,\beta^2\Delta_a^2+3\right\},$$

we get

$$\frac{d}{dt}W_1\le\left(-M_1\|Z\|^2-M_2\|Y\|^2\right)e^{-\frac{1}{\rho_0}\int_0^t\|\theta\|ds}.$$

From (H_1) we obtain

$$e^{-\frac{1}{\rho_0}\int_0^t\|\theta\|ds}<e^{-\frac{1}{\rho_0}N},$$

for some positive constant N. Hence

$$\frac{d}{dt}W_1\le -D_2\left(\|Y\|^2+\|Z\|^2\right)e^{-\frac{1}{\rho_0}N},$$

where $D_2=\min\{M_1,M_2\}$. Consequently, it follows that $\frac{d}{dt}W(X_t,Y_t,Z_t)=0$ if and only if $X=Y=Z=0$, and $\frac{d}{dt}W(\Phi)<0$ for, $\Phi\neq 0$. Thus, all the conditions of Lemma 3.4 are satisfied. This shows that the zero solution of system (3.3) is uniformly asymptotically stable. □

Example 3.6. As a special case of the following equation

$$\Big(G\big(X(t)\big)X'(t)\Big)'' + AX''(t) + BX'(t) + H\Big(X\big(t-r\big)\Big) = P\left(t\right) \tag{3.12}$$

with $P(t) = 0$, let us take n = 2 such that

$$\begin{aligned} X(t) &= \begin{pmatrix} x\left(t\right) \\ y\left(t\right) \end{pmatrix}, \quad G\big(X(t)\big) = \begin{pmatrix} g_{11}(x(t)) & 0 \\ 0 & g_{22}(y(t)) \end{pmatrix}, \\ A &= \begin{pmatrix} 2 & 0 \\ 0 & \frac{5}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 80 & 0 \\ 0 & 81 \end{pmatrix} \end{aligned}$$

where

$$g_{11}(x(t)) = \frac{\sin(x(t))}{(1+x^2(t))} + 3, \qquad g_{22}(y(t)) = \frac{\cos(y(t))}{(1+y^2(t))} + 5,$$

and

$$H\left(X\left(t-r\left(t\right)\right)\right) = \begin{pmatrix} Nx\left(t-r\left(t\right)\right)e^{-x^2(t-r(t))+1} + Mx\left(t-r\left(t\right)\right) \\ Ny\left(t-r\left(t\right)\right)e^{-y^2(t-r(t))+1} + My\left(t-r\left(t\right)\right) \end{pmatrix}$$

Thus

$$\begin{aligned} &J_h\left(X(t)\right) \\ &= \begin{pmatrix} M + Ne^{1-x^2(t)} - 2Nx^2(t)e^{1-x^2(t)} & 0 \\ 0 & M + Ne^{1-y^2(t)} - 2Ny^2(t)e^{1-y^2(t)} \end{pmatrix}, \end{aligned}$$

with

$$N = \frac{1}{5\left(e + 2e^{-\frac{1}{2}}\right)} = 5.087\,3 \times 10^{-2}$$

and

$$M = \frac{1}{10}\frac{e}{e+2e^{-\frac{1}{2}}} + \frac{3}{5}\frac{e^{-\frac{1}{2}}}{e+2e^{-\frac{1}{2}}} = 0.161\,71.$$

Clearly, $G\left(X\right), A, B$ and $J_h\left(X\right)$ are diagonal matrices, hence they are symmetric and commute pairwise. Then, by an easy calculation, we obtain eigenvalues of the matrices G, A, B and $J_h(X)$ as follows:

$$\begin{aligned} \lambda_1\left(G\right) &= \frac{\sin x}{(1+x^2)} + 3, \quad \lambda_2\left(G\right) = \frac{\cos x}{(1+x^2)} + 5, \\ \lambda_1\left(A\right) &= 2, \quad \lambda_2\left(A\right) = \frac{5}{2}, \quad \lambda_1\left(B\right) = 80, \quad \lambda_2\left(B\right) = 81, \\ \lambda_1\left(J_h(X)\right) &= M + Ne^{1-x^2(t)} - 2Nx^2\left(t\right)e^{1-x^2(t)}, \\ \lambda_2\left(J_h(X)\right) &= M + Ne^{1-y^2(t)} - 2Ny^2\left(t\right)e^{1-y^2(t)}. \end{aligned}$$

It follows easily that

$$\delta_g = 2,\ \Delta_g = 6,\ \delta_a = 2,\ \Delta_a = 2.5,\ \delta_b = 80,\ \Delta_b = 81,\ \delta_h = \frac{1}{10},\ \Delta_h = \frac{3}{10}.$$

Choose $\beta = 14.5$, we have $\frac{1}{2} = \frac{1}{\delta_a} < \beta < \frac{\Delta_g^{-2}\delta_b}{\Delta_h \delta_g^{-1}} = 400$. If we take $\xi = \frac{1}{2}$, we must have that $\gamma < \min\left\{25.747, 1.064\,2 \times 10^{-2}\right\}$. A trivial verification shows that G is nonsingular matrix and we have

$$\frac{d}{dt}G(X(t)) = \begin{pmatrix} \frac{d}{dt}g_{11}(x(t)) & 0 \\ 0 & \frac{d}{dt}g_{22}(y(t)) \end{pmatrix},$$

where

$$\begin{aligned}
\frac{d}{dt}g_{11}(x(t)) &= \left(\frac{\cos(x(t))}{10\,(1+x^2(t))} - \frac{2x(t)\sin(x(t))}{10\,(1+x^2(t))^2}\right)x'(t),\\
\frac{d}{dt}g_{22}(y(t)) &= \left(\frac{-\sin(y(t))}{10\,(1+y^2(t))} - \frac{2y\cos(y(t))}{10\,(1+y^2(t))^2}\right)y'(t).
\end{aligned}$$

Thus

$$\|\frac{d}{dt}G(X(t))\| = \max\left\{\left|\frac{d}{dt}g_{11}(x(t))\right|, \left|\frac{d}{dt}g_{22}(y(t))\right|\right\} = D(t),$$

and

$$\begin{aligned}
\|\theta(t)\| &= \|G^{-1}\big(X(t)\big)\frac{d}{dt}G\big(X(t)\big)G^{-1}\big(X(t)\big)\|\\
&\leq \|G^{-2}\big(X(t)\big)\|\|\frac{d}{dt}G\big(X(t)\big)\| = \left(\frac{1}{\delta_G}\right)^2 D(t).
\end{aligned}$$

For $t \in [0, +\infty)$ a straightforward calculation give

$$\begin{aligned}
&\int_0^t \|\theta(s)\|ds \leq \frac{1}{4}\int_0^t D(s)ds\\
&= \frac{1}{4}\int_0^t \max\left\{\left|\frac{d}{ds}g_{11}(x(s))\right|, \left|\frac{d}{ds}g_{22}(y(s))\right|\right\}ds\\
&\leq \frac{1}{4}\left(\int_0^t \left|\left(\frac{\cos x}{1+x^2} - \frac{2x\sin x}{(1+x^2)^2}\right)x'(s)\right| + \left|\left(\frac{-\sin y}{1+y^2} - \frac{2y\cos y}{(1+y^2)^2}\right)y'(s)\right| ds\right)\\
&\leq \frac{1}{4}\left(\int_{\omega_1(t)}^{\omega_2(t)} \left|\left(\frac{\cos u}{1+u^2} - \frac{2u\sin u}{(1+u^2)^2}\right)du\right| + \int_{\varphi_1(t)}^{\varphi_2(t)} \left|\left(\frac{-\sin v}{1+v^2} - \frac{2v\cos v}{(1+v^2)^2}\right)dv\right|\right)\\
&< \frac{1}{4}\left(\int_{-\infty}^{+\infty} \left|\frac{1+u^2+2u}{(1+u^2)^2}\right| du + \int_{-\infty}^{+\infty} \left|\frac{1+u^2+2u}{(1+u^2)^2}\right| du\right)\\
&= \frac{1}{2}(\pi+2).
\end{aligned}$$

where $\omega_1(t) = \min\{x(0), x(t)\}$, $\omega_2(t) = \max\{x(0), x(t)\}$ and $\varphi_1(t) = \min\{y(0), y(t)\}$, $\varphi_2(t) = \max\{y(0), y(t)\}$. We take $r(t) = \exp(-t^2)$, then $0 \le r(t) \le \gamma$, $(\gamma > 0)$, and $r'(t) = -2t\exp\left(-t^2\right) \le \xi$ for $0 < \xi < 1$. Thus, all the conditions of Theorem 3.5 are satisfied.

4 Boundedness

First, consider a system of delay differential equations

$$x' = F(t, x_t), \quad x_t(\theta) = x(t+\theta), \ -r \le \theta \le 0, \ t \ge 0, \tag{4.1}$$

where $F\colon \mathbb{R} \times C_H \longrightarrow \mathbb{R}^n$ is a continuous mapping and takes bounded set into bounded sets.
The following lemma is a well-known result obtained by Burton [6].

Lemma 4.1. [3, 6, 7, 20, 21] *Let $V(t, \phi)\colon \mathbb{R} \times C_H \longrightarrow \mathbb{R}$ be a continuous and local Lipschitz in ϕ. If*
(i) $W(|x(t)|) \le V(t, x_t) \le W_1(|x(t)|) + W_2\left(\int_{t-r(t)}^{t} W_3(|x(s)|)\, ds\right)$,
(ii) $V'_{(4.1)} \le W_3(|x(s)|) + M$.
for some $M > 0$, where $W(r), W_i\,(i = 1, 2, 3)$ are wedges, then the solutions of (4.1) are uniformly bounded and uniformly ultimately bounded for bound B.

To study the boundedness of solutions of (1.2) for which $P(t) \ne 0$, we would need to write (1.2) in the form

$$\begin{cases} X' = G^{-1}(X)Y; \quad Y' = Z; \\ Z' = -AG^{-1}(X)Z - \left[A(G^{-1})' + BG^{-1}\right]Y - H(X) \\ \qquad + \int_{t-r(t)}^{t} J_h(X)G^{-1}(X)Y\, ds + P(t). \end{cases} \tag{4.2}$$

Thus our main theorem in this section is stated with respect to (4.2) as follows:

Theorem 4.2. *Assume that all the conditions of Theorem 3.5 are satisfied and*

$$\|P(t)\| \le m, \tag{4.3}$$

where m is a positive constant. Then all solutions of (4.2) are uniformly bounded and uniformly ultimately bounded provided that

$$\gamma < \min\left\{\eta_1, \ \eta_2, \ \eta_3\right\}, \tag{4.4}$$

where

$$\begin{aligned} \eta_1 &= \frac{\Delta_g^{-2}\delta_b - \beta\Delta_h\delta_g^{-1}}{\left(\omega + \Delta_h\delta_g^{-2} + \frac{1}{2}\Delta_h\Delta_a^2\delta_g^{-2}\right)}, \\ \eta_2 &= \frac{2\left(\beta\delta_a - 1\right)\Delta_g^{-1} - \left(\delta_g^{-1} + \Delta_a\right)^2}{\beta\Delta_h\delta_g^{-1}}, \\ \eta_3 &= \frac{2\left(\delta_a\delta_b - \Delta_h\right)\delta_h - \left(\delta_g^{-1} - \delta_a\right)\Delta_h^2}{\left(\Delta_a\Delta_b - \Delta_h\right)\Delta_h\delta_g^{-1}}. \end{aligned}$$

Proof. Consider the function

$$V(X_t, Y_t, Z_t) = V_1(X_t, Y_t, Z_t) + V_2(X_t, Y_t, Z_t), \tag{4.5}$$

where $V_1(X_t, Y_t, Z_t)$ is defined as 3.6 (with replacing ω_0 by ω_1.) and $V_2(X_t, Y_t, Z_t)$ defined as

$$\begin{aligned} &V_2(X_t, Y_t, Z_t) \\ &= 2\int_0^1 \langle AH(\sigma X), AX\rangle\, d\sigma + \langle B(AB - \Delta_h I)X, X\rangle + 2\left\langle G^{-1}Y, H(X)\right\rangle \\ &\quad + \left\langle \Delta_h G^{-1}Y, Y\right\rangle + 2\left\langle (AB - \Delta_h I)X, Z + AG^{-1}Y\right\rangle \\ &\quad + \left\langle A\left(Z + AG^{-1}Y\right), Z + AG^{-1}Y\right\rangle. \end{aligned} \tag{4.6}$$

We observe that the function V_2 can be rewritten as follows

$$\begin{aligned} &V_2(X_t, Y_t, Z_t) \\ &= 2\int_0^1 \langle AH(\sigma X), AX\rangle\, d\sigma + \left\langle G^{-1}(\Delta_h Y + H(X)), Y + \Delta_h^{-1}H(X)\right\rangle \\ &\quad - \left\langle G^{-1}H(X), \Delta_h^{-1}H(X)\right\rangle + \left\langle \Delta_h A^{-1}(AB - \Delta_h I)X, X\right\rangle \\ &\quad + \Big\langle (AB - \Delta_h I)X + A\left(Z + AG^{-1}Y\right), A^{-1}(AB - \Delta_h I)X + Z + AG^{-1}Y\Big\rangle. \end{aligned}$$

However,

$$\langle H(X), H(X)\rangle = 2\int_0^1\int_0^1 \sigma\,\langle J_h(\sigma X)\, J_h(\sigma\tau X)\, X, X\rangle\, d\tau d\sigma.$$

Using Lemma 2.1, Lemma 2.2 and Lemma 2.4, we obtain

$$\begin{aligned} V_2(X_t, Y_t, Z_t) &\geq 2\int_0^1\int_0^1 \left\langle \sigma\left\{\delta_a^2 - \delta_g^{-1}\right\} J_h(\tau\sigma X)\, X, X\right\rangle d\sigma d\tau \\ &\quad + \Delta_h\Delta_g^{-1}\left\|Y + \Delta_h^{-1}AH(X)\right\|^2 + \Delta_h\Delta_a^{-1}(\delta_a\delta_b - \Delta_h)\|X\|^2 \\ &\quad + \left\|A^{-\frac{1}{2}}(AB - \Delta_h I)X + A^{\frac{1}{2}}Z + A^{\frac{3}{2}}G^{-1}Y\right\|^2 \\ &\geq \Delta_h\Delta_g^{-1}\left\|Y + \Delta_h^{-1}AH(X)\right\|^2 + \Delta_h\Delta_a^{-1}(\delta_a\delta_b - \Delta_h)\|X\|^2 \\ &\quad + \left\|A^{-\frac{1}{2}}(AB - \Delta_h I)X + A^{\frac{1}{2}}Z + A^{\frac{3}{2}}G^{-1}Y\right\|^2. \end{aligned}$$

It follows that $V_2(X_t, Y_t, Z_t)$ is positive definite. From (3.9), (3.10), (3.11), (4.2) and Lemma 2.3, we find

$$\begin{aligned}
&\frac{d}{dt}V_1(X_t,Y_t,Z_t)\\
&\le -2\left[\Delta_g^{-2}\delta_b-\beta\Delta_h\delta_g^{-1}-\omega_1\gamma-\Delta_h\delta_g^{-2}\gamma\right]\|Y\|^2\\
&\quad-\left[2\left(\beta\delta_a-1\right)\Delta_g^{-1}-\beta\Delta_h\delta_g^{-1}\gamma\right]\|Z\|^2\\
&\quad+\left[\beta\Delta_h\delta_g^{-1}+\Delta_h\delta_g^{-2}-\omega_1\left(1-\xi\right)\right]\int_{t-r(t}^{t}\langle Y\left(s\right),Y\left(s\right)\rangle\,ds\\
&\quad+\left(\frac{\beta^2\Delta_a^2+\beta\Delta_b+3}{k_1}\right)\|\theta\|\,V+\left[2\beta\|Z\|+\delta_g^{-1}\|Y\|\right]m.
\end{aligned}\tag{4.7}$$

Also from (4.2), (4.6) and Lemma 2.3 we obtain,

$$\begin{aligned}
&\frac{d}{dt}V_2(X_t,Y_t,Z_t)\\
&=-2\int_0^1\langle\left(AB-\Delta_hI\right)J_h\left(\sigma X\right)X,H\left(X\right)\rangle\,d\sigma\\
&\quad+2\left\langle\left(J_h\left(X\right)-\Delta_hI\right)G^{-1}Y,AG^{-1}Y\right\rangle+2\int_{t-r(t)}^{t}\left\langle A^2G^{-1}Y,J_h\left(X\right)G^{-1}Y\right\rangle ds\\
&\quad+2\int_{t-r(t)}^{t}\left\langle\left(AB-\Delta_hI\right)X,J_h\left(X\right)G^{-1}Y\right\rangle ds+2\left\langle\left(G^{-1}-A\right)Z,H\left(X\right)\right\rangle\\
&\quad+\langle\Delta_h\theta Y,Y\rangle+2\langle\theta Y,H\left(X\right)\rangle+2\left\langle A\left(Z+AG^{-1}Y\right),P\left(t\right)\right\rangle\\
&\quad+\langle 2\left(AB-\Delta_hI\right)X,P\left(t\right)\rangle.
\end{aligned}\tag{4.8}$$

Therefore, from (4.5), (4.7) and (4.8), we obtain

$$\begin{aligned}
&\frac{d}{dt}V(X_t,Y_t,Z_t)\\
&\le-\left[2\left(\delta_a\delta_b-\Delta_h\right)\delta_h-\left(\Delta_a\Delta_b-\Delta_h\right)\Delta_h\delta_g^{-1}r\left(t\right)-\left(\delta_g^{-1}-\delta_a\right)\Delta_h^2\right]\|X\|^2\\
&\quad-2\left[\Delta_g^{-2}\delta_b-\beta\Delta_h\delta_g^{-1}-\omega_1\gamma-\Delta_h\delta_g^{-2}\gamma-\frac{1}{2}\Delta_h\Delta_a^2\delta_g^{-2}r\left(t\right)\right]\|Y\|^2\\
&\quad-\left[2\left(\beta\delta_a-1\right)\Delta_g^{-1}-\beta\Delta_h\delta_g^{-1}\gamma-\left(\delta_g^{-1}-\delta_a\right)\right]\|Z\|^2\\
&\quad+\left[\eta_4-\omega_1\left(1-\xi\right)\right]\int_{t-r(t)}^{t}\langle Y\left(s\right),Y\left(s\right)\rangle\,ds+k_3\|\theta\|\,V\\
&\quad+\left[2\left(\beta+\Delta_a\right)\|Z\|+\left(\delta_g^{-1}+\delta_g^{-1}\Delta_a^2\right)\|Y\|+\left(\Delta_a\Delta_b-\Delta_h\right)\|X\|\right]m,
\end{aligned}$$

where

$$\eta_4=\Delta_h\Delta_a^2\delta_g^{-2}+\Delta_h\delta_g^{-1}\delta_a\delta_b-\delta_g^{-1}\Delta_h^2+\beta\Delta_h\delta_g^{-1}+\Delta_h\delta_g^{-2},$$

and

$$k_3=\left(\frac{\beta^2\Delta_a^2+\beta\Delta_b+3}{k_1}+\frac{\left(\Delta_h+1\right)+\Delta_h^2}{k_2}\right).$$

If we choose $\omega_1 = \dfrac{\eta_4}{(1-\xi)}$, then

$$\frac{d}{dt}V(X_t,Y_t,Z_t) \leq -\left[\eta_3-\gamma\right]\left(\Delta_a\Delta_b-\Delta_h\right)\Delta_h\delta_g^{-1}\left\|X\right\|^2$$
$$-2\left[\eta_1-\gamma\right]\left(\omega_1+\Delta_h\delta_g^{-2}+\frac{1}{2}\Delta_h\Delta_a^2\delta_g^{-2}\right)\left\|Y\right\|^2-\left[\eta_2-\gamma\right]\beta\Delta_h\delta_g^{-1}\left\|Z\right\|^2$$
$$+k_3\left\|\theta\right\|V+\left[2\left(\beta+\Delta_a\right)\left\|Z\right\|+\left(\delta_g^{-1}+\delta_g^{-1}\Delta_a^2\right)\left\|Y\right\|+\left(\Delta_a\Delta_b-\Delta_h\right)\left\|X\right\|\right]m.$$

Consider the function W defined by

$$W(X_t,Y_t,Z_t)=V(X_t,Y_t,Z_t)e^{-\frac{1}{\rho_1}\int_0^t\|\theta(s)\|ds}.$$

W is positive definite and we have

$$\frac{d}{dt}W(X_t,Y_t,Z_t)=\left(\frac{d}{dt}V(X_t,Y_t,Z_t)-\frac{1}{\rho_1}\left\|\theta\right\|V\right)e^{-\frac{1}{\rho_1}\int_0^t\|\theta(s)\|ds}.$$

By taking $k_3=\frac{1}{\rho_1}$ and using the fact that γ satisfies (4.4), it follows that there exists a positive constant D such that

$$\begin{aligned}\frac{d}{dt}W(X_t,Y_t,Z_t) &\leq -D\left[\left\|X\right\|^2+\left\|Y\right\|^2+\left\|Z\right\|^2\right]+\omega_1 D\left[\left\|X\right\|+\left\|Y\right\|+\left\|Z\right\|\right]\\ &\leq -\frac{1}{2}D\left\|X\right\|^2-\frac{1}{2}D\left(\left(\left\|X\right\|-\omega_1\right)^2-\omega_1^2\right)-D\left[\left\|Y\right\|^2+\left\|Z\right\|^2\right]\\ &\quad +\omega_1 D\left[\left\|Y\right\|+\left\|Z\right\|\right].\end{aligned}$$

By the equality

$$-D\left\|X\right\|^2+\omega_1 D\left\|X\right\|=-\frac{1}{2}D\left\|X\right\|^2-\frac{1}{2}D\left(\left(\left\|X\right\|-\omega_1\right)^2-\omega_1^2\right),$$

we obtain

$$\begin{aligned}&\frac{d}{dt}W(X_t,Y_t,Z_t)\\ &\leq -\frac{1}{2}D\left\|X\right\|^2-\frac{1}{2}D\left(\left\|X\right\|-\omega_1\right)^2+\frac{1}{2}D\omega_1^2-\frac{1}{2}D\left\|Z\right\|^2-\frac{1}{2}D\left(\left\|Y\right\|-\omega_1\right)^2\\ &\quad +\frac{1}{2}D\omega_1^2-\frac{1}{2}D\left\|Z\right\|^2-\frac{1}{2}D\left(\left\|Z\right\|-\omega_1\right)^2+\frac{1}{2}D\omega_1^2\\ &\leq -\frac{1}{2}D\left(\left\|X\right\|^2+\left\|Y\right\|^2+\left\|Z\right\|^2\right)\\ &\quad -D\left(\left(\left\|X\right\|-\omega_1\right)^2+\left(\left\|Y\right\|-\omega_1\right)^2+\left(\left\|Z\right\|-\omega_1\right)^2\right)+\frac{3}{2}D\omega_1^2\\ &\leq -\frac{1}{2}D\left(\left\|X\right\|^2+\left\|Y\right\|^2+\left\|Z\right\|^2\right)+\frac{3}{2}D\omega_1^2 \text{ for } \quad D,\omega_1>0.\end{aligned}$$

This completes the proof. □

Example 4.3. Now, as a special case of system (3.3) (for $P(t) \neq 0$), let us take $n = 2$ that G, A, B, $H(X(t - r(t)))$ defined in Example 1 hold. If we take $r(t) = \exp(-t^2)$, then $0 \leq r(t) \leq \gamma$, $(\gamma > 0)$, and that $r'(t) = -2t\exp(-t^2) \leq \xi$ for $0 < \xi < 1$. Let

$$P(t) = \begin{pmatrix} \frac{\cos t}{1+t^2} \\ \frac{\sin t}{1+t^2} \end{pmatrix}, \quad \beta = 14.5 \quad \text{and} \quad \xi = \frac{1}{2}.$$

We have that

$$\|P(t)\| \leq \frac{2}{1+t^2} \leq 2 \quad \text{and} \quad \gamma < \left\{0.153\,26, 8.800\,7 \times 10^{-4}, 1.057\,5\right\}.$$

Thus, all the conditions of Theorem 4.2 are satisfied.

5 Conclusion

The problem of the stability and boundedness of solutions of differential equations is very important in the theory and applications of differential equations. In the present work we consider a nonlinear vector differential equation of third order with variable delay which include many particular cases of delay differential equations. We discuss the stability of zero solution and uniformly boundedness and uniformly ultimately boundedness of all solutions. Using the Lyapunov–Krasovskiĭ functional approach, two new results are given on the topic and an example is given for the related illustrations.

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References

[1] Afuwape, A. U.: *Ultimate boundedness results for a certain system of third-order nonlinear differential equations*. J. Math. Anal. Appl. **97** (1983), 140–150.

[2] Afuwape, A. U., Omeike, M. O.: *Further ultimate boundedness of solutions of some system of third-order nonlinear ordinary differential equations*. Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math. **43** (2004), 7–20.

[3] Afuwape, A. U., Omeike, M. O.: *On the stability and boundedness of solutions of a kind of third order delay differential equations*. Applied Mathematics and Computation **200** (2000), 444–451.

[4] Afuwape, A. U., Carvajal, Y. E.: *Stability and ultimate boundedness of solutions of a certain third order nonlinear vector differential equation*. J. Nigerian Math. Soc. **31** (2012), 69–80.

[5] Burton, T. A.: Volterra Integral and Differential Equations. 2nd ed., Mathematics in Science and Engineering, *Elsevier*, New York, 2005.

[6] Burton, T. A.: Stability and Periodic Solutions of Ordinary and Functional Differential Equations. *Academic Press*, Orlando, 1985.

[7] Burton, T. A., Zhang, S.: *Unified boundedness, periodicity and stability in ordinary and functional differential equations*. Ann. Math. Pura Appl. **145** (1986), 129–158.

[8] Ezeilo, J. O. C.: *n-dimensional extensions of boundedness and stability theorems for some third-order differential equations*. J. Math. Anal. Appl. **18** (1967), 395–416.

[9] Ezeilo, J. O. C., Tejumola, H. O.: *Boundedness and periodicity of solutions of a certain system of third-order nonlinear differential equations*. Ann. Math. Pura Appl. **74** (1966), 283–316.

[10] Ezeilo, J. O. C., Tejumola, H. O.: *Further results for a system of third-order ordinary differential equations*. Atti. Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **58** (1975), 143–151, 283–316.

[11] Graef, J. R., Beldjerd, D., Remili, M.: *On stability, ultimate boundedness, and existence of periodic solutions of certain third order differential equations with delay*. PanAmerican Mathematical Journal **25** (2015), 82–94.

[12] Graef, J. R., Oudjedi, D., Remili, M.: *Stability and square integrability of solutions of nonlinear third order differential equations*. Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis **22** (2015), 313–324.

[13] Hale, J. K.: Theory of Functional Differential Equations. *Springer Verlag*, New York, 1977.

[14] Mahmoud, A. M., Tunç, C.: *Stability and boundedness of solutions of a certain n-dimensional nonlinear delay differential system of third-order*. Adv. Pure Appl. Math. **7**, 1 (2016), 1–11.

[15] Meng, F. W.: *Ultimate boundedness results for a certain system of third-order nonlinear differential equations*. J. Math. Anal. Appl. **177** (1993), 496–509.

[16] Omeike, M. O.: *Stability and boundedness of solutions of a certain system of third-order nonlinear delay differential equations*. Acta Univ. Palacki. Olomuc., Fac. rer. nat., Math. **54**, 1 (2015), 109–119.

[17] Remili, M., Beldjerd, D.: *On the asymptotic behavior of the solutions of third order delay differential equations*. Rend. Circ. Mat. Palermo, **63**, 3 (2014), 447–455.

[18] Remili, M., Oudjedi, D. L.: *Stability and boundedness of the solutions of non autonomous third order differential equations with delay*. Acta Univ. Palacki. Olomuc., Fac. Rer. Nat., Math. **53**, 2 (2014), 139–147.

[19] Remili, M., Oudjedi, D. L.: *Uniform stability and boundedness of a kind of third order delay differential equations*. Bull. Comput. Appl. Math., **2**, 1 (2014), 25–35.

[20] Sadek, A. I.: *Stability and boundedness of a kind of third-order delay differential system*. Applied Mathematics Letters, **16** (2003), 657–662.

[21] Sadek, A. I.: *On the stability of solutions of certain fourth order delay differential equations*. Applied Mathematics and Computation, **148** (2004), 587–597.

[22] Tiryaki, A.: *Boundedness and periodicity results for a certain system of third-order nonlinear differential equations*. Indian J. Pure Appl. Math., **30**, 4 (1999), 361–372.

[23] Tunç, C., Gozen, M.: *Convergence of solutions to a certain vector differential equation of third order*. Abstract and Applied Analysis, **2014**, ID 424512 (2014), 1–6.

[24] Tunç, C.: *On the boundedness of solutions of certain nonlinear vector differential equations of third order*. Bull. Math. Soc. Sci. Math. Roumanie (N.S.), **49(97)**, 3 (2006), 291–300.

[25] Tunç, C.: *On the stability and boundedness of solutions of nonlinear vector differential equations of third order*. Nonlinear Anal., **70**, 6 (2009), 2232–2236.

[26] Tunç, C.: *On the qualitative properties of differential equations of third order with retarded argument*. Proyecciones, **33**, 3 (2014), 325–347.

[27] Tunç, C.: *New ultimate boundedness and periodicity results for certain third-order nonlinear vector differential equations*. Math. J. Okayama Univ., **48** (2006), 159–172.

[28] Yoshizawa, T.: Stability Theory by Liapunov's Second Method. *The Mathematical Society of Japan*, Tokyo, 1996.

[29] Zhu, Y.: *On stability, boundedness and existence of periodic solution of a kind of thirdorder nonlinear delay differential system*. Ann. Diff. Eqns., **8**, 2 (1992), 249–259.