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Conformal Ricci Soliton in Lorentzian α-Sasakian Manifolds

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Abstract

In this paper we have studied conformal curvature tensor, conharmonic curvature tensor, projective curvature tensor in Lorentzian α-Sasakian manifolds admitting conformal Ricci soliton. We have found that a Weyl conformally semi symmetric Lorentzian α-Sasakian manifold admitting conformal Ricci soliton is η-Einstein manifold. We have also studied conharmonically Ricci symmetric Lorentzian α-Sasakian manifold admitting conformal Ricci soliton. Similarly we have proved that a Lorentzian α-Sasakian manifold M with projective curvature tensor admitting conformal Ricci soliton is η-Einstein manifold. We have also established an example of 3-dimensional Lorentzian α-Sasakian manifold.

Key words: Conformal Ricci soliton, conformal curvature tensor, conharmonic curvature tensor, Lorentzian α-Sasakian manifolds, projective curvature tensor.

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1 Introduction

In 1982 Hamilton [11] introduced the concept of Ricci flow and proved its existence. This concept was developed to answer Thurston's geometric conjecture

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which says that each closed three manifold admits a geometric decomposition. Hamilton also [12] classified all compact manifolds with positive curvature operator in dimension four. Since then, the Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature.

The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S \tag{1.1}$$

on a compact Riemannian manifold M with Riemannian metric g. Ricci soliton emerges as the limit of the solutions of Ricci flow. A solution to the Ricci flow is called a Ricci soliton if it moves only by a one-parameter group of diffeomorphism and scaling. Ramesh Sharma [28] started the study of Ricci soliton in contact manifolds and after him M. M. Tripathi [31], Bejan, Crasmareanu [4] studied Ricci soliton in contact metric manifolds. The Ricci soliton equation is given by

$$\pounds_X g + 2S + 2\lambda g = 0, \tag{1.2}$$

where $\pounds_X$ is the Lie derivative, S is Ricci tensor, g is Riemannian metric, X is a vector field and λ is a scalar. The $\varphi-$ vector fields are special type Ricci soliton studied in [14, 15].

In 2005, A.E. Fischer [9] introduced a new concept called conformal Ricci flow which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. Since the conformal geometry plays an important role to constrain the scalar curvature and the equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equations are named as the conformal Ricci flow equations. These new equations are given by

$$\frac{\partial g}{\partial t} + 2\left(S + \frac{g}{n}\right) = -pg \tag{1.3}$$

and $R(g) = -1$, where p is a scalar non-dynamical field(time dependent scalar field), $R(g)$ is the scalar curvature of the manifold and n is the dimension of manifold.

In 2015, N. Basu and A. Bhattacharyya [3] introduced the notion of conformal Ricci soliton and the equation is as follows

$$\pounds_X g + 2S = \left[2\lambda - \left(p + \frac{2}{n}\right)\right] g. \tag{1.4}$$

The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

A Riemannian manifold is said to be locally symmetric if its curvature tensor R satisfies $\nabla R = 0$, where ∇ is Levi-Civita connection on the Riemannian manifold. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and their generalization. A Riemannian manifold is said to be semi symmetric if its curvature tensor R satisfies

$R(X,Y).R = 0$ for all $X, Y \in TM$, where $R(X,Y)$ acts on R as a derivation. N. S. Sinyukov, J. Mikeš, I. Hinterleitner and others studied geodesic mappings of symmetric and semisymmetric spaces [29, 10, 18, 13, 19, 17, 22, 23, 24, 25, 16]. K. Sekigawa [27], Z. I. Szabo [30] studied Riemannian manifolds or hypersurfaces of such manifold satisfying the condition $R(X,Y).R = 0$ or condition similar to it. It is easy to see that $R(X,Y).R = 0$ implies $R(X,Y).C = 0$. So it is meaningful to undertake the study of manifolds satisfying such type of conditions.

1.1 Definition of Einstein manifold

An Einstein manifold is a Riemannian or pseudo-Riemannian manifold with Ricci tensor is proportional to the metric. If M is the underlying n-dimensional manifold and g is its metric tensor then the Einstein condition means that

$$S(X,Y) = \lambda g(X,Y),$$

for some constant λ, where S denotes the Ricci tensor of g. Einstein manifolds with $\lambda = 0$ are called Ricci-flat manifolds.

1.2 Definition of η-Einstein manifold

A trans-Sasakian manifold M^n is said to be η-Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

where a, b are smooth functions.

2 Basic concepts of Lorentzian α-Sasakian manifolds

A differentiable manifold of dimension $(2n+1)$ is called Lorentzian $\alpha-$Sasakian manifold [1] if it admits a $(1,1)$ tensor field φ, a vector field ξ and 1-form η and Lorentzian metric g which satisfy on M respectively such that

$$\varphi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \eta \circ \varphi = 0, \quad \varphi\xi = 0, \tag{2.1}$$

$$g(\varphi X, \varphi Y) = g(X,Y) + \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2.2}$$

$$\nabla_X \xi = \alpha\varphi X, \quad (\nabla_X \eta)Y = \alpha g(\varphi X, Y), \tag{2.3}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g on M. Geometry of Sasakian spaces was studied in [21, 20, 26, 19].

On an Lorentzian α-Sasakian manifold M the following relations hold [1]:

$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.4}$$

$$R(\xi,X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X], \tag{2.5}$$

$$S(X,\xi) = 2n\alpha^2\eta(X), \tag{2.6}$$

$$Q\xi = 2n\alpha^2\xi, \tag{2.7}$$

$$S(\xi,\xi) = -2n\alpha^2, \tag{2.8}$$

where α is some constant, R is the Riemannian curvature, S is the Ricci tensor and Q is the Ricci operator given by $S(X,Y) = g(QX,Y)$ for all $X,Y \in \chi(M)$.

Now from definition of Lie derivative we have

$$\begin{aligned}(\pounds_\xi g)(X,Y) &= (\nabla_\xi g)(X,Y) + g(\alpha\varphi X,Y) + g(X,\alpha\varphi Y)\\ &= 2\alpha g(\varphi X,Y),\ [\because g(X,\varphi Y) = g(\varphi X,Y)].\end{aligned} \tag{2.9}$$

Applying (2.9) in (1.4) we get

$$\begin{aligned}S(X,Y) &= \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right] g(X,Y) - \alpha g(\varphi X,Y)\\ &= Ag(X,Y) - \alpha g(\varphi X,Y),\end{aligned} \tag{2.10}$$

where

$$A = \frac{1}{2}\left[2\lambda - \left(p + \frac{2}{n}\right)\right].$$

Since $S(X,Y) = g(QX,Y)$ for the Ricci operator Q, we have

$$g(QX,Y) = Ag(X,Y) - \alpha g(\varphi X,Y)$$

i.e.

$$QX = AX - \alpha\varphi X, \quad \forall Y. \tag{2.11}$$

Also

$$S(Y,\xi) = A\eta(Y), \quad S(\xi,\xi) = -A, \quad Q\xi = A\xi. \tag{2.12}$$

If we put $X = Y = e_i$ in (2.10), where $\{e_i\}$ is orthonormal basis of the tangent space TM where TM is a tangent bundle of M and summing over i, we get

$$R(g) = An - \alpha g(\varphi e_i, e_i)$$

As $R = -1$, we have

$$-1 = An - \alpha.(\operatorname{tr}\varphi) \quad \text{i.e.} \quad A = \frac{1}{n}(\alpha.(\operatorname{tr}\varphi) - 1).$$

2.1 Example of a 3-dimensional Lorentzian α-Sasakian manifold

In this section we construct an example of a 3-dimensional Lorentzian α-Sasakian manifold.To construct this, we consider the three dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$ where (x, y, z) are the standard coordinates in R^3. The vector fields

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \quad e_2 = e^{-z}\frac{\partial}{\partial y}, \quad e_3 = -e^{-z}\frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Lorentzian metric defined by

$$g(e_1, e_1) = 1, \quad g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1,$$
$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let η be the 1-form which satisfies the relation $\eta(e_3) = -1$. Let φ be the $(1, 1)$ tensor field defined by $\varphi(e_1) = -e_1$, $\varphi(e_2) = -e_2$, $\varphi(e_3) = 0$. Then we have

$$\varphi^2(Z) = Z + \eta(Z)e_3,$$
$$g(\varphi Z, \varphi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M^3)$. Thus for $e_3 = \xi$, (φ, ξ, η, g) defines an almost contact metric structure on M. Now, after calculating we have

$$[e_1, e_3] = -e^{-z}e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_3] = -e^{-z}e_2.$$

The Riemannian connection ∇ of the metric is given by the Koszul's formula which is

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Y g(Z, X) - Zg(X, Y) \\ &-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{2.13}$$

By Koszul's formula we get

$$\nabla_{e_1} e_1 = -e^{-z}e_3, \quad \nabla_{e_2} e_1 = 0, \quad \nabla_{e_3} e_1 = 0,$$
$$\nabla_{e_1} e_2 = 0, \quad \nabla_{e_2} e_2 = -e^{-z}e_3, \quad \nabla_{e_3} e_2 = 0,$$
$$\nabla_{e_1} e_3 = -e^{-z}e_1, \quad \nabla_{e_2} e_3 = -e^{-z}e_2, \quad \nabla_{e_3} e_3 = 0.$$

From the above we have found that $\alpha = e^{-z}$ and it can be easily shown that $M^3(\varphi, \xi, \eta, g)$ is a Lorentzian α-Sasakian manifold.

3 Lorentzian α-Sasakian manifold admitting conformal Ricci soliton and $R(\xi, X).\tilde{C} = 0$

Let M be an $(2n + 1)$ dimensional Lorentzian α-Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). The conformal curvature tensor $\tilde{C}$ on M is defined by [2]

$$\tilde{C}(X,Y)Z = R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] + \frac{R}{2n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{3.1}$$

where R is scalar curvature.

Now we prove the following theorem:

Theorem 3.1. *If a Lorentzian α-Sasakian manifold admits conformal Ricci soliton and is Weyl conformally semi summetric i.e. $R(\xi,X).\tilde{C}=0$, then the manifold is η-Einstein manifold where $\tilde{C}$ is Conformal curvature tensor and $R(\xi,X)$ is derivation of tensor algebra of the tangent space of the manifold.*

Proof. Let M be an $(2n+1)$ dimensional Lorentzian α-Sasakian manifold admitting a conformal Ricci soliton (g,V,λ). So we have $R=-1$ [9].

After putting $R=-1$ and $Z=\xi$ in (3.1) we have

$$\tilde{C}(X,Y)\xi = R(X,Y)\xi - \frac{1}{2n-1}[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY] - \frac{1}{2n(n-1)}[g(Y,\xi)X - g(X,\xi)Y]. \tag{3.2}$$

Using (2.2), (2.4), (2.11) and (2.12) in (3.2) we get

$$\tilde{C}(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y] - \frac{1}{2n-1}[A\eta(Y)X - A\eta(X)Y + \eta(Y)(AX - \alpha\varphi X) - \eta(X)(AY - \alpha\varphi Y)] - \frac{1}{2n(n-1)}[\eta(Y)X - \eta(X)Y]. \tag{3.3}$$

Using (3.1) and after a brief simplification we obtain

$$\tilde{C}(X,Y)\xi = [\alpha^2 - \frac{2A}{2n-1} - \frac{1}{2n(n-1)}](\eta(Y)X - \eta(X)Y). \tag{3.4}$$

Considering

$$B = \alpha^2 - \frac{2A}{2n-1} - \frac{1}{2n(n-1)},$$

(3.4) becomes

$$\tilde{C}(X,Y)\xi = B[\eta(Y)X - \eta(X)Y] \tag{3.5}$$

and

$$g(\tilde{C}(X,Y)\xi, Z) = B[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)],$$

which implies

$$-\eta(\tilde{C}(X,Y)Z) = B[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]. \tag{3.6}$$

Now we consider that the Lorentzian α-Sasakian manifold M admits conformal Ricci soliton and is Weyl conformally semi symmetric i.e. $R(\xi,X).\tilde{C} = 0$ holds in M (the manifold is locally isometric to the hyperbolic space $H^{n+1}(-\alpha^2)$ [32]), which implies

$$\begin{aligned} R(\xi,X)(\tilde{C}(Y,Z)W) \;-\; & \tilde{C}(R(\xi,X)Y,Z)W - \tilde{C}(Y,R(\xi,X)Z)W \\ -\; & \tilde{C}(Y,Z)R(\xi,X)W = 0, \end{aligned} \tag{3.7}$$

for all vector fields X,Y,Z,W on M.

Using (2.5) in (3.7) and putting $W = \xi$ we get

$$\begin{aligned} &g(X,\tilde{C}(Y,Z)\xi)\xi - \eta(\tilde{C}(Y,Z)\xi)X - g(X,Y)\tilde{C}(\xi,Z)\xi \\ &\qquad + \eta(Y)\tilde{C}(X,Z)\xi - g(X,Z)\tilde{C}(Y,\xi)\xi + \eta(Z)\tilde{C}(Y,X)\xi \\ &\qquad\qquad - g(X,\xi)\tilde{C}(Y,Z)\xi + \eta(\xi)\tilde{C}(Y,Z)X = 0. \end{aligned} \tag{3.8}$$

Taking inner product with ξ in (3.8) and using (2.1) we obtain

$$\begin{aligned} &- g(X,\tilde{C}(Y,Z)\xi) - g(X,Y)\eta(\tilde{C}(\xi,Z)\xi) \\ &\qquad + \eta(Y)\eta(\tilde{C}(X,Z)\xi) - g(X,Z)\eta(\tilde{C}(Y,\xi)\xi) + \eta(Z)\eta(\tilde{C}(Y,X)\xi) \\ &\qquad\qquad - \eta(X)\eta(\tilde{C}(Y,Z)\xi) - \eta(\tilde{C}(Y,Z)X) = 0. \end{aligned} \tag{3.9}$$

Using (3.5) in (3.9) we have

$$-B\eta(Z)g(X,Y) + B\eta(Y)g(X,Z) - \eta(\tilde{C}(Y,Z)X) = 0. \tag{3.10}$$

Putting $Z = \xi$ in (3.10) and using (2.1) we get

$$Bg(X,Y) + B\eta(Y)\eta(X) - \eta(\tilde{C}(Y,\xi)X) = 0. \tag{3.11}$$

Now from (3.1) we can write

$$\begin{aligned} &\tilde{C}(Y,\xi)X \\ &\quad = R(Y,\xi)X - \frac{1}{2n-1}[S(\xi,X)Y - S(Y,X)\xi + g(\xi,X)QY - g(Y,X)Q\xi] \\ &\qquad\qquad - \frac{1}{2n(n-1)}[g(\xi,X)Y - g(Y,X)\xi]. \end{aligned} \tag{3.12}$$

Taking inner product with ξ and using (2.1), (2.5), (2.12) in (3.12) we get

$$\begin{aligned} &\eta(\tilde{C}(Y,\xi)X) = \alpha^2\eta(X)\eta(Y) + \alpha^2 g(X,Y) \\ &\qquad - \frac{A}{2n-1}\eta(X)\eta(Y) - \frac{1}{2n-1}S(X,Y) - \frac{A}{2n-1}\eta(X)\eta(Y) \\ &\qquad - \frac{A}{2n-1}g(X,Y) - \frac{1}{2n(n-1)}\eta(X)\eta(Y) - \frac{1}{2n(n-1)}g(X,Y). \end{aligned} \tag{3.13}$$

After putting (3.13) in (3.11) the equation reduces to

$$\begin{aligned} & Bg(X,Y) + B\eta(Y)\eta(X) - \alpha^2\eta(X)\eta(Y) - \alpha^2 g(X,Y) \\ & + \frac{A}{2n-1}\eta(X)\eta(Y) + \frac{1}{2n-1}S(X,Y) + \frac{A}{2n-1}\eta(X)\eta(Y) + \frac{A}{2n-1}g(X,Y) \\ & \qquad + \frac{1}{2n(n-1)}\eta(X)\eta(Y) + \frac{1}{2n(n-1)}g(X,Y) = 0. \quad (3.14) \end{aligned}$$

Simplifying (3.14) we have

$$\begin{aligned} & g(X,Y)\left[B - \alpha^2 + \frac{A}{2n-1} + \frac{1}{2n(n-1)}\right] \\ & + \eta(X)\eta(Y)\left[B - \alpha^2 + \frac{2A}{2n-1} + \frac{1}{2n(n-1)}\right] + \frac{1}{2n-1}S(X,Y) = 0, \quad (3.15) \end{aligned}$$

which can be written in the form

$$S(X,Y) = \rho g(X,Y) + \sigma\eta(X)\eta(Y), \tag{3.16}$$

where

$$\rho = (2n-1)\left(\alpha^2 - B - \frac{A}{2n-1} - \frac{1}{2n(n-1)}\right)$$

and

$$\sigma = (2n-1)\left(\alpha^2 - B - \frac{2A}{2n-1} - \frac{1}{2n(n-1)}\right).$$

So from (3.16) we conclude that the manifold becomes η-Einstein manifold. □

4 Lorentzian α-Sasakian manifold admitting conformal Ricci soliton and $K(\xi, X).S = 0$

Let M be an $(2n+1)$ dimensional Lorentzian α-Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). The conharmonic curvature tensor K on M is defined by [8]

$$\begin{aligned} K(X,Y)Z = \; & R(X,Y)Z - \frac{1}{2n-1}[S(Y,Z)X - S(X,Z)Y \\ & + g(Y,Z)QX - g(X,Z)QY]. \quad (4.1) \end{aligned}$$

for all $X, Y, Z \in \chi(M)$, R is the curvature tensor and Q is the Ricci operator.

Now we prove the following theorem:

Theorem 4.1. *If a Lorentzian α-Sasakian manifold admits conformal Ricci soliton and the manifold is conharmonically Ricci symmetric i.e. $K(\xi, X).S = 0$ then the Ricci operator Q satisfies the quadratic equation $FQ^2 + Q - D = 0$ for all $X \in \chi(M)$ where F, D are constants, K is conharmonic curvature tensor and S is a Ricci tensor.*

Proof. Let M be an $(2n+1)$ dimensional Lorentzian α-Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). From (4.1) we can write

$$\begin{aligned} K(\xi, X)Y &= R(\xi, X)Y \\ &- \frac{1}{2n-1}[S(X,Y)\xi - S(\xi,Y)X + g(X,Y)Q\xi - g(\xi,Y)QX]. \end{aligned} \tag{4.2}$$

Using (2.5), (2.12) in (4.2) we have

$$\begin{aligned} K(\xi, X)Y &= \alpha^2[g(X,Y)\xi - \eta(Y)X] \\ &- \frac{1}{2n-1}[S(X,Y)\xi - A\eta(Y)X + Ag(X,Y)\xi - \eta(Y)QX]. \end{aligned} \tag{4.3}$$

Similarly from (4.2) we get

$$\begin{aligned} K(\xi, X)Z &= R(\xi, X)Z - \frac{1}{2n-1}[S(X,Z)\xi \\ &- S(\xi,Z)X + g(X,Z)Q\xi - g(\xi,Z)QX] = \alpha^2[g(X,Z)\xi - \eta(Z)X] \\ &- \frac{1}{2n-1}[S(X,Z)\xi - A\eta(Z)X + Ag(X,Z)\xi - \eta(Z)QX]. \end{aligned} \tag{4.4}$$

Now we consider that the tensor derivative of S by $K(\xi, X)$ is zero i.e. $K(\xi, X).S = 0$. Then the Lorentzian α-Sasakian manifold admitting conformal Ricci soliton is conharmonically Ricci symmetric (the manifold is locally isometric to the hyperbolic space $H^{n+1}(-\alpha^2)$ [32]). It gives

$$S(K(\xi, X)Y, Z) + S(Y, K(\xi, X)Z) = 0. \tag{4.5}$$

Using (4.3) and (4.4) in (4.5) we get

$$\begin{aligned} &S(\alpha^2 g(X,Y)\xi - \alpha^2\eta(Y)X \\ &- \frac{1}{2n-1}S(X,Y)\xi + \frac{A}{2n-1}\eta(Y)X - \frac{A}{2n-1}g(X,Y)\xi + \frac{\eta(Y)}{2n-1}QX, Z) \\ &+ S(\alpha^2 g(X,Z)\xi - \alpha^2\eta(Z)X - \frac{1}{2n-1}S(X,Z)\xi + \frac{A}{2n-1}\eta(Z)X \\ &- \frac{A}{2n-1}g(X,Z)\xi + \frac{\eta(Z)}{2n-1}QX, Y) = 0. \end{aligned} \tag{4.6}$$

Putting $Z = \xi$ and using (2.1), (2.12) in (4.6) we get

$$\left(\frac{A^2}{2n-1} - A\alpha^2\right) g(X,Y) + \alpha^2 S(X,Y) - \frac{1}{2n-1}S(QX,Y) = 0$$

which implies

$$Eg(X,Y) + \frac{1}{2n-1}S(QX,Y) = -\alpha^2 S(X,Y), \tag{4.7}$$

where $E = \frac{A^2}{2n-1} - A\alpha^2$.

From (4.7) we can write

$$S(X,Y) = Dg(X,Y) - \frac{1}{\alpha^2(2n-1)}S(QX,Y), \tag{4.8}$$

where $D = -\frac{1}{\alpha^2}E$, which implies

$$QX = DX - FQ^2X \quad \forall Y \in \chi(M), \tag{4.9}$$

where $F = \frac{1}{\alpha^2(2n-1)}$, i.e.

$$FQ^2 + Q - D = 0 \quad \forall X. \tag{4.10}$$

□

5 Lorentzian α-Sasakian manifold admitting conformal Ricci soliton and $P(\xi, X).\tilde{C} = 0$

Let M be an $(2n+1)$ dimensional Lorentzian α-Sasakian manifold admitting a conformal Ricci soliton (g, V, λ). The Weyl projective curvature tensor P on M is given by [2]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y].$$

Now we prove the following theorem:

Theorem 5.1. *If a Lorentzian α-Sasakian manifold M admits conformal Ricci soliton and $P(\xi, X).\tilde{C} = 0$ holds, then the manifold becomes η-Einstein manifold, where P is projective curvature tensor and $\tilde{C}$ is conformal curvature tensor.*

Proof. We know from (3.1) that

$$\begin{aligned}\tilde{C}(\xi,X)Y &= R(\xi,X)Y \\ &\quad - \frac{1}{2n-1}[S(X,Y)\xi - S(\xi,Y)X + g(X,Y)Q\xi - g(\xi,Y)QX] \\ &\quad - \frac{1}{2n(n-1)}[g(X,Y)\xi - g(\xi,Y)X], \end{aligned} \tag{5.1}$$

since for conformal Ricci soliton the scalar curvature $R = -1$ [9].

From (2.5), (2.12) and taking inner product with ξ on (5.1) we have

$$\begin{aligned}
\eta(\tilde{C}(\xi,X)Y) &= \alpha^2 g(X,Y)\eta(\xi) - \alpha^2\eta(Y)\eta(X) \\
- \frac{1}{2n-1}S(X,Y)\eta(\xi) &+ \frac{A}{2n-1}\eta(Y)\eta(X) - \frac{A}{2n-1}\eta(\xi)g(X,Y) \\
+ \frac{1}{2n-1}\eta(Y)\eta(QX) &- \frac{1}{2n(n-1)}[g(X,Y)\eta(\xi) - \eta(Y)\eta(X)] \\
&= g(X,Y)\left[\frac{A}{2n-1} - \alpha^2 + \frac{1}{2n(n-1)}\right] \\
&+ \eta(Y)\eta(X)\left[\frac{2A}{2n-1} - \alpha^2 + \frac{1}{2n(n-1)}\right] \\
+ \frac{1}{2n-1}S(X,Y) &= Fg(X,Y) + G\eta(Y)\eta(X) + TS(X,Y),
\end{aligned}$$

where

$$F = \frac{A}{2n-1} - \alpha^2 + \frac{1}{2n(n-1)},$$

$$G = \frac{2A}{2n-1} - \alpha^2 + \frac{1}{2n(n-1)}$$

and

$$T = \frac{1}{2n-1}.$$

Also

$$\eta(\tilde{C}(X,Y)\xi) = B[\eta(Y)\eta(X) - \eta(X)\eta(Y)] = 0$$

and

$$\eta(\tilde{C}(Y,\xi)\xi) = B[\eta(Y)\eta(\xi) - \eta(\xi)\eta(Y)] = 0.$$

Now

$$P(\xi,X)Y = R(\xi,X)Y - \frac{1}{2n}[S(X,Y)\xi - S(\xi,Y)X]. \tag{5.2}$$

Using (2.5), (2.12) in (5.2) we get

$$P(\xi,X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X] - \frac{1}{2n}[S(X,Y)\xi - A\eta(Y)X]. \tag{5.3}$$

Here we consider that the tensor derivative of $\tilde{C}$ by $P(\xi,X)$ is zero i.e. conformally symmetric with respect to projective curvature tensor i.e. $P(\xi,X).\tilde{C} = 0$ holds (the manifold is locally isometric to the hyperbolic space $H^{n+1}(-\alpha^2)$ [32]).

So

$$\begin{aligned}
P(\xi,X)\tilde{C}(Y,Z)W &- \tilde{C}(P(\xi,X)Y,Z)W - \tilde{C}(Y,P(\xi,X)Z)W \\
&- \tilde{C}(Y,Z)P(\xi,X)W = 0,
\end{aligned} \tag{5.4}$$

for all vector fields X, Y, Z, W on M.

Using (5.3) in (5.4) and putting $W = \xi$ we have

$$\begin{aligned}\alpha^2 g(X, \tilde{C}(Y,Z)\xi)\xi - \alpha^2\eta(\tilde{C}(Y,Z)\xi)X \\ - \frac{1}{2n}S(X,\tilde{C}(Y,Z)\xi)\xi + \frac{A}{2n}\eta(\tilde{C}(Y,Z)\xi)X - \alpha^2 g(X,Y)\tilde{C}(\xi,Z)\xi \\ + \alpha^2\eta(Y)\tilde{C}(X,Z)\xi + \frac{1}{2n}S(X,Y)\tilde{C}(\xi,Z)\xi - \frac{A}{2n}\eta(Y)\tilde{C}(X,Z)\xi \\ - \alpha^2 g(X,Z)\tilde{C}(Y,\xi)\xi + \alpha^2\eta(Z)\tilde{C}(Y,X)\xi + \frac{1}{2n}S(X,Z)\tilde{C}(Y,\xi)\xi \\ - \frac{A}{2n}\eta(Z)\tilde{C}(Y,X)\xi - \alpha^2 g(X,\xi)\tilde{C}(Y,Z)\xi + \alpha^2\eta(\xi)\tilde{C}(Y,Z)X \\ + \frac{1}{2n}S(X,\xi)\tilde{C}(Y,Z)\xi - \frac{A}{2n}\eta(\xi)\tilde{C}(Y,Z)X = 0. \quad (5.5)\end{aligned}$$

Taking inner product with ξ on (5.5) we get

$$-\alpha^2 g(X,\tilde{C}(Y,Z)\xi) + \frac{1}{2n}S(X,\tilde{C}(Y,Z)\xi) = 0. \quad (5.6)$$

From (3.2) and (5.6) we have

$$-\alpha^2 B\eta(Z)g(X,Y) + \alpha^2\eta(Y)Bg(X,Z) + \frac{B}{2n}\eta(Z)S(X,Y) - \frac{B}{2n}\eta(Y)S(X,Z) = 0. \quad (5.7)$$

Putting $z = \xi$ in (5.7) and using (2.1), (2.12) we obtain

$$\alpha^2 Bg(X,Y) + B\alpha^2\eta(Y)\eta(X) - \frac{B}{2n}S(X,Y) - \frac{AB}{2n}\eta(Y)\eta(X) = 0,$$

which implies

$$S(X,Y) = 2n\alpha^2 g(X,Y) + 2n(\alpha^2 - \frac{A}{2n})\eta(Y)\eta(X). \quad (5.8)$$

So the manifold becomes η-Einstein manifold. □

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