Conformal Ricci Soliton in Lorentzian $\alpha$-Sasakian Manifolds

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Abstract

In this paper we have studied conformal curvature tensor, conharmonic curvature tensor, projective curvature tensor in Lorentzian $\alpha$-Sasakian manifolds admitting conformal Ricci soliton. We have found that a Weyl conformally semi symmetric Lorentzian $\alpha$-Sasakian manifold admitting conformal Ricci soliton is $\eta$-Einstein manifold. We have also studied conharmonically Ricci symmetric Lorentzian $\alpha$-Sasakian manifold admitting conformal Ricci soliton. Similarly we have proved that a Lorentzian $\alpha$-Sasakian manifold $M$ with projective curvature tensor admitting conformal Ricci soliton is $\eta$-Einstein manifold. We have also established an example of 3-dimensional Lorentzian $\alpha$-Sasakian manifold.

Key words: Conformal Ricci soliton, conformal curvature tensor, conharmonic curvature tensor, Lorentzian $\alpha$-Sasakian manifolds, projective curvature tensor.

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1 Introduction

In 1982 Hamilton [11] introduced the concept of Ricci flow and proved its existence. This concept was developed to answer Thurston’s geometric conjecture.
which says that each closed three manifold admits a geometric decomposition. Hamilton also [12] classified all compact manifolds with positive curvature operator in dimension four. Since then, the Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature.

The Ricci flow equation is given by

$$\frac{\partial g}{\partial t} = -2S$$

(1.1)
on a compact Riemannian manifold $M$ with Riemannian metric $g$. Ricci soliton emerges as the limit of the solutions of Ricci flow. A solution to the Ricci flow is called a Ricci soliton if it moves only by a one-parameter group of diffeomorphism and scaling. Ramesh Sharma [28] started the study of Ricci soliton in contact manifolds and after him M. M. Tripathi [31], Bejan, Crasmareanu [4] studied Ricci soliton in contact metric manifolds. The Ricci soliton equation is given by

$$\mathcal{L}_X g + 2S + 2\lambda g = 0,$$

(1.2)
where $\mathcal{L}_X$ is the Lie derivative, $S$ is Ricci tensor, $g$ is Riemannian metric, $X$ is a vector field and $\lambda$ is a scalar. The $\varphi-$ vector fields are special type Ricci soliton studied in [14, 15].

In 2005, A.E. Fischer [9] introduced a new concept called conformal Ricci flow which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. Since the conformal geometry plays an important role to constrain the scalar curvature and the equations are the vector field sum of a conformal flow equation and a Ricci flow equation, the resulting equations are named as the conformal Ricci flow equations. These new equations are given by

$$\frac{\partial g}{\partial t} + 2 \left( \frac{S + \frac{g}{n}}{n} \right) = -pg$$

(1.3)
and $R(g) = -1$, where $p$ is a scalar non-dynamical field(time dependent scalar field), $R(g)$ is the scalar curvature of the manifold and $n$ is the dimension of manifold.

In 2015, N. Basu and A. Bhattacharyya [3] introduced the notion of conformal Ricci soliton and the equation is as follows

$$\mathcal{L}_X g + 2S = \left[ 2\lambda - \left( p + \frac{2}{n} \right) \right] g.$$  

(1.4)
The equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

A Riemannian manifold is said to be locally symmetric if its curvature tensor $R$ satisfies $\nabla R = 0$, where $\nabla$ is Levi-Civita connection on the Riemannian manifold. As a generalization of locally symmetric spaces, many geometers have considered semi-symmetric spaces and their generalization. A Riemannian manifold is said to be semi symmetric if its curvature tensor $R$ satisfies
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$R(X,Y).R = 0$ for all $X, Y \in TM$, where $R(X,Y)$ acts on $R$ as a derivation. N. S. Sinyukov, J. Mikeš, I. Hinterleitner and others studied geodesic mappings of symmetric and semisymmetric spaces [29, 10, 18, 13, 19, 17, 22, 23, 24, 25, 16]. K. Sekigawa [27], Z. I. Szabo [30] studied Riemannian manifolds or hypersurfaces of such manifold satisfying the condition $R(X,Y).R = 0$ or condition similar to it. It is easy to see that $R(X,Y).R = 0$ implies $R(X,Y).C = 0$. So it is meaningful to undertake the study of manifolds satisfying such type of conditions.

1.1 Definition of Einstein manifold

An Einstein manifold is a Riemannian or pseudo-Riemannian manifold with Ricci tensor is proportional to the metric. If $M$ is the underlying $n$-dimensional manifold and $g$ is its metric tensor then the Einstein condition means that

$$S(X, Y) = \lambda g(X, Y),$$

for some constant $\lambda$, where $S$ denotes the Ricci tensor of $g$. Einstein manifolds with $\lambda = 0$ are called Ricci-flat manifolds.

1.2 Definition of $\eta$-Einstein manifold

A trans-Sasakian manifold $M^n$ is said to be $\eta$-Einstein manifold if its Ricci tensor $S$ is of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y),$$

where $a, b$ are smooth functions.

2 Basic concepts of Lorentzian $\alpha$-Sasakian manifolds

A differentiable manifold of dimension $(2n + 1)$ is called Lorentzian $\alpha$–Sasakian manifold [1] if it admits a $(1, 1)$ tensor field $\varphi$, a vector field $\xi$ and 1-form $\eta$ and Lorentzian metric $g$ which satisfy on $M$ respectively such that

$$\varphi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \eta \circ \varphi = 0, \quad \varphi \xi = 0,$$

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X),$$

$$\nabla_X \xi = \alpha \varphi X, \quad (\nabla_X \eta) Y = \alpha g(\varphi X, Y),$$

where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$ on $M$. Geometry of Sasakian spaces was studied in [21, 20, 26, 19].
On an Lorentzian $\alpha$-Sasakian manifold $M$ the following relations hold [1]:

\[
R(X, Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y],
\]

\[
R(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X],
\]

\[
S(X, \xi) = 2n\alpha^2\eta(X),
\]

\[
Q\xi = 2n\alpha^2\xi,
\]

\[
S(\xi, \xi) = -2n\alpha^2,
\]

where $\alpha$ is some constant, $R$ is the Riemannian curvature, $S$ is the Ricci tensor and $Q$ is the Ricci operator given by $S(X, Y) = g(QX, Y)$ for all $X, Y \in \chi(M)$.

Now from definition of Lie derivative we have

\[
(L_\xi g)(X, Y) = (\nabla_\xi g)(X, Y) + g(\alpha\varphi X, Y) + g(X, \alpha\varphi Y)
\]

\[
= 2\alpha g(\varphi X, Y), \quad [ : g(X, \varphi Y) = g(\varphi X, Y)].
\]

Applying (2.9) in (1.4) we get

\[
S(X, Y) = \frac{1}{2} \left[ 2\lambda - \left( p + \frac{2}{n} \right) \right] g(X, Y) - \alpha g(\varphi X, Y)
\]

\[
= A g(X, Y) - \alpha g(\varphi X, Y),
\]

where

\[
A = \frac{1}{2} \left[ 2\lambda - \left( p + \frac{2}{n} \right) \right].
\]

Since $S(X, Y) = g(QX, Y)$ for the Ricci operator $Q$, we have

\[
g(QX, Y) = Ag(X, Y) - \alpha g(\varphi X, Y)
\]

i.e.

\[
QX = AX - \alpha\varphi X, \quad \forall Y.
\]

Also

\[
S(Y, \xi) = A\eta(Y), \quad S(\xi, \xi) = -A, \quad Q\xi = A\xi.
\]

If we put $X = Y = e_i$ in (2.10), where $\{e_i\}$ is orthonormal basis of the tangent space $TM$ where $TM$ is a tangent bundle of $M$ and summing over $i$, we get

\[
R(g) = An - \alpha g(\varphi e_i, e_i)
\]

As $R = -1$, we have

\[
-1 = An - \alpha (\text{tr } \varphi) \quad \text{i.e.} \quad A = \frac{1}{n}(\alpha.(\text{tr } \varphi) - 1).
\]
2.1 Example of a 3-dimensional Lorentzian $\alpha$-Sasakian manifold

In this section we construct an example of a 3-dimensional Lorentzian $\alpha$-Sasakian manifold. To construct this, we consider the three dimensional manifold $M = \{(x, y, z) \in R^3 : z \neq 0\}$ where $(x, y, z)$ are the standard coordinates in $R^3$. The vector fields

$$e_1 = e^{-z} \frac{\partial}{\partial x}, \quad e_2 = e^{-z} \frac{\partial}{\partial y}, \quad e_3 = -e^{-z} \frac{\partial}{\partial z}$$

are linearly independent at each point of $M$.

Let $g$ be the Lorentzian metric defined by

$$g(e_1, e_1) = 1, \quad g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1,$$

$$g(e_1, e_2) = g(e_2, e_3) = g(e_3, e_1) = 0.$$

Let $\eta$ be the 1-form which satisfies the relation $\eta(e_3) = -1$. Let $\varphi$ be the $(1, 1)$ tensor field defined by $\varphi(e_1) = -e_1, \varphi(e_2) = -e_2, \varphi(e_3) = 0$. Then we have

$$\varphi^2(Z) = Z + \eta(Z)e_3,$$

$$g(\varphi Z, \varphi W) = g(Z, W) + \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M^3)$. Thus for $e_3 = \xi$, $(\varphi, \xi, \eta, g)$ defines an almost contact metric structure on $M$. Now, after calculating we have

$$[e_1, e_3] = -e^{-z}e_1, \quad [e_1, e_2] = 0, \quad [e_2, e_3] = -e^{-z}e_2.$$

The Riemannian connection $\nabla$ of the metric is given by the Koszul’s formula which is

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y)$$

$$-g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \quad (2.13)$$

By Koszul’s formula we get

$$\nabla_{e_1} e_1 = -e^{-z}e_3,$$

$$\nabla_{e_2} e_2 = 0,$$

$$\nabla_{e_3} e_3 = -e^{-z}e_1.$$

From the above we have found that $\alpha = e^{-z}$ and it can be easily shown that $M^3(\varphi, \xi, \eta, g)$ is a Lorentzian $\alpha$-Sasakian manifold.

3 Lorentzian $\alpha$-Sasakian manifold admitting conformal Ricci soliton and $R(\xi, X).\tilde{C} = 0$

Let $M$ be an $(2n + 1)$ dimensional Lorentzian $\alpha$-Sasakian manifold admitting a conformal Ricci soliton $(g, V, \lambda)$. The conformal curvature tensor $\tilde{C}$ on $M$ is
defined by [2]
\[
\tilde{C}(X, Y) Z = R(X, Y)Z - \frac{1}{2n-1} [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX
\]
\[
- g(X, Z)QY] + \frac{R}{2n(n-1)} [g(Y, Z)X - g(X, Z)Y],
\] (3.1)

where \( R \) is scalar curvature.

Now we prove the following theorem:

**Theorem 3.1.** If a Lorentzian \( \alpha \)-Sasakian manifold admits conformal Ricci soliton and is Weyl conformally semi symmetric i.e. \( R(\xi, X)\tilde{C} = 0 \), then the manifold is \( \eta \)-Einstein manifold where \( \tilde{C} \) is Conformal curvature tensor and \( R(\xi, X) \) is derivation of tensor algebra of the tangent space of the manifold.

**Proof.** Let \( M \) be an \( (2n+1) \) dimensional Lorentzian \( \alpha \)-Sasakian manifold admitting a conformal Ricci soliton \( (g, V, \lambda) \). So we have \( R = -1 \) [9].

After putting \( R = -1 \) and \( Z = \xi \) in (3.1) we have
\[
\tilde{C}(X, Y) \xi = R(X, Y)\xi - \frac{1}{2n-1} [S(Y, \xi)X - S(X, \xi)Y + g(Y, \xi)QX - g(X, \xi)QY]
\]
\[
- \frac{1}{2n(n-1)} [g(Y, \xi)X - g(X, \xi)Y].
\] (3.2)

Using (2.2), (2.4), (2.11) and (2.12) in (3.2) we get
\[
\tilde{C}(X, Y) \xi = \alpha^2 [\eta(Y)X - \eta(X)Y] - \frac{1}{2n-1} [A\eta(Y)X - A\eta(X)Y]
\]
\[
+ \eta(Y)(AX - \alpha\varphi X) - \eta(X)(AY - \alpha\varphi Y)] - \frac{1}{2n(n-1)} [\eta(Y)X - \eta(X)Y].
\] (3.3)

Using (3.1) and after a brief simplification we obtain
\[
\tilde{C}(X, Y) \xi = [\alpha^2 - \frac{2A}{2n-1} - \frac{1}{2n(n-1)}](\eta(Y)X - \eta(X)Y).
\] (3.4)

Considering
\[
B = \alpha^2 - \frac{2A}{2n-1} - \frac{1}{2n(n-1)},
\]
(3.4) becomes
\[
\tilde{C}(X, Y) \xi = B[\eta(Y)X - \eta(X)Y]
\] (3.5)
and
\[
g(\tilde{C}(X, Y) \xi, Z) = B[\eta(Y)g(X, Z) - \eta(X)g(Y, Z)].
\]
which implies
\[ -\eta(\tilde{C}(X,Y)Z) = B[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]. \] (3.6)

Now we consider that the Lorentzian \(\alpha\)-Sasakian manifold \(M\) admits conformal Ricci soliton and is Weyl conformally semi symmetric i.e. \(R(\xi,X)\tilde{C} = 0\) holds in \(M\) (the manifold is locally isometric to the hyperbolic space \(H^{n+1}(\alpha^2)\) [32]), which implies
\[
R(\xi,X)(\tilde{C}(Y,Z)W) - \tilde{C}(R(\xi,X)Y,Z)W - \tilde{C}(Y,R(\xi,X)Z)W
- \tilde{C}(Y,Z)R(\xi,X)W = 0, \tag{3.7}
\]
for all vector fields \(X,Y,Z,W\) on \(M\).

Using (2.5) in (3.7) and putting \(W = \xi\) we get
\[
g(X,\tilde{C}(Y,Z)\xi)\xi - \eta(\tilde{C}(Y,Z)\xi)X - g(X,Y)\tilde{C}(\xi,Z)\xi
+ \eta(Y)\tilde{C}(X,Z)\xi - g(X,Z)\tilde{C}(Y,\xi)\xi + \eta(Z)\tilde{C}(\xi,X)\xi
- g(X,\xi)\tilde{C}(Y,Z)\xi + \eta(\xi)\tilde{C}(Y,Z)X = 0. \tag{3.8}
\]
Taking inner product with \(\xi\) in (3.8) and using (2.1) we obtain
\[
- g(X,\tilde{C}(Y,Z)\xi) - g(X,Y)\eta(\tilde{C}(\xi,Z)\xi)
+ \eta(Y)\eta(\tilde{C}(X,Z)\xi) - g(X,Z)\eta(\tilde{C}(Y,\xi)\xi) + \eta(Z)\eta(\tilde{C}(\xi,X)\xi)
- \eta(X)\eta(\tilde{C}(Y,Z)\xi) - \eta(\tilde{C}(Y,Z)X) = 0. \tag{3.9}
\]
Using (3.5) in (3.9) we have
\[
-B\eta(Z)g(X,Y) + B\eta(Y)g(X,Z) - \eta(\tilde{C}(Y,Z)X) = 0. \tag{3.10}
\]
Putting \(Z = \xi\) in (3.10) and using (2.1) we get
\[
Bg(X,Y) + B\eta(Y)\eta(X) - \eta(\tilde{C}(Y,\xi)X) = 0. \tag{3.11}
\]
Now from (3.1) we can write
\[
\tilde{C}(Y,\xi)X
= R(Y,\xi)X - \frac{1}{2n-1}[S(\xi,X)Y - S(Y,X)\xi + g(\xi,X)QY - g(Y,X)Q\xi]
- \frac{1}{2n(n-1)}[g(\xi,X)Y - g(Y,X)\xi]. \tag{3.12}
\]
Taking inner product with \(\xi\) and using (2.1), (2.5), (2.12) in (3.12) we get
\[
\eta(\tilde{C}(Y,\xi)X) = \alpha^2\eta(\xi)\eta(Y) + \alpha^2g(X,Y)
- \frac{A}{2n-1}\eta(\xi)\eta(Y) - \frac{1}{2n-1}S(X,Y) - \frac{A}{2n-1}\eta(\xi)\eta(Y)
- \frac{A}{2n-1}g(X,Y) - \frac{1}{2n(n-1)}\eta(\xi)\eta(Y) - \frac{1}{2n(n-1)}g(X,Y). \tag{3.13}
\]
After putting (3.13) in (3.11) the equation reduces to

\[ Bg(X, Y) + B\eta(Y)\eta(X) - \alpha^2\eta(X)\eta(Y) - \alpha^2 g(X, Y) \]
\[ + \frac{A}{2n-1}\eta(X)\eta(Y) + \frac{1}{2n-1}S(X, Y) + \frac{A}{2n-1}\eta(X)\eta(Y) + \frac{A}{2n-1}g(X, Y) \]
\[ + \frac{1}{2n(n-1)}\eta(X)\eta(Y) + \frac{1}{2n(n-1)}g(X, Y) = 0. \quad (3.14) \]

Simplifying (3.14) we have

\[ g(X, Y) \left[ B - \alpha^2 + \frac{A}{2n-1} + \frac{1}{2n(n-1)} \right] \]
\[ + \eta(X)\eta(Y) \left[ B - \alpha^2 + \frac{2A}{2n-1} + \frac{1}{2n(n-1)} \right] + \frac{1}{2n-1}S(X, Y) = 0, \quad (3.15) \]

which can be written in the form

\[ S(X, Y) = \rho g(X, Y) + \sigma\eta(X)\eta(Y), \quad (3.16) \]

where

\[ \rho = (2n-1) \left( \alpha^2 - B - \frac{A}{2n-1} - \frac{1}{2n(n-1)} \right) \]

and

\[ \sigma = (2n-1) \left( \alpha^2 - B - \frac{2A}{2n-1} - \frac{1}{2n(n-1)} \right). \]

So from (3.16) we conclude that the manifold becomes \( \eta \)-Einstein manifold. ☐

4 Lorentzian \( \alpha \)-Sasakian manifold admitting conformal Ricci soliton and \( K (\xi, X)_S = 0 \)

Let \( M \) be an \( (2n+1) \) dimensional Lorentzian \( \alpha \)-Sasakian manifold admitting a conformal Ricci soliton \( (g, V, \lambda) \). The conharmonic curvature tensor \( K \) on \( M \) is defined by [8]

\[ K(X, Y)Z = R(X, Y)Z - \frac{1}{2n-1}[S(Y, Z)X - S(X, Z)Y \]
\[ + g(Y, Z)QX - g(X, Z)QY]. \quad (4.1) \]

for all \( X, Y, Z \in \chi(M) \), \( R \) is the curvature tensor and \( Q \) is the Ricci operator.

Now we prove the following theorem:

**Theorem 4.1.** If a Lorentzian \( \alpha \)-Sasakian manifold admits conformal Ricci soliton and the manifold is conharmonically Ricci symmetric i.e. \( K (\xi, X)_S = 0 \) then the Ricci operator \( Q \) satisfies the quadratic equation \( FQ^2 + Q - D = 0 \) for all \( X \in \chi(M) \) where \( F, D \) are constants, \( K \) is conharmonic curvature tensor and \( S \) is a Ricci tensor.
Proof. Let $M$ be an $(2n+1)$ dimensional Lorentzian $\alpha$-Sasakian manifold admitting a conformal Ricci soliton $(g, V, \lambda)$. From (4.1) we can write

$$K(\xi, X)Y = R(\xi, X)Y - \frac{1}{2n-1}[S(X, Y)\xi - S(\xi, Y)X + g(X, Y)Q\xi - g(\xi, Y)QX].$$

(4.2)

Using (2.5), (2.12) in (4.2) we have

$$K(\xi, X)Y = \alpha^2[g(X, Y)\xi - \eta(Y)X] - \frac{1}{2n-1}[S(X, Y)\xi - A\eta(Y)X + A\eta(X, Y)\xi - \eta(Y)QX].$$

(4.3)

Similarly from (4.2) we get

$$K(\xi, X)Z = R(\xi, X)Z - \frac{1}{2n-1}[S(X, Z)\xi - S(\xi, Z)X + g(X, Z)Q\xi - g(\xi, Z)QX] = \alpha^2[g(X, Z)\xi - \eta(Z)X] - \frac{1}{2n-1}[S(X, Z)\xi - A\eta(Z)X + A\eta(X, Z)\xi - \eta(Z)QX].$$

(4.4)

Now we consider that the tensor derivative of $S$ by $K(\xi, X)$ is zero i.e. $K(\xi, X)S = 0$. Then the Lorentzian $\alpha$-Sasakian manifold admitting conformal Ricci soliton is conharmonically Ricci symmetric (the manifold is locally isometric to the hyperbolic space $H^{n+1}(\alpha^2)$ [32]). It gives

$$S(K(\xi, X)Y, Z) + S(Y, K(\xi, X)Z) = 0.$$ 

(4.5)

Using (4.3) and (4.4) in (4.5) we get

$$S(\alpha^2g(X, Y)\xi - \alpha^2\eta(Y)X$$

$$- \frac{1}{2n-1}S(X, Y)\xi + \frac{A}{2n-1}\eta(Y)X - \frac{A}{2n-1}g(X, Y)\xi + \frac{\eta(Y)}{2n-1}QX, Z)$$

$$+ S(\alpha^2g(X, Z)\xi - \alpha^2\eta(Z)X - \frac{1}{2n-1}S(X, Z)\xi + \frac{A}{2n-1}\eta(Z)X$$

$$- \frac{A}{2n-1}g(X, Z)\xi + \frac{\eta(Z)}{2n-1}QX, Y) = 0.$$ 

(4.6)

Putting $Z = \xi$ and using (2.1), (2.12) in (4.6) we get

$$\left(\frac{A^2}{2n-1} - A\alpha^2\right)g(X, Y) + \alpha^2S(X, Y) - \frac{1}{2n-1}S(QX, Y) = 0.$$
which implies
\[ Eg(X, Y) + \frac{1}{2n - 1}S(QX, Y) = -\alpha^2 S(X, Y), \quad (4.7) \]
where \( E = \frac{A^2}{2n - 1} - A\alpha^2 \).

From (4.7) we can write
\[ S(X, Y) = Dg(X, Y) - \frac{1}{\alpha^2(2n - 1)}S(QX, Y), \quad (4.8) \]
where \( D = -\frac{1}{\alpha^2}E \), which implies
\[ QX = DX - FQ^2X \quad \forall Y \in \chi(M), \quad (4.9) \]
where \( F = \frac{1}{\alpha^2(2n - 1)} \), i.e.
\[ FQ^2 + Q - D = 0 \quad \forall X. \quad (4.10) \]

\[ \square \]

5 Lorentzian \( \alpha \)-Sasakian manifold admitting conformal Ricci soliton and \( P(\xi, X).\tilde{C} = 0 \)

Let \( M \) be an \((2n + 1)\) dimensional Lorentzian \( \alpha \)-Sasakian manifold admitting a conformal Ricci soliton \((g, V, \lambda)\). The Weyl projective curvature tensor \( P \) on \( M \) is given by [2]
\[ P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y]. \]

Now we prove the following theorem:

**Theorem 5.1.** If a Lorentzian \( \alpha \)-Sasakian manifold \( M \) admits conformal Ricci soliton and \( P(\xi, X).\tilde{C} = 0 \) holds, then the manifold becomes \( \eta \)-Einstein manifold, where \( P \) is projective curvature tensor and \( \tilde{C} \) is conformal curvature tensor.

**Proof.** We know from (3.1) that
\[
\tilde{C}(\xi, X)Y = R(\xi, X)Y - \frac{1}{2n - 1}[S(X, Y)\xi - S(\xi, Y)X + g(X, Y)Q\xi - g(\xi, Y)QX] \\
- \frac{1}{2n(n - 1)}[g(X, Y)\xi - g(\xi, Y)X], \quad (5.1)
\]
since for conformal Ricci soliton the scalar curvature \( R = -1 \) [9].
From (2.5), (2.12) and taking inner product with $\xi$ on (5.1) we have

$$\eta(\tilde{C}(\xi, X)Y) = \alpha^2 g(X, Y)\eta(\xi) - \alpha^2 \eta(Y)\eta(X)$$

$$- \frac{1}{2n-1} S(X, Y)\eta(\xi) + \frac{A}{2n-1} \eta(Y)\eta(X) - \frac{A}{2n-1} \eta(\xi)g(X, Y)$$

$$+ \frac{1}{2n-1} \eta(Y)\eta(QX) - \frac{1}{2n(n-1)}[g(X, Y)\eta(\xi) - \eta(Y)\eta(X)]$$

$$= g(X, Y)\left[ \frac{A}{2n-1} - \alpha^2 + \frac{1}{2n(n-1)} \right]$$

$$+ \eta(Y)\eta(X)\left[ \frac{2A}{2n-1} - \alpha^2 + \frac{1}{2n(n-1)} \right]$$

$$+ \frac{1}{2n-1} S(X, Y) = F g(X, Y) + G \eta(Y)\eta(X) + TS(X, Y),$$

where

$$F = \frac{A}{2n-1} - \alpha^2 + \frac{1}{2n(n-1)},$$

$$G = \frac{2A}{2n-1} - \alpha^2 + \frac{1}{2n(n-1)}$$

and

$$T = \frac{1}{2n-1}.$$

Also

$$\eta(\tilde{C}(X, Y)\xi) = B[\eta(Y)\eta(X) - \eta(X)\eta(Y)] = 0$$

and

$$\eta(\tilde{C}(Y, \xi)\xi) = B[\eta(Y)\eta(\xi) - \eta(\xi)\eta(Y)] = 0.$$

Now

$$P(\xi, X)Y = R(\xi, X)Y - \frac{1}{2n}[S(X, Y)\xi - S(\xi, Y)X]. \quad (5.2)$$

Using (2.5), (2.12) in (5.2) we get

$$P(\xi, X)Y = \alpha^2 [g(X, Y)\xi - \eta(Y)X] - \frac{1}{2n}[S(X, Y)\xi - A\eta(Y)X]. \quad (5.3)$$

Here we consider that the tensor derivative of $\tilde{C}$ by $P(\xi, X)$ is zero i.e. conformally symmetric with respect to projective curvature tensor i.e. $P(\xi, X)\tilde{C} = 0$ holds (the manifold is locally isometric to the hyperbolic space $H^{n+1}(-\alpha^2)$ [32]).

So

$$P(\xi, X)\tilde{C}(Y, Z)W - \tilde{C}(P(\xi, X)Y, Z)W - \tilde{C}(Y, P(\xi, X)Z)W$$

$$- \tilde{C}(Y, Z)P(\xi, X)W = 0, \quad (5.4)$$

for all vector fields $X, Y, Z, W$ on $M$. 
Using (5.3) in (5.4) and putting \( W = \xi \) we have

\[
\begin{align*}
\alpha^2 g(X, \tilde{C}(Y, Z)\xi) & - \alpha^2 \eta(\tilde{C}(Y, Z)\xi) X \\
- \frac{1}{2n} S(X, \tilde{C}(Y, Z)\xi) & + \frac{A}{2n} \eta(\tilde{C}(Y, Z)\xi) X - \alpha^2 g(X, Y) \tilde{C}(\xi, Z) \xi \\
+ \alpha^2 \eta(Y) \tilde{C}(X, Z) \xi & + \frac{1}{2n} S(X, Y) \tilde{C}(\xi, Z) \xi - \frac{A}{2n} \eta(Y) \tilde{C}(X, Z) \xi \\
- \alpha^2 g(X, Z) \tilde{C}(Y, \xi) \xi & + \frac{1}{2n} S(X, Y) \tilde{C}(Y, \xi) \xi - \frac{A}{2n} \eta(Y) \tilde{C}(Y, Z) \xi \\
- \frac{1}{2n} S(X, \xi) \tilde{C}(Y, Z) \xi & + \frac{1}{2n} \eta(\xi) \tilde{C}(Y, Z) X - \frac{A}{2n} \eta(\xi) \tilde{C}(Y, Z) X = 0.
\end{align*}
\]

(5.5)

Taking inner product with \( \xi \) on (5.5) we get

\[
- \alpha^2 g(X, \tilde{C}(Y, Z)\xi) + \frac{1}{2n} S(X, \tilde{C}(Y, Z)\xi) = 0.
\]

(5.6)

From (3.2) and (5.6) we have

\[
- \alpha^2 B\eta(Z) g(X, Y) + \alpha^2 \eta(Y) Bg(X, Z) + \frac{B}{2n} \eta(Z) S(X, Y) - \frac{B}{2n} \eta(Y) S(X, Z) = 0.
\]

(5.7)

Putting \( z = \xi \) in (5.7) and using (2.1), (2.12) we obtain

\[
\alpha^2 Bg(X, Y) + B\alpha^2 \eta(Y) \eta(X) - \frac{B}{2n} S(X, Y) - \frac{AB}{2n} \eta(Y) \eta(X) = 0,
\]

which implies

\[
S(X, Y) = 2n\alpha^2 g(X, Y) + 2n(\alpha^2 - \frac{A}{2n}) \eta(Y) \eta(X).
\]

(5.8)

So the manifold becomes \( \eta \)-Einstein manifold.

\[ \square \]

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**References**


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