Characterization on Mixed Generalized Quasi-Einstein Manifold

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Abstract

In the present paper we study characterizations of odd and even dimensional mixed generalized quasi-Einstein manifold. Next we prove that a mixed generalized quasi-Einstein manifold is a generalized quasi-Einstein manifold under a certain condition. Then we obtain three and four dimensional examples of mixed generalized quasi-Einstein manifold to ensure the existence of such manifold. Finally we establish the examples of warped product on mixed generalized quasi-Einstein manifold.

Key words: Einstein manifold, quasi-Einstein manifold, generalized quasi-Einstein manifold, mixed generalized quasi-Einstein manifold, super quasi-Einstein manifold, warped product.

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1 Introduction

A Riemannian manifold $(M, g)$ with dimension $(n \geq 2)$ is said to be an Einstein manifold if the Ricci tensor satisfies the condition $S(X, Y) = \frac{r}{n} g(X, Y)$, holds on $M$, here $S$ and $r$ denote the Ricci tensor and the scalar curvature of $(M, g)$ respectively. According to [3] the above equation is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry,

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as well as in general theory of relativity. The notion of quasi-Einstein manifold was defined in [9]. A non-flat Riemannian manifold \((M, g), (n \geq 2)\) is said to be an quasi Einstein manifold if the condition

\[
S(X, Y) = \alpha g(X, Y) + \beta \rho(X)\rho(Y),
\]

is fulfilled on \(M\), where \(\alpha\) and \(\beta\) are scalars of which \(\beta \neq 0\) and \(\rho\) is non-zero 1-form such that \(g(X, \xi) = \rho(X)\) for all vector field \(X\) and \(\xi\) is a unit vector field.

Note that the subprojective manifolds by Kagan have the Ricci tensor with the same properties [14, 19].

In [8], U. C. De and G. C. Ghosh introduced generalized quasi-Einstein manifold, denoted by \(G(QE)_n\) where the Ricci tensor \(S\) of type \((0, 2)\) which is not identically zero satisfies the condition

\[
S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \varrho B(X)B(Y), \tag{1.1}
\]

where \(\alpha, \beta, \varrho\) are scalars such that \(\beta, \varrho\) are nonzero and \(A, B\) are two nonzero 1-forms such that

\[
g(X, \xi_1) = A(X), \ g(X, \xi_2) = B(X), \ \forall X, \tag{1.2}
\]

\(\xi_1, \xi_2\) being unit vectors which are orthogonal, i.e., \(g(\xi_1, \xi_2) = 0\).

Here \(\alpha, \beta, \gamma, \delta\) are called the associated scalars, and \(A, B\) are called the associated main and auxiliary 1-forms respectively, \(\xi_1, \xi_2\) are main and auxiliary generators of the manifold.

In [6], M. C. Chaki introduced super quasi-Einstein manifold, denoted by \(S(QE)_n\) and gave an example of a 4-dimensional semi Riemannian super quasi-Einstein manifold, where the Ricci tensor \(S\) of type \((0, 2)\) which is not identically zero satisfies the condition

\[
S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma \left[ A(X)B(Y) + A(Y)B(X) \right] + \delta D(X, Y), \tag{1.3}
\]

where \(\alpha, \beta, \gamma, \delta\) are scalars such that \(\beta, \gamma, \delta\) are nonzero and \(A, B\) are two nonzero 1-forms such that \(g(X, \xi_1) = A(X)\) and \(g(X, \xi_2) = B(X)\), \(\xi_1, \xi_2\) being unit vectors which are orthogonal, i.e., \(g(\xi_1, \xi_2) = 0\) and \(D\) is symmetric \((0, 2)\) tensor with zero trace which satisfies the condition \(D(X, \xi_1) = 0, \ \forall X \in \chi(M)\).

Here \(\alpha, \beta, \gamma, \delta\) are called the associated scalars, and \(A, B\) are called the associated main and auxiliary 1-forms respectively, \(\xi_1, \xi_2\) are main and auxiliary generators and \(D\) is called the associated tensor of the manifold.

In [4], A. Bhattacharyya and T. De introduced the notion of mixed generalized quasi-Einstein manifold, denoted by \(MG(QE)_n\). A non-flat Riemannian manifold \((M, g), (n \geq 3)\) is called if its the Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies the condition

\[
S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \varrho B(X)B(Y) + \gamma \left[ A(X)B(Y) + A(Y)B(X) \right], \tag{1.4}
\]
where \( \alpha, \beta, \varrho, \gamma \) are scalars such that \( \beta, \varrho, \gamma, \delta \) are nonzero and \( A, B \) are two nonzero 1-forms such that

\[
g(X, \xi_1) = A(X), \quad g(X, \xi_2) = B(X), \quad g(\xi_1, \xi_2) = 0, \quad \forall X, \tag{1.5}
\]

\( \xi_1, \xi_2 \) being unit vectors which are orthogonal.

Here \( \alpha, \beta, \varrho, \gamma \) are called the associated scalars, and \( A, B \) are called the associated main and auxiliary 1-forms respectively, \( \xi_1, \xi_2 \) are main and auxiliary generators of the manifold.

Let \( M \) be an \( m \)-dimensional, \( m \geq 3 \), Riemannian manifold and \( p \in M \). Denote by \( K(\pi) \) or \( K(U \wedge V) \) the sectional curvature of \( M \) associated with a plane section \( \pi \subseteq T_p M \), where \( \{U, V\} \) is an orthonormal basis of \( \pi \). For a \( n \)-dimensional subspace \( L \subseteq T_p M, 2 \leq n \leq m \), its scalar curvature \( \tau(L) \) is denoted by \( \tau(L) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j) \), where \( \{e_1, e_2, \ldots, e_n\} \) is any orthonormal basis of \( L \) ([9]).

The notion of warped product generalizes that of a surface of revolution. It was introduced in [5], for studying manifolds of negative curvature. Let \((B, g_B), (F, g_F)\) be two Riemannian manifolds with \( \dim B = m > 0, \dim F = k > 0 \) and \( f: B \to (0, \infty), f \in C^\infty(B) \). The warped product \( M = B \times_f F \) is the Riemannian manifold \( B \times F \) furnished with the metric \( g_M = g_B + f^2 g_F \). \( B \) is called the base of \( M \), \( F \) is the fibre and the warped product is called a simply Riemannian product if \( f \) is a constant function. The function \( f \) is called the warping function of the warped product[15].

Singer and Thorpe gave the well-known characterization of 4-dimensional Einstein spaces in [20]. Later we have seen that in [7] Chen obtained the generalization of 4-dimensional Einstein spaces. In [10] the result for odd dimensional Einstein spaces was obtained by Dumitru. Also in [2] Bejan generalized these results (both odd and even dimensions) to quasi Einstein manifold. Also characterization of super quasi-Einstein manifold for both of odd and even dimensions was studied in [12]. From above studies, we have given characterization of mixed generalized quasi-Einstein manifold for both of odd and even dimensions with three and four dimensional examples of mixed generalized quasi-Einstein manifold to ensure the existence of such manifold. Next we obtain that a mixed generalized quasi-Einstein manifold is generalized quasi-Einstein manifold if either of generators is parallel vector field. In the last section we have given examples of warped product on mixed generalized quasi-Einstein manifold.

Geodesic mappings of Einstein spaces were studied in [18, 16, 11, 13, 19], and others. In [11, 17, 19] there are metrics of Einstein spaces.

\section{2 Characterization of mixed generalized quasi-Einstein manifold}

In this section we establish the characterization of odd and even dimensional \( MG(QE)_n \).

**Theorem 2.1.** A Riemannian manifold of dimension \((2n+1)\) with \( n \geq 2 \) is mixed generalized quasi-Einstein manifold if and only if the Ricci operator \( Q \)
has eigen vector fields $\xi_1$ and $\xi_2$ such that at any point $p \in M$, there exist three real numbers $a, b$ and $c$ satisfying

$$\tau(P) + a = \tau(P\perp) ; \quad \xi_1, \xi_2 \in T_pP\perp,$$

$$\tau(N) + b = \tau(N\perp) ; \quad \xi_1 \in T_pN, \xi_2 \in T_pN\perp,$$

$$\tau(R) + c = \tau(R\perp) ; \quad \xi_1 \in T_pR, \xi_2 \in T_pR\perp,$$

for any $n$-plane sections $P$, $N$ and $(n + 1)$-plane section $R$ where $P\perp$, $N\perp$ and $R\perp$ denote the orthogonal complements of $P$, $N$ and $R$ in $T_pM$ respectively and

$$a = \{\alpha + \beta + \varrho\}/2, \quad b = \{\alpha - \beta + \varrho\}/2, \quad c = \{\varrho - \alpha - \beta\}/2,$$

where $\alpha$, $\beta$, $\varrho$ are scalars.

**Proof.** First suppose that $M$ is a $(2n + 1)$ dimensional mixed generalized quasi-Einstein manifold, so

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \varrho B(X)B(Y) + \gamma [A(X)B(Y) + A(Y)B(X)], \quad (2.1)$$

where $\alpha$, $\beta$, $\varrho$, $\gamma$ are scalars such that $\beta$, $\varrho$, $\gamma$ are nonzero and $A$, $B$ are two nonzero 1-forms such that $g(X, \xi_1) = A(X)$ and $g(X, \xi_2) = B(X)$, $\forall X \in \chi(M)$, $\xi_1, \xi_2$ being unit vectors which are orthogonal, i.e., $g(\xi_1, \xi_2) = 0$.

Let $P \subseteq T_pM$ be an $n$-dimensional plane orthogonal to $\xi_1$, $\xi_2$ and let $\{e_1, e_2, \ldots, e_n\}$ be orthonormal basis of it. Since $\xi_1$ and $\xi_2$ are orthogonal to $P$, we can take orthonormal basis $\{e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}$ of $P\perp$ such that $e_{2n} = \xi_1$ and $e_{2n+1} = \xi_2$. Thus $\{e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}$ is an orthonormal basis of $T_pM$. Then we can take $X = Y = e_i$ in (2.1), we have

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} 
\alpha, & \text{for } 1 \leq i \leq 2n - 1 \\
\alpha + \beta, & \text{for } i = 2n \\
\alpha + \varrho, & \text{for } i = 2n + 1 
\end{cases}$$

By use of (2.1) for any $1 \leq i \leq 2n + 1$, we can write

$$S(e_1, e_1) = K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \cdots + K(e_1 \wedge e_{2n-1}) + K(e_1 \wedge \xi_1) + K(e_1 \wedge \xi_2) = \alpha,$$

$$S(e_2, e_2) = K(e_2 \wedge e_1) + K(e_2 \wedge e_3) + \cdots + K(e_2 \wedge e_{2n-1}) + K(e_2 \wedge \xi_1) + K(e_2 \wedge \xi_2) = \alpha,$$

$$S(e_{2n-1}, e_{2n-1}) = K(e_{2n-1} \wedge e_1) + K(e_{2n-1} \wedge e_2) + K(e_{2n-1} \wedge e_3) + \cdots + K(e_{2n-1} \wedge \xi_2) = \alpha,$$

$$S(\xi_1, \xi_1) = K(\xi_1 \wedge e_1) + K(\xi_1 \wedge e_2) + \cdots + K(\xi_1 \wedge e_{2n-1}) + K(\xi_1 \wedge \xi_2) = \alpha + \beta,$$

$$S(\xi_2, \xi_2) = K(\xi_2 \wedge e_1) + K(\xi_2 \wedge e_2) + \cdots + K(\xi_2 \wedge e_{2n-1}) + K(\xi_2 \wedge \xi_1) = \alpha + \varrho.$$

Adding first $n$-equations, we get

$$2\tau(P) + \sum_{1 \leq i < j \leq 2n+1} K(e_i \wedge e_j) = n\alpha. \quad (2.2)$$
Then adding the last \((n + 1)\) equations, we have
\[
2\tau(P^\perp) + \sum_{1\leq j \leq n < i \leq 2n+1} K(e_i \wedge e_j) = (n + 1)\alpha + \beta + \varrho \tag{2.3}
\]

Then, by substracting the equation (2.2) and (2.3), we obtain
\[
\tau(P^\perp) - \tau(P) = \{\alpha + \beta + \varrho\}.
\]

Therefore \(\tau(P) + a = \tau(P^\perp)\), where \(a = \{\alpha + \beta + \varrho\}/2\). Similarly, Let \(N \subseteq T_pM\) be an \(n\)-dimensional plane orthogonal to \(\xi_2\) and let \(\{e_1, e_2, \ldots, e_n\}\) be orthonormal basis of it. Since \(\xi_2\) is orthogonal to \(N\), we can take an orthonormal basis \(\{e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}\) of \(N^\perp\) orthogonal to \(\xi_1\), such that \(e_n = \xi_1\) and \(e_{2n+1} = \xi_2\), respectively. Thus, \(\{e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}\) is an orthonormal basis of \(T_pM\). Then we can take \(X = Y = e_i\) in (2.1) to have
\[
S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} 
\alpha, & 1 \leq i \leq n-1 \\
\alpha + \beta, & i = n \\
\alpha, & n + 1 \leq i \leq 2n \\
\alpha + \varrho, & i = 2n+1
\end{cases}
\]

Adding first \(n\)-equations, we get
\[
2\tau(N) + \sum_{1\leq i \leq n < j \leq 2n+1} K(e_i \wedge e_j) = n\alpha + \beta, \tag{2.4}
\]

and adding the last \((n + 1)\) equations, we have
\[
2\tau(N^\perp) + \sum_{1\leq j \leq n < i \leq 2n+1} K(e_i \wedge e_j) = (n + 1)\alpha + \varrho. \tag{2.5}
\]

Then, by substracting the equation (2.4) and (2.5), we obtain
\[
\tau(N^\perp) - \tau(N) = \{\alpha - \beta + \varrho\}/2.
\]

Therefore \(\tau(N) + b = \tau(N^\perp)\), where \(b = \{\alpha - \beta + \varrho\}/2\). Analogously, Let \(R \subseteq T_pM\) be an \((n + 1)\)-plane orthogonal to \(\xi_2\) and let \(\{e_1, e_2, \ldots, e_{n+1}\}\) be orthonormal basis of it. Since \(\xi_2\) is orthogonal to \(R\), we can take an orthonormal basis \(\{e_{n+2}, e_{n+3}, \ldots, e_{2n}, e_{2n+1}\}\) of \(R^\perp\) orthogonal to \(\xi_1\), such that \(e_{n+1} = \xi_1\) and \(e_{2n+1} = \xi_2\). Thus, \(\{e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}\) is an orthonormal basis of \(T_pM\). Then we can take \(X = Y = e_i\) in (2.1) to have
\[
S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} 
\alpha, & 1 \leq i \leq n \\
\alpha + \beta, & i = n + 1 \\
\alpha, & n + 2 \leq i \leq 2n \\
\alpha + \varrho, & i = 2n+1
\end{cases}
\]

Adding the first \(n + 1\)-equations, we get
\[
2\tau(R) + \sum_{1\leq i \leq n+1 < j \leq 2n+1} K(e_i \wedge e_j) = (n + 1)\alpha + \beta, \tag{2.6}
\]
and adding the last \( n \) equations, we have
\[
2\tau(R^\perp) + \sum_{1 \leq j \leq n+1 < i \leq 2n+1} K(e_i \wedge e_j) = n\alpha + g. \tag{2.7}
\]
Then, by substracting the equation (2.6) and (2.7), we obtain
\[
\tau(R^\perp) - \tau(R) = \{g - \alpha - \beta\}/2.
\]
Therefore \( \tau(R) + c = \tau(R^\perp) \), where \( c = \{g - \alpha - \beta\}/2 \).

Conversely, let \( V \) be an arbitrary unit vector of \( T_pM \), at \( p \in M \), orthogonal to \( \xi_1 \) and \( \xi_2 \). We take an orthonormal basis \( \{e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\} \) of \( T_pM \) such that \( V = e_1, e_{n+1} = \xi_1 \) and \( e_{2n+1} = \xi_2 \). We consider \( n \)-plane section \( N \) and \((n+1)\)-plane section \( R \) in \( T_pM \) as follows \( N = \text{span}\{e_1, \ldots, e_n, e_{n+1}\} \) and \( R = \text{span}\{e_1, e_2, \ldots, e_n, e_{n+1}\} \) respectively. Then we have
\[
N^\perp = \text{span}\{e_1, e_{n+2}, \ldots, e_2n, e_{2n+1}\} \quad \text{and} \quad R^\perp = \text{span}\{e_{n+2}, \ldots, e_{2n}\}
\]
respectively. Now
\[
S(V, V) = [K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \cdots + K(e_1 \wedge e_{n+1})]
+ [K(e_1 \wedge e_{n+2}) + K(e_1 \wedge e_{2n}) + K(e_1 \wedge e_{2n+1})]
= [\tau(R) - \sum_{2 \leq i < j \leq n+1} K(e_i \wedge e_j)] + [\tau(N^\perp) - \sum_{n+2 \leq i < j \leq 2n+1} K(e_i \wedge e_j)]
= \tau(R) - \tau(N) + \tau(R^\perp) - \tau(N^\perp)
= [\tau(R) - \tau(N)] + [b + \tau(N) - c - \tau(R)] = b - c.
\]
Therefore, \( S(V, V) = b - c \), for any unit vector \( V \in T_pM \), orthogonal to \( \xi_1 \) and \( \xi_2 \). Then we can write for any \( 1 \leq i \leq 2n+1 \), \( S(e_i, e_i) = b - c \), since \( S(V, V) = (b-c)g(V, V) \). It follows that
\[
S(X, X) = (b - c)g(X, X) + K_1A(X)A(X)
\]
and
\[
S(Y, Y) = (b - c)g(Y, Y) + K_2B(Y)B(Y) + K_3[A(Y)B(Y) + B(Y)A(Y)]
\]
for any \( X \in [\text{span}\{\xi_1\}]^\perp \) and \( Y \in [\text{span}\{\xi_2\}]^\perp \), where \( A, B \) are the dual forms of \( \xi_1 \) and \( \xi_2 \) with respect to \( g \), respectively and \( K_1, K_2, K_3 \) are scalars, such that \( K_1 \neq 0, K_2 \neq 0, K_3 \neq 0 \).

Now from the above equations, we get from symmetry that \( S \) with tensors \((b-c)g + K_1(A \otimes A)\) and \((b-c) + K_2(B \otimes B) + K_3[(A \otimes B) + (A \otimes B)]\) must coincide on the complement of \( \xi_1 \) and \( \xi_2 \), respectively, that is
\[
S(X, Y) = (b - c)g(X, Y)
+ K_1A(X)A(Y) + K_2B(X)B(Y) + K_3[A(X)B(Y) + B(X)A(Y)],
\]
for any $X, Y \in [\text{span}\{\xi_1, \xi_2\}]^\perp$. Since $\xi_1$ and $\xi_2$ are eigenvector fields of $Q$, we also have $S(X, \xi_1) = 0$ and $S(Y, \xi_2) = 0$ for any $X, Y \in T_pM$ orthogonal to $\xi_1$ and $\xi_2$. Thus, we can extend the above equation to

$$S(X, Z) = (b - c)g(X, Z) + K_1 A(X)A(Z) + K_2 B(X)B(Z) + K_3 [A(X)B(Z) + A(Z)B(X)], \quad (2.8)$$

for any $X \in [\text{span}\{\xi_1, \xi_2\}]^\perp$ and $Z \in T_pM$, where $K_1, K_2, K_3$ are scalars and $K_1 \neq 0, K_2 \neq 0, K_3 \neq 0$. Now, let us consider the $n$-plane section $P$ and $(n + 1)$-plane section $R$ in $T_pM$ as follows $P = \text{span}\{e_1, e_2, \ldots, e_n\}$ and $R = \text{span}\{e_1, e_2, \ldots, e_n, \xi_1\}$. Then we have $P^\perp = \text{span}\{e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}$ and $R^\perp = \text{span}\{e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}$ respectively. Now

$$S(\xi_1, \xi_1) = [K(\xi_1 \wedge e_1) + K(\xi_1 \wedge e_2) + \cdots + K(\xi_1 \wedge e_n)]$$

$$+ [K(\xi_1 \wedge e_{n+2}) + \cdots + K(\xi_1 \wedge e_{2n+1})]$$

$$= [\tau(R) - \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)] + [\tau(P^\perp) - \sum_{n+2 \leq i < j \leq 2n+1} K(e_i \wedge e_j)]$$

$$= \tau(R) - \tau(P) + \tau(P^\perp) - \tau(R^\perp) = [\tau(R) - \tau(P)] + [a + \tau(P) - c - \tau(R)] = a - c$$

Therefore we can write

$$S(\xi_1, \xi_1) = (b - c)g(\xi_1, \xi_1) + (a - b)A(\xi_1)A(\xi_1). \quad (2.9)$$

Analogously, let us consider the $n$-plane section $P$ and $N \in T_pM$ as follows $P = \text{span}\{e_1, e_2, \ldots, e_n\}$ and $N = \text{span}\{e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$ respectively. Then we have $P^\perp = \text{span}\{e_{n+1}, e_{n+2}, \ldots, e_{2n}, \xi_2\}$ and $N^\perp = \text{span}\{e_1, \ldots, e_n, \xi_2\}$ respectively. Now, we have

$$S(\xi_2, \xi_2) = [K(\xi_2 \wedge e_1) + K(\xi_2 \wedge e_2) + \cdots + K(\xi_2 \wedge e_n)]$$

$$+ [K(\xi_2 \wedge e_{n+1}) + K(\xi_2 \wedge e_{n+2}) + \cdots + K(e_2 \wedge e_{2n})]$$

$$= [\tau(N^\perp) - \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)] + [\tau(P^\perp) - \sum_{n+1 \leq i < j \leq 2n} K(e_i \wedge e_j)]$$

$$= \tau(N^\perp) - \tau(P) + \tau(P^\perp) - \tau(N) = [\tau(N) + b - \tau(P)] + [a + \tau(P) - \tau(N)] = a + b.$$
Theorem 2.2. A Riemannian manifold of dimension $2n$ with $n \geq 2$ is mixed generalized quasi-Einstein manifold if and only if the Ricci operator $Q$ has eigen vector fields $\xi_1$ and $\xi_2$ such that at any point $p \in M$, there exist three real numbers $a$, $b$ and $c$ satisfying

$$
\tau(P) + a = \tau(P^\perp); \quad \xi_1, \xi_2 \in T_p P^\perp,
$$
$$
\tau(N) + b = \tau(N^\perp); \quad \xi_1 \in T_p N, \xi_2 \in T_p N^\perp,
$$
$$
\tau(R) + c = \tau(R^\perp); \quad \xi_1 \in T_p R, \xi_2 \in T_p R^\perp,
$$

for any $n$-plane section $P$, $N$ and $(n+1)$-plane section $R$ where $P^\perp$, $N^\perp$ and $R^\perp$ denote the orthogonal complements of $P$, $N$ and $R$ in $T_p M$ respectively and $a = (\beta + \varrho)/2$, $b = (2\alpha - \beta + \varrho)/2$, $c = (\varrho - \beta)/2$,

where $\alpha$, $\beta$, $\varrho$ are scalars.

Proof. Let $P$ and $R$ be $n$-plane sections and $N$ be an $(n-1)$-plane section such that, $P = \text{span}\{e_1, e_2, \ldots, e_n\}$, $R = \text{span}\{e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$ and $N = \text{span}\{e_2, e_3, \ldots, e_n\}$ respectively. Therefore the orthogonal complements of these sections can be written as $P^\perp = \text{span}\{e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$, $R^\perp = \text{span}\{e_1, e_2, \ldots, e_n\}$ and $N^\perp = \text{span}\{e_1, e_{n+1}, \ldots, e_{2n}\}$.

Then rest of the proof is similar to the proof of Theorem 2.1. \qed

3 \quad MG(QE)$_n$ with the parallel vector field generators

Theorem 3.1. A mixed generalized quasi-Einstein manifold is generalized quasi-Einstein manifold if either of generators is parallel vector field.

Proof. By the definition of the Riemannian curvature tensor, if $\xi_1$ is parallel vector field, then we find that

$$
R(X, Y)\xi_1 = \nabla_X \nabla_Y \xi_1 - \nabla_Y \nabla_X \xi_1 - \nabla_{[X,Y]} \xi_1 = 0,
$$

and consequently we get

$$
S(X, \xi_1) = 0. \quad (3.1)
$$

Again, put $Y = \xi_1$ in the equation (1.2) and applying (1.3) and (1.4), we get

$$
S(X, \xi_1) = (\alpha + \beta)g(X, \xi_1) + \gamma g(X, \xi_2).
$$

So, if $\xi_1$ is a parallel vector field, by (3.1), we get

$$
(\alpha + \beta)g(X, \xi_1) + \gamma g(X, \xi_2) = 0. \quad (3.2)
$$

Now, putting $X = \xi_2$ in the equation (3.2) and using (1.3) we get $\gamma = 0$. So, if $\xi_1$ is parallel vector field in a mixed generalized quasi-Einstein manifold, then the manifold is generalized quasi Einstein manifold.
Again, if $\xi_2$ is parallel vector field, then $R(X, Y)\xi_2 = 0$. Contracting, we get

$$S(Y, \xi_2) = 0. \quad (3.3)$$

Putting $X = \xi_2$ in the equation (1.2) and applying (1.3), we get

$$S(Y, \xi_2) = (\alpha + \varrho)g(Y, \xi_2) + \gamma g(Y, \xi_1).$$

If, $\xi_2$ is a parallel vector field, by (3.3), we get

$$(\alpha + \varrho)g(Y, \xi_2) + \gamma g(Y, \xi_1) = 0. \quad (3.4)$$

Putting $Y = \xi_1$ and using (3.4), (1.3), (1.4), we get $\gamma = 0$, i.e., the manifold is generalized quasi-Einstein manifold.

4 Examples of 3-dimensional and 4-dimensional mixed generalized quasi-Einstein manifold

Example 4.1. Let us consider a Riemannian metric $g$ on $\mathbb{R}^3$ by

$$ds^2 = g_{ij}dx^i dx^j = (x^3)^{4/3}[(dx^1)^2 + (dx^2)^2] + (dx^3)^2,$$

$(i, j = 1, 2, 3)$ and $x^3 \neq 0$. Then the only non-vanishing components of Christoffel symbols, the curvature tensors and the Ricci tensors are

$$\Gamma^1_{13} = \Gamma^2_{23} = \frac{2}{3x^3}, \quad \Gamma^3_{11} = \Gamma^3_{22} = -\frac{2}{3}(x^3)^{1/3}$$

$$R_{1331} = R_{2332} = -\frac{2}{9(x^3)^{2/3}}, \quad R_{1221} = \frac{4}{9}(x^3)^{2/3}$$

$$R_{11} = R_{22} = \frac{2}{9(x^3)^{2/3}}, \quad R_{33} = -\frac{4}{9(x^3)^2}$$

Let us consider the associated scalars $\alpha$, $\beta$, $\varrho$, $\gamma$ as follows:

$$\alpha = -\frac{4}{9(x^3)^2}, \quad \beta = \frac{6(x^3)^{1/3}}{9}, \quad \varrho = \frac{12}{9(x^3)^{2/3}}, \quad \gamma = -\frac{6}{9(x^3)^{2/3}},$$

and the 1-forms

$$A_i(x) = \begin{cases} \frac{1}{x^3} & \text{for } i = 1, 2, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad B_i(x) = \begin{cases} (x^3)^{1/3} & \text{for } i = 2, \\ 0 & \text{otherwise} \end{cases}$$

Then we have

(i) $R_{11} = \alpha g_{11} + \beta A_1 A_1 + \varrho B_1 B_1 + \gamma [A_1 B_1 + A_1 B_1]$  
(ii) $R_{22} = \alpha g_{22} + \beta A_2 A_2 + \varrho B_2 B_2 + \gamma [A_2 B_2 + A_2 B_2]$  
(iii) $R_{33} = \alpha g_{33} + \beta A_3 A_3 + \varrho B_3 B_3 + \gamma [A_3 B_3 + A_3 B_3]$
Since all the cases other than \((i)-(iii)\) are trivial, we can say that
\[
R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma [A_i B_j + A_j B_i] \quad \text{for } i, j = 1, 2, 3.
\]
Thus if \((R^3, g)\) is a Riemannian manifold endowed with the metric given by
\[
ds^2 = g_{ij}dx^i dx^j = (x^3)^{4/3}[(dx^1)^2 + (dx^2)^2] + (dx^3)^2,
\]
\((i, j = 1, 2, 3)\) and \(x^3 \neq 0\), then \((R^3, g)\) is an MG(QE)_3.

Next we consider the Lorentzian metric \(g\) on \(R^3\) by
\[
ds^2 = g_{ij}dx^i dx^j = -(x^3)^{4/3}(dx^1)^2 + (x^3)^{4/3}(dx^2)^2 + (dx^3)^2,
\]
\((i, j = 1, 2, 3)\) and \(x^3 \neq 0\).

Now, by similar way, after some construction of associated scalars and associated 1-forms, we can say that the manifold is a mixed generalized quasi-Einstein manifold. Therefore we get another example of MG(QE)_3.

**Example 4.2.** \((R^3, g)\) is a Lorentzian manifold endowed with the metric given by
\[
ds^2 = g_{ij}dx^i dx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2]
\]
\((i, j = 1, 2, 3, 4)\) and \(p = \frac{\varepsilon^4}{k^2}, k\) is constant, then the only non-vanishing components of Christoffel symbols, the curvature tensors and the Ricci tensors are
\[
\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{p}{1 + 2p}, \quad \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{p}{1 + 2p},
\]
\[
R_{1221} = R_{1331} = R_{1441} = \frac{p}{1 + 2p}, \quad R_{2332} = R_{2442} = R_{3443} = \frac{p^2}{1 + 2p},
\]
\[
R_{11} = \frac{3p}{(1 + 2p)^2}, \quad R_{22} = R_{33} = R_{44} = \frac{p}{(1 + 2p)}.
\]
It can be easily seen that the scalar curvature \(r\) of the given manifold \((R^4, g)\) is
\[
r = \frac{6p(1 + p)}{(1 + 2p)^3},
\]
which is non-vanishing and non-constant.

Let us consider the associated scalars \(\alpha, \beta, \gamma, \delta\) as follows:
\[
\alpha = \frac{p}{(1 + 2p)^2}, \quad \beta = \frac{2p}{(1 + 2p)^3}, \quad \gamma = \frac{p}{(1 + 2p)^3}, \quad \delta = -\frac{p}{2(1 + 2p)^2}.
\]
Then we have
\[ R_{i1} = \alpha g_{i1} + \beta A_i A_1 + \gamma B_i B_1 + \delta[A_1 B_1 + A_1 B_1] \]
\[ R_{i2} = \alpha g_{i2} + \beta A_i A_2 + \gamma B_i B_2 + \delta[A_2 B_2 + A_2 B_2] \]
\[ R_{i3} = \alpha g_{i3} + \beta A_i A_3 + \gamma B_i B_3 + \delta[A_3 B_3 + A_3 B_3] \]
\[ R_{i4} = \alpha g_{i4} + \beta A_i A_4 + \gamma B_i B_4 + \delta[A_4 B_4 + A_4 B_4] \]

Since all the cases other than (i)–(iv) are trivial, we can say that

\[ R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j + \delta[A_i B_j + A_j B_i], \quad \text{for } i, j = 1, 2, 3, 4. \]

So if \((R^4, g)\) be a Riemannian manifold endowed with the metric given by

\[ ds^2 = g_{ij} dx^i dx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2] \]

\((i, j = 1, 2, 3, 4)\) and \(p = \frac{\varepsilon_i}{k^2}, k \text{ is constant},\) then \((R^4, g)\) is a mixed generalized quasi Einstein manifold with non-zero and non-constant scalar curvature.

If we consider the Lorentzian metric \(g\) on \(R^3\) by

\[ ds^2 = g_{ij} dx^i dx^j = -(1 + 2p)(dx^1)^2 + (1 + 2p)[(dx^2)^2 + (dx^3)^2 + (dx^4)^2] \]

\((i, j = 1, 2, 3, 4)\) and \(p = \frac{\varepsilon_i}{k^2}, k \text{ is constant}.\)

Now, by similar way after some construction of associated scalars and associated 1-forms, we can say that the manifold is a mixed generalized quasi-Einstein manifold. Therefore we get another example of \(MG(QE)_4.\)

**Example 4.4.** Let \((R^4, g)\) be a Lorentzian manifold endowed with the metric given by

\[ ds^2 = g_{ij} dx^i dx^j = -(1 + 2p)(dx^1)^2 + (1 + 2p)[(dx^2)^2 + (dx^3)^2 + (dx^4)^2] \]

\((i, j = 1, 2, 3, 4)\) and \(p = \frac{\varepsilon_i}{k^2}, k \text{ is constant}.\) Then \((R^4, g)\) is an \(MG(QE)_4\) with non-zero and non-constant scalar curvature.

## 5 Examples of warped product on mixed generalized quasi-Einstein manifold

**Example 5.1.** Here we consider the Example 4.1, a 3-dimensional example of mixed generalized quasi-Einstein manifold. Let \((R^3, g)\) be a Riemannian manifold endowed with the metric given by

\[ ds^2 = g_{ij} dx^i dx^j = (x^3)^{4/3}[(dx^1)^2 + (dx^2)^2] + (dx^3)^2, \]

where \((i, j = 1, 2, 3)\) and \(x^3 \neq 0.\)
To define warped product on $MG(QE)_3$, we consider the warping function $f: R \setminus 0 \to (0, \infty)$ by $f(x^3) = (x^3)^{\frac{2}{3}}$ and observe that $f = (x^3)^{\frac{2}{3}} > 0$ is a smooth function. The line element defined on $R \setminus \{0\} \times R^2$ which is of the form $B \times_f F$, where $B = R \setminus \{0\}$ is the base and $F = R^2$ is the fibre.

Therefore the metric $ds^2_M$ can be expressed as $ds^2_B + f^2 ds^2_F$, i.e.,

$$ds^2 = g_{ij}dx^idx^j = (dx^3)^2 + \{(x^3)^{2/3}\}^2[(dx^1)^2 + (dx^2)^2],$$

which is the example of Riemannian warped product on $MG(QE)_3$.

**Example 5.2.** We consider the example 4.3, a 4-dimensional example of mixed generalized quasi-Einstein manifold. Let $(R^4, g)$ be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^idx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

where $(i, j = 1, 2, 3, 4)$, $p = \frac{e^{x^1}}{k^2}$, $k$ is constant.

To define warped product on $MG(QE)_4$, we consider the warping function $f: R^3 \to (0, \infty)$ by $f(x^1, x^2, x^3) = \sqrt{1+2p}$ and we observe that $f > 0$ is a smooth function. The line element defined on $R^3 \times R$ which is of the form $B \times_f F$, where $B = R^3$ is the base and $F = R$ is the fibre.

Therefore the metric $ds^2_M$ can be expressed as $ds^2_B + f^2 ds^2_F$, i.e.,

$$ds^2 = g_{ij}dx^idx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + [(1+2p)^2(dx^4)^2],$$

which is the example of Riemannian warped product on $MG(QE)_4$.

Finally we note that the similar metrics were obtained in [1].

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**References**


Characterization on mixed generalized quasi-Einstein manifold


