Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **55**, 2 (2016) 143–155

Characterization on Mixed Generalized Quasi-Einstein Manifold

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(Received February 23, 2016)

Abstract

In the present paper we study characterizations of odd and even dimensional mixed generalized quasi-Einstein manifold. Next we prove that a mixed generalized quasi-Einstein manifold is a generalized quasi-Einstein manifold under a certain condition. Then we obtain three and four dimensional examples of mixed generalized quasi-Einstein manifold to ensure the existence of such manifold. Finally we establish the examples of warped product on mixed generalized quasi-Einstein manifold.

Key words: Einstein manifold, quasi-Einstein manifold, generalized quasi-Einstein manifold, mixed generalized quasi-Einstein manifold, super quasi-Einstein manifold, warped product.

2010 Mathematics Subject Classification: 53C25

1 Introduction

A Riemannian manifold (M,g) with dimension $(n \geq 2)$ is said to be an Einstein manifold if the Ricci tensor satisfies the condition $S(X,Y) = \frac{r}{n}g(X,Y)$, holds on M, here S and r denote the Ricci tensor and the scalar curvature of (M,g) respectively. According to [3] the above equation is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry,

*The first author is supported by UGC JRF of India, Ref. No: 23/06/2013(i)EU-V.

as well as in general theory of relativity. The notion of quasi-Einstien manifold was defined in [9]. A non-flat Riemannian manifold (M, g), $(n \geq 2)$ is said to be an quasi Einstein manifold if the condition

$$S(X, Y) = \alpha g(X, Y) + \beta\rho(X)\rho(Y),$$

is fulfilled on M, where α and β are scalars of which $\beta \neq 0$ and ρ is non-zero 1-form such that $g(X, \xi) = \rho(X)$ for all vector field X and ξ is a unit vector field.

Note that the subprojective manifolds by Kagan have the Ricci tensor with the same properties [14, 19].

In [8], U. C. De and G. C. Ghosh introduced generalized quasi-Einstein manifold, denoted by $G(QE)_n$ where the Ricci tensor S of type $(0, 2)$ which is not identically zero satisfies the condition

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \varrho B(X)B(Y), \tag{1.1}$$

where α, β, ϱ are scalars such that β, ϱ are nonzero and A, B are two nonzero 1-forms such that

$$g(X, \xi_1) = A(X), \; g(X, \xi_2) = B(X), \; \forall X, \tag{1.2}$$

ξ_1, ξ_2 being unit vectors which are orthogonal, i.e., $g(\xi_1, \xi_2) = 0$.

Here $\alpha, \beta, \gamma, \delta$ are called the associated scalars, and A, B are called the associated main and auxiliary 1-forms respectively, ξ_1, ξ_2 are main and auxiliary generators of the manifold.

In [6], M. C. Chaki introduced super quasi-Einstein manifold, denoted by $S(QE)_n$ and gave an example of a 4-dimensional semi Riemannian super quasi-Einstein manifold, where the Ricci tensor S of type $(0, 2)$ which is not identically zero satisfies the condition

$$\begin{aligned} S(X, Y) = \alpha g(X, \; Y) &+ \beta A(X)A(Y) \\ &+ \gamma[A(X)B(Y) + A(Y)B(X)] + \delta D(X, Y), \end{aligned} \tag{1.3}$$

where α, β, γ are scalars such that β, γ, δ are nonzero and A, B are two nonzero 1-forms such that $g(X, \xi_1) = A(X)$ and $g(X, \xi_2) = B(X)$, ξ_1, ξ_2 being unit vectors which are orthogonal, i.e., $g(\xi_1, \xi_2) = 0$ and D is symmetric $(0, 2)$ tensor with zero trace which satisfies the condition $D(X, \xi_1) = 0$, $\forall X \in \chi(M)$.

Here α, β, γ, δ are called the associated scalars, and A, B are called the associated main and auxiliary 1-forms respectively, ξ_1, ξ_2 are main and auxiliary generators and D is called the associated tensor of the manifold.

In [4], A. Bhattacharyya and T. De introduced the notion of mixed generalized quasi-Einstein manifold, denoted by $MG(QE)_n$. A non-flat Riemannian manifold(M, g),$(n \geq 3)$ is called if its the Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$\begin{aligned} S(X, Y) = \alpha g(X, Y) &+ \beta A(X)A(Y) \\ &+ \varrho B(X)B(Y) + \gamma[A(X)B(Y) + A(Y)B(X)], \end{aligned} \tag{1.4}$$

where α, β, ϱ, γ are scalars such that β, ϱ, γ, δ are nonzero and A, B are two nonzero 1-forms such that

$$g(X, \xi_1) = A(X), \quad g(X, \xi_2) = B(X), \quad g(\xi_1, \xi_2) = 0, \quad \forall X, \tag{1.5}$$

ξ_1, ξ_2 being un,it vectors which are orthogonal.

Here α, β, ϱ, γ are called the associated scalars, and A, B are called the associated main and auxiliary 1-forms respectively, ξ_1, ξ_2 are main and auxiliary generators of the manifold.

Let M be an m-dimensional, $m \geq 3$, Riemannian manifold and $p \in M$. Denote by $K(\pi)$ or $K(U \wedge V)$ the sectional curvature of M associated with a plane section $\pi \subseteq T_pM$, where $\{U, V\}$ is an orthonormal basis of π. For a n-dimensional subspace $L \subseteq T_pM$, $2 \leq n \leq m$, its scalar cuvrvature $\tau(L)$ is denoted by $\tau(L) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$, where $\{e_1, e_2, \ldots, e_n\}$ is any orthonormal basis of L ([9]).

The notion of warped product generalizes that of a surface of revolution. It was introduced in [5], for studying manifolds of negative curvature. Let (B, g_B), (F, g_F) be two Riemannian manifolds with dim $B = m > 0$, dim$F = k > 0$ and $f\colon B \to (0, \infty)$, $f \in C^\infty(B)$. The warped product $M = B \times_f F$ is the Riemannian manifold $B \times F$ furnished with the metric $g_M = g_B + f^2 g_F$. B is called the base of M, F is the fibre and the warped product is called a simply Riemannian product if f is a constant function. The function f is called the warping function of the warped product[15].

Singer and Thorpe gave the well-known characterization of 4-dimensional Einstein spaces in [20]. Later we have seen that in [7] Chen obtained the generalization of 4-dimensional Einstein spaces. In [10] the result for odd dimensional Einstein spaces was obtained by Dumitru. Also in [2] Bejan generalized these results (both odd and even dimensions)to quasi Einstein manifold. Also characterization of super quasi-Einstein manifold for both of odd and even dimensions was studied in [12]. From above studies, we have given characterization of mixed generalized quasi-Einstein manifold for both of odd and even dimensions with three and four dimensional examples of mixed generalized quasi-Einstein manifold to ensure the existence of such manifold. Next we obtain that a mixed generalized quasi-Einstein manifold is generalized quasi-Einstein manifold if either of generators is parallel vector field. In the last section we have given examples of warped product on mixed generalized quasi-Einstein manifold.

Geodesic mappings of Einstein spaces were studied in [18, 16, 11, 13, 19], and others. In [11, 17, 19] there are metrics of Einstein spaces.

2 Characterization of mixed generalized quasi-Einstein manifold manifold

In this section we establish the characterization of odd and even dimensional $MG(QE)_n$.

Theorem 2.1. *A Riemannian manifold of dimension (2n+1) with $n \geq 2$ is mixed generalized quasi-Einstein manifold if and only if the Ricci operator Q*

has eigen vector fields ξ_1 *and* ξ_2 *such that at any point* $p \in M$*, there exist three real numbers* a, b *and* c *satisfying*

$$\tau(P) + a = \tau(P^{\perp}); \quad \xi_1, \xi_2 \in T_pP^{\perp},$$
$$\tau(N) + b = \tau(N^{\perp}); \quad \xi_1 \in T_pN, \xi_2 \in T_pN^{\perp},$$
$$\tau(R) + c = \tau(R^{\perp}); \quad \xi_1 \in T_pR, \xi_2 \in T_pR^{\perp},$$

for any n*-plane sections* P*,* N *and* $(n+1)$*-plane section* R *where* $P^{\perp}$*,* $N^{\perp}$ *and* $R^{\perp}$ *denote the orthogonal complements of* P*,* N *and* R *in* T_pM *respectively and*

$$a = \{\alpha + \beta + \varrho\}/2, \quad b = \{\alpha - \beta + \varrho\}/2, \quad c = \{\varrho - \alpha - \beta\}/2,$$

where α*,* β*,* ϱ *are scalars.*

Proof. First suppose that M is a $(2n+1)$ dimensional mixed generalized quasi-Einstein manifold, so

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y) + \varrho B(X)B(Y) + \gamma[A(X)B(Y) + A(Y)B(X)], \quad (2.1)$$

where α, β, ϱ, γ are scalars such that β, ϱ, γ are nonzero and A, B are two nonzero 1-forms such that $g(X, \xi_1) = A(X)$ and $g(X, \xi_2) = B(X)$, $\forall X \in \chi(M)$, ξ_1, ξ_2 being unit vectors which are orthogonal, i.e., $g(\xi_1, \xi_2) = 0$.

Let $P \subseteq T_pM$ be an n-dimensional plane orthogonal to ξ_1, ξ_2 and let $\{e_1, e_2, \dots, e_n\}$ be orthonormal basis of it. Since ξ_1 and ξ_2 are orthogonal to P, we can take orthonormal basis $\{e_{n+1}, e_{n+2}, \dots, e_{2n+1}\}$ of $P^{\perp}$ such that $e_{2n} = \xi_1$ and $e_{2n+1} = \xi_2$. Thus $\{e_1, e_2, \dots, e_n, e_{n+1}, e_{n+2}, \dots, e_{2n+1}\}$ is an orthonormal basis of T_pM. Then we can take $X = Y = e_i$ in (2.1), we have

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} \alpha, & \text{for } 1 \le i \le 2n-1 \\ \alpha + \beta, & \text{for } i = 2n \\ \alpha + \varrho, & \text{for } i = 2n+1 \end{cases}$$

By use of (2.1) for any $1 \le i \le 2n+1$, we can write

$$S(e_1, e_1) = K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \dots + K(e_1 \wedge e_{2n-1}) + K(e_1 \wedge \xi_1) + K(e_1 \wedge \xi_2) = \alpha,$$
$$S(e_2, e_2) = K(e_2 \wedge e_1) + K(e_2 \wedge e_3) + \dots + K(e_2 \wedge e_{2n-1}) + K(e_2 \wedge \xi_1) + K(e_2 \wedge \xi_2) = \alpha,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$S(e_{2n-1}, e_{2n-1}) = K(e_{2n-1} \wedge e_1) + K(e_{2n-1} \wedge e_2) + K(e_{2n-1} \wedge e_3) + \dots + K(e_{2n-1} \wedge \xi_2) = \alpha,$$
$$S(\xi_1, \xi_1) = K(\xi_1 \wedge e_1) + K(\xi_1 \wedge e_2) + \dots + K(\xi_1 \wedge e_{2n-1}) + K(\xi_1 \wedge \xi_2) = \alpha + \beta,$$
$$S(\xi_2, \xi_2) = K(\xi_2 \wedge e_1) + K(\xi_2 \wedge e_2) + \dots + K(\xi_2 \wedge e_{2n-1}) + K(\xi_2 \wedge \xi_1) = \alpha + \varrho.$$

Adding first n-equations, we get

$$2\tau(P) + \sum_{1 \le i \le n < j \le 2n+1} K(e_i \wedge e_j) = n\alpha. \quad (2.2)$$

Then adding the last $(n+1)$ equations, we have

$$2\tau(P^{\perp}) + \sum_{1\leq j\leq n<i\leq 2n+1} K(e_i \wedge e_j) = (n+1)\alpha + \beta + \varrho \tag{2.3}$$

Then, by substracting the equation (2.2) and (2.3), we obtain

$$\tau(P^{\perp}) - \tau(P) = \{\alpha + \beta + \varrho\}.$$

Therefore $\tau(P) + a = \tau(P^{\perp})$, where $a = \{\alpha+\beta+\varrho\}/2$. Similarly, Let $N \subseteq T_pM$ be an n-dimensional plane orthogonal to ξ_2 and let $\{e_1, e_2, \ldots, e_n\}$ be orthonormal basis of it. Since ξ_2 is orthogonal to N, we can take an orthonormal basis $\{e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}$ of $N^{\perp}$ orthogonal to ξ_1, such that $e_n = \xi_1$ and $e_{2n+1} = \xi_2$, respectively. Thus, $\{e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}$ is an orthonormal basis of T_pM. Then we can take $X = Y = e_i$ in (2.1) to have

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} \alpha, & 1 \leq i \leq n-1 \\ \alpha+\beta, & i = n \\ \alpha, & n+1 \leq i \leq 2n \\ \alpha+\varrho, & i = 2n+1 \end{cases}$$

Adding first n-equations, we get

$$2\tau(N) + \sum_{1\leq i\leq n<j\leq 2n+1} K(e_i \wedge e_j) = n\alpha + \beta, \tag{2.4}$$

and adding the last $(n+1)$ equations, we have

$$2\tau(N^{\perp}) + \sum_{1\leq j\leq n<i\leq 2n+1} K(e_i \wedge e_j) = (n+1)\alpha + \varrho. \tag{2.5}$$

Then, by substracting the equation (2.4) and (2.5), we obtain

$$\tau(N^{\perp}) - \tau(N) = \{\alpha - \beta + \varrho\}/2.$$

Therefore $\tau(N) + b = \tau(N^{\perp})$, where $b = \{\alpha - \beta + \varrho\}/2$. Analogously, Let $R \subseteq T_pM$ be an $(n+1)$-plane orthogonal to ξ_2 and let $\{e_1, e_2, \ldots, e_{n+1}\}$ be orthonormal basis of it. Since ξ_2 is orthogonal to R, we can take an orthonormal basis $\{e_{n+2}, e_{n+3}, \ldots, e_{2n}, e_{2n+1}\}$ of $R^{\perp}$ orthogonal to ξ_1, such that $e_{n+1} = \xi_1$ and $e_{2n+1} = \xi_2$. Thus, $\{e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}$ is an orthonormal basis of T_pM. Then we can take $X = Y = e_i$ in (2.1) to have

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} \alpha, & 1 \leq i \leq n \\ \alpha+\beta, & i = n+1 \\ \alpha, & n+2 \leq i \leq 2n \\ \alpha+\varrho, & i = 2n+1 \end{cases}$$

Adding the first $n+1$-equations, we get

$$2\tau(R) + \sum_{1\leq i\leq n+1<j\leq 2n+1} K(e_i \wedge e_j) = (n+1)\alpha + \beta, \tag{2.6}$$

and adding the last n equations, we have

$$2\tau(R^{\perp}) + \sum_{1\leq j\leq n+1<i\leq 2n+1} K(e_i \wedge e_j) = n\alpha + \varrho. \tag{2.7}$$

Then, by substracting the equation (2.6) and (2.7), we obtain

$$\tau(R^{\perp}) - \tau(R) = \{\varrho - \alpha - \beta\}/2.$$

Therefore $\tau(R) + c = \tau(R^{\perp})$, where $c = \{\varrho - \alpha - \beta\}/2$.

Conversely, let V be an arbitrary unit vector of T_pM, at $p \in M$, orthogonal to ξ_1 and ξ_2. We take an orthonormal basis $\{e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}$ of T_pM such that $V = e_1, e_{n+1} = \xi_1$ and $e_{2n+1} = \xi_2$. We consider n-plane section N and $(n+1)$-plane section R in T_pM as follows $N = \text{span}\{e_2, \ldots, e_n, e_{n+1}\}$ and $R = \text{span}\{e_1, e_2, \ldots, e_n, e_{n+1}\}$ respectively. Then we have

$$N^{\perp} = \text{span}\{e_1, e_{n+2}, \ldots, e_{2n}, e_{2n+1}\} \quad \text{and} \quad R^{\perp} = \text{span}\{e_{n+2}. \ldots, e_{2n}\}$$

respectively. Now

$$\begin{aligned} S(V,V) &= [K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \cdots + K(e_1 \wedge e_{n+1})] \\ &\quad + [K(e_1 \wedge e_{n+2}) + \vdots + K(e_1 \wedge e_{2n}) + K(e_1 \wedge e_{2n+1})] \\ &= [\tau(R) - \sum_{2\leq i<j\leq n+1} K(e_i \wedge e_j)] + [\tau(N^{\perp}) - \sum_{n+2\leq i<j\leq 2n+1} K(e_i \wedge e_j)] \\ &= \tau(R) - \tau(N) + \tau(R^{\perp}) - \tau(N^{\perp}) = [\tau(R) - \tau(N)] + [b + \tau(N) - c - \tau(R)] = b - c. \end{aligned}$$

Therefore, $S(V,V) = b - c$, for any unit vector $V \in T_pM$, orthogonal to ξ_1 and ξ_2. Then we can write for any $1 \leq i \leq 2n+1$, $S(e_i, e_i) = b - c$, since $S(V,V) = (b - c)g(V,V)$. It follows that

$$S(X,X) = (b - c)g(X,X) + K_1 A(X)A(X)$$

and

$$S(Y,Y) = (b - c)g(Y,Y) + K_2 B(Y)B(Y) + K_3[A(Y)B(Y) + B(Y)A(Y)]$$

for any $X \in [\text{span}\{\xi_1\}]^{\perp}$ and $Y \in [\text{span}\{\xi_2\}]^{\perp}$, where A, B are the dual forms of ξ_1 and ξ_2 with respect to g, respectively and K_1, K_2, K_3 are scalars, such that $K_1 \neq 0$, $K_2 \neq 0$, $K_3 \neq 0$.

Now from the above equations, we get from symmetry that S with tensors $(b - c)g + K_1(A \otimes A)$ and $(b - c) + K_2(B \otimes B) + K_3[(A \otimes B) + (A \otimes B)]$ must coincide on the complement of ξ_1 and ξ_2, respectively, that is

$$\begin{aligned} S(X,Y) &= (b - c)g(X,Y) \\ &\quad + K_1 A(X)A(Y) + K_2 B(X)B(Y) + K_3[A(X)B(Y) + B(X)A(Y)], \end{aligned}$$

for any $X, Y \in [\text{span}\{\xi_1, \xi_2\}]^\perp$. Since ξ_1 and ξ_2 are eigenvector fields of Q, we also have $S(X, \xi_1) = 0$ and $S(Y, \xi_2) = 0$ for any $X, Y \in T_pM$ orthogonal to ξ_1 and ξ_2. Thus, we can extend the above equation to

$$S(X, Z) = (b - c)g(X, Z) + K_1 A(X)A(Z) + K_2 B(X)B(Z) + K_3[A(X)B(Z) + A(Z)B(X)], \quad (2.8)$$

for any $X \in [\text{span}\{\xi_1, \xi_2\}]^\perp$ and $Z \in T_pM$, where K_1, K_2, K_3 are scalars and $K_1 \neq 0, K_2 \neq 0, K_3 \neq 0$. Now, let us consider the n-plane section P and $(n + 1)$-plane section R in T_pM as follows $P = \text{span}\{e_1, e_2, \ldots, e_n\}$ and $R = \text{span}\{e_1, e_2, \ldots, e_n, \xi_1\}$. Then we have $P^\perp = \text{span}\{\xi_1, e_{n+2}, \ldots, e_{2n+1}\}$ and $R^\perp = \text{span}\{e_{n+2}, \ldots, e_{2n}, e_{2n+1}\}$ respectively. Now

$$\begin{aligned} S(\xi_1, \xi_1) &= [K(\xi_1 \wedge e_1) + K(\xi_1 \wedge e_2) + \cdots + K(\xi_1 \wedge e_n)] \\ &\quad + [K(\xi_1 \wedge e_{n+2}) + \cdots + K(\xi_1 \wedge e_{2n}) + K(e_1 \wedge e_{2n+1})] \\ &= [\tau(R) - \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)] + [\tau(P^\perp) - \sum_{n+2 \leq i < j \leq 2n+1} K(e_i \wedge e_j)] \\ &= \tau(R) - \tau(P) + \tau(P^\perp) - \tau(R^\perp) = [\tau(R) - \tau(P)] + [a + \tau(P) - c - \tau(R)] = a - c \end{aligned}$$

Therefore we can write

$$S(\xi_1, \xi_1) = (b - c)g(\xi_1, \xi_1) + (a - b)A(\xi_1)A(\xi_1). \quad (2.9)$$

Analogously, let us consider the n-plane section P and $N \in T_pM$ as follows $P = \text{span}\{e_1, e_2, \ldots, e_n\}$ and $N = \text{span}\{e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$ respectively. Then we have $P^\perp = \text{span}\{e_{n+1}, e_{n+2}, \ldots, e_{2n}, \xi_2\}$ and $N^\perp = \text{span}\{e_1, \ldots, e_n, \xi_2\}$ respectively. Now, we have

$$\begin{aligned} S(\xi_2, \xi_2) &= [K(\xi_2 \wedge e_1) + K(\xi_2 \wedge e_2) + \cdots + K(\xi_2 \wedge e_n)] \\ &\quad + [K(\xi_2 \wedge e_{n+1}) + K(\xi_2 \wedge e_{n+2}) + \cdots + K(e_2 \wedge e_{2n})] \\ &= [\tau(N^\perp) - \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)] + [\tau(P^\perp) - \sum_{n+1 \leq i < j \leq 2n} K(e_i \wedge e_j)] \\ &= \tau(N^\perp) - \tau(P) + \tau(P^\perp) - \tau(N) = [\tau(N) + b - \tau(P)] + [a + \tau(P) - \tau(N)] = a + b. \end{aligned}$$

Then, we get

$$S(\xi_2, \xi_2) = (b - c)g(\xi_2, \xi_2) + (a + c)B(\xi_2)B(\xi_2) + K_3[A(\xi_2)B(\xi_2) + A(\xi_2)B(\xi_2)]. \quad (2.10)$$

Now from (2.8), (2.9) and (2.10) we can write the Ricci tensor by

$$S(X,\ Y) = \mu_1 g(X,\ Y) + K_1 A(X)A(Y) + K_2 B(X)B(Y) + K_3[A(X)B(Y) + A(Y)B(X)], \quad (2.11)$$

for any $X, Y \in T_pM$. From (2.11) it follows that M is a mixed generalized quasi-Einstein manifold, where μ_1, K_1, K_2, K_3 are scalars and $K_1 \neq 0$, $K_2 \neq 0$, $K_3 \neq 0$. Hence the theorem is proved. □

Theorem 2.2. *A Riemannian manifold of dimension $2n$ with $n \geq 2$ is mixed generalized quasi-Einstein manifold if and only if the Ricci operator Q has eigen vector fields ξ_1 and ξ_2 such that at any point $p \in M$, there exist three real numbers a, b and c satisfying*

$$\tau(P) + a = \tau(P^{\perp}); \quad \xi_1, \xi_2 \in T_pP^{\perp},$$
$$\tau(N) + b = \tau(N^{\perp}); \quad \xi_1 \in T_pN, \xi_2 \in T_pN^{\perp},$$
$$\tau(R) + c = \tau(R^{\perp}); \quad \xi_1 \in T_pR, \xi_2 \in T_pR^{\perp},$$

for any n-plane section P, N and $(n+1)$-plane section R where $P^{\perp}$, $N^{\perp}$ and $R^{\perp}$ denote the orthogonal complements of P, N and R in T_pM respectively and

$$a = \{\beta + \varrho\}/2, \quad b = \{2\alpha - \beta + \varrho\}/2, \quad c = \{\varrho - \beta\}/2,$$

where α, β, ϱ are scalars.

Proof. Let P and R be n-plane sections and N be an $(n-1)$-plane section such that, $P = \text{span}\{e_1, e_2, \dots, e_n\}$, $R = \text{span}\{e_{n+1}, e_{n+2}, \dots, e_{2n}\}$ and $N = \text{span}\{e_2, e_3, \dots, e_n\}$ respectively. Therefore the orthogonal complements of these sections can be written as $P^{\perp} = \text{span}\{e_{n+1}, e_{n+2}, \dots, e_{2n}\}$, $R^{\perp} = \text{span}\{e_1, e_2, \dots, e_n\}$ and $N^{\perp} = \text{span}\{e_1, e_{n+1}, \dots, e_{2n}\}$.

Then rest of the proof is similar to the proof of Theorem 2.1. □

3 $MG(QE)_n$ with the parallel vector field generators

Theorem 3.1. *A mixed generalized quasi-Einstein manifold is generalized quasi-Einstein manifold if either of generators is parallel vector field.*

Proof. By the definition of the Riemannian curvature tensor, if ξ_1 is parallel vector field, then we find that

$$R(X,Y)\xi_1 = \nabla_X \nabla_Y \xi_1 - \nabla_Y \nabla_X \xi_1 - \nabla_{[X,Y]}\xi_1 = 0,$$

and consequently we get

$$S(X, \xi_1) = 0. \tag{3.1}$$

Again, put Y = ξ_1 in the equation (1.2) and applying (1.3) and (1.4), we get

$$S(X, \xi_1) = (\alpha + \beta)g(X, \xi_1) + \gamma g(X, \xi_2).$$

So, if ξ_1 is a parallel vector field, by (3.1), we get

$$(\alpha + \beta)g(X, \xi_1) + \gamma g(X, \xi_2) = 0. \tag{3.2}$$

Now, putting X = ξ_2 in the equation (3.2) and using (1.3) we get $\gamma = 0$. So, if ξ_1 is parallel vector field in amixed generalized quasi-Einstein manifold, then the manifold is generalized quasi Einstein manifold.

Again, if ξ_2 is parallel vector field, then $R(X,Y)\xi_2 = 0$. Contracting, we get

$$S(Y, \xi_2) = 0. \tag{3.3}$$

Putting $X = \xi_2$ in the equation (1.2) and applying (1.3), we get

$$S(Y, \xi_2) = (\alpha + \varrho)g(Y, \xi_2) + \gamma g(Y, \xi_1).$$

If, ξ_2 is a parallel vector field, by (3.3), we get

$$(\alpha + \varrho)g(Y, \xi_2) + \gamma g(Y, \xi_1) = 0. \tag{3.4}$$

Putting Y $= \xi_1$ and using (3.4), (1.3), (1.4), we get $\gamma = 0$, i.e., the manifold is generalized quasi-Einstein manifold. □

4 Examples of 3-dimensional and 4-dimensional mixed generalized quasi-Einstein manifold

Example 4.1. Let us consider a Riemannian metric g on R^3 by

$$ds^2 = g_{ij}dx^i dx^j = (x^3)^{4/3}[(dx^1)^2 + (dx^2)^2)] + (dx^3)^2,$$

$(i, j = 1, 2, 3)$ and $x^3 \neq 0$. Then the only non-vanishing components of Christofell symbols, the curvature tensors and the Ricci tensors are

$$\Gamma^1_{13} = \Gamma^2_{23} = \frac{2}{3x^3}, \quad \Gamma^3_{11} = \Gamma^3_{22} = -\frac{2}{3}(x^3)^{\frac{1}{3}}$$

$$R_{1331} = R_{2332} = -\frac{2}{9(x^3)^{\frac{2}{3}}}, \quad R_{1221} = \frac{4}{9}(x^3)^{\frac{2}{3}}$$

$$R_{11} = R_{22} = \frac{2}{9(x^3)^{\frac{2}{3}}}, \quad R_{33} = -\frac{4}{9(x^3)^2}$$

Let us consider the associated scalars α, β, ϱ, γ as follows:

$$\alpha = -\frac{4}{9(x^3)^2}, \quad \beta = \frac{6(x^3)^{\frac{4}{3}}}{9}, \quad \varrho = \frac{12}{9(x^3)^2}, \quad \gamma = -\frac{6}{9(x^3)^{\frac{1}{3}}},$$

and the 1-forms

$$A_i(x) = \begin{cases} \frac{1}{x^3} & \text{for } i = 1, 2 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad B_i(x) = \begin{cases} (x^3)^{\frac{2}{3}} & \text{for } i = 2 \\ 0 & \text{otherwise} \end{cases}$$

Then we have

(i) $$R_{11} = \alpha g_{11} + \beta A_1 A_1 + \varrho B_1 B_1 + \gamma[A_1 B_1 + A_1 B_1]$$

(ii) $$R_{22} = \alpha g_{22} + \beta A_2 A_2 + \varrho B_2 B_2 + \gamma[A_2 B_2 + A_2 B_2]$$

(iii) $$R_{33} = \alpha g_{33} + \beta A_3 A_3 + \varrho B_3 B_3 + \gamma[A_3 B_3 + A_3 B_3]$$

Since all the cases other than (i)–(iii) are trivial, we can say that

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \varrho B_i B_j + \gamma[A_i B_j + A_j B_i] \quad \text{for } i,j = 1,2,3.$$

Thus if (R^3, g) is a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = (x^3)^{4/3}[(dx^1)^2 + (dx^2)^2)] + (dx^3)^2,$$

$(i,j = 1,2,3)$ and $x^3 \neq 0$, then (R^3, g) is an $MG(QE)_3$.

Next we consider the Lorentzian metric g on R^3 by

$$ds^2 = g_{ij}dx^i dx^j = -(x^3)^{4/3}(dx^1)^2 + (x^3)^{4/3}(dx^2)^2) + (dx^3)^2,$$

$(i,j = 1,2,3)$ and $x^3 \neq 0$.

Now, by similar way, after some construction of associated scalars and associated 1-forms, we can say that the manifold is a mixed generalized quasi-Einstein manifold. Therefore we get another example of $MG(QE)_3$.

Example 4.2. (R^3, g) is a Lorentzian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = -(x^3)^{4/3}(dx^1)^2 + (x^3)^{4/3}(dx^2)^2) + (dx^3)^2,$$

$(i,j = 1,2,3)$ and $x^3 \neq 0$, then (R^3, g) is an $MG(QE)_3$.

Example 4.3. Let us consider a Riemannian metric g on R^4 by

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

$(i,j = 1,2,3,4)$ and $p = \frac{e^{x^1}}{k^2}$, k is constant, then the only non-vanishing components of Christofell symbols, the curvature tensors and the Ricci tensors are

$$\Gamma^1_{22} = \Gamma^1_{33} = \Gamma^1_{44} = -\frac{p}{1+2p}, \quad \Gamma^1_{11} = \Gamma^2_{12} = \Gamma^3_{13} = \Gamma^4_{14} = \frac{p}{1+2p}$$

$$R_{1221} = R_{1331} = R_{1441} = \frac{p}{1+2p}, \quad R_{2332} = R_{2442} = R_{3443} = \frac{p^2}{1+2p}$$

$$R_{11} = \frac{3p}{(1+2p)^2}, \quad R_{22} = R_{33} = R_{44} = \frac{p}{(1+2p)}$$

It can be easily seen that the scalar curvature r of the given manifold (R^4, g) is

$$r = \frac{6p(1+p)}{(1+2p)^3},$$

which is non-vanishing and non-constant.

Let us consider the associated scalars α, β, γ, δ as follows:

$$\alpha = \frac{p}{(1+2p)^2}, \quad \beta = \frac{2p}{(1+2p)^3}, \quad \gamma = \frac{p}{(1+2p)^3}, \quad \delta = -\frac{p}{2(1+2p)^2},$$

and the 1-form

$$A_i(x) = \begin{cases} \sqrt{1+2p} & \text{for } i = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad B_i(x) = \begin{cases} \sqrt{1+2p} & \text{for } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then we have

(*i*) $$R_{11} = \alpha g_{11} + \beta A_1 A_1 + \gamma B_1 B_1 + \delta[A_1 B_1 + A_1 B_1]$$

(*ii*) $$R_{22} = \alpha g_{22} + \beta A_2 A_2 + \gamma B_2 B_2 + \delta[A_2 B_2 + A_2 B_2]$$

(*iii*) $$R_{33} = \alpha g_{33} + \beta A_3 A_3 + \gamma B_3 B_3 + \delta[A_3 B_3 + A_3 B_3]$$

(*iv*) $$R_{44} = \alpha g_{44} + \beta A_4 A_4 + \gamma B_4 B_4 + \delta[A_4 B_4 + A_4 B_4]$$

Since all the cases other than (i)–(iv) are trivial, we can say that

$$R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma B_i B_j + \delta[A_i B_j + A_j B_i], \quad \text{for } i, j = 1, 2, 3, 4.$$

So if (R^4, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij} dx^i dx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

$(i, j = 1, 2, 3, 4)$ and $p = \frac{e^{x^1}}{k^2}$, k is constant, then (R^4, g) is a mixed generalized quasi Einstein manifold with non-zero and non-constant scalar curvature.

If we consider the Lorentzian metric g on R^3 by

$$ds^2 = g_{ij} dx^i dx^j = -(1 + 2p)(dx^1)^2 + (1 + 2p)[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

$(i, j = 1, 2, 3, 4)$ and $p = \frac{e^{x^1}}{k^2}$, k is constant.

Now, by similar way after some construction of associated scalars and associated 1-forms, we can say that the manifold is a mixed generalized quasi-Einstein manifold. Therefore we get another example of $MG(QE)_4$.

Example 4.4. Let (R^4, g) be a Lorentzian manifold endowed with the metric given by

$$ds^2 = g_{ij} dx^i dx^j = -(1 + 2p)(dx^1)^2 + (1 + 2p)[(dx^2)^2 + (dx^3)^2 + (dx^4)^2]$$

$(i, j = 1, 2, 3, 4)$ and $p = \frac{e^{x^1}}{k^2}$, k is constant. Then (R^4, g) is an $MG(QE)_4$ with non-zero and non-constant scalar curvature.

5 Examples of warped product on mixed generalized quasi-Einstein manifold

Example 5.1. Here we consider the Example 4.1, a 3-dimensional example of mixed generalized quasi-Einstein manifold. Let (R^3, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij} dx^i dx^j = (x^3)^{4/3}[(dx^1)^2 + (dx^2)^2)] + (dx^3)^2,$$

where $(i, j = 1, 2, 3)$ and $x^3 \neq 0$.

To define warped product on $MG(QE)_3$, we consider the warping function $f\colon R\backslash 0 \to (0,\infty)$ by $f(x^3) = (x^3)^{\frac{2}{3}}$ and observe that $f = (x^3)^{\frac{2}{3}} > 0$ is a smooth function. The line element defined on $R\setminus\{0\}\times R^2$ which is of the form $B\times_f F$, where $B = R\setminus\{0\}$ is the base and $F = R^2$ is the fibre.

Therefore the metric ds^2_M can be expressed as $ds^2_B + f^2 ds^2_F$ i.e.,

$$ds^2 = g_{ij}dx^i dx^j = (dx^3)^2 + \{(x^3)^{2/3}\}^2[(dx^1)^2 + (dx^2)^2],$$

which is the example of Riemannian warped product on $MG(QE)_3$.

Example 5.2. We consider the example 4.3, a 4-dimensional example of mixed generalized quasi-Einstein manifold. Let (R^4, g) be a Riemannian manifold endowed with the metric given by

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2],$$

where $(i,j = 1,2,3,4)$, $p = \frac{e^{x^1}}{k^2}$, k is constant.

To define warped product on $MG(QE)_4$, we consider the warping function $f\colon R^3 \to (0,\infty)$ by $f(x^1,x^2,x^3) = \sqrt{(}1+2p)$ and we observe that $f > 0$ is a smooth function. The line element defined on $R^3\times R$ which is of the form $B\times_f F$, where $B = R^3$ is the base and $F = R$ is the fibre.

Therefore the metric ds^2_M can be expressed as $ds^2_B + f^2 ds^2_F$ i.e.,

$$ds^2 = g_{ij}dx^i dx^j = (1+2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2] + [\sqrt{(}1+2p)]^2(dx^4)^2,$$

which is the example of Riemannian warped product on $MG(QE)_4$.

Finally we note that the similar metrics were obtained in [1].

Acknowledgement. The authors wish to express their sincere thanks and gratitude to the referee for valuable suggestions towards the improvement of the paper.

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