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On a Class of Generalized quasi-Einstein Manifolds with Applications to Relativity

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Abstract

Quasi Einstein manifold is a simple and natural generalization of Einstein manifold. The object of the present paper is to study some properties of generalized quasi Einstein manifolds. We also discuss $G(QE)_4$ with space-matter tensor and some properties related to it. Two non-trivial examples have been constructed to prove the existence of generalized quasi Einstein spacetimes.

Key words: Einstein manifolds, quasi Einstein manifolds, generalized quasi Einstein manifolds, quasi-conformal curvature tensor, space-matter tensor.

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1 Introduction

A Riemannian or a semi-Riemannian manifold (M^n, g), $n = \dim M \geq 2$, is said to be an Einstein manifold if the following condition

$$S = \frac{r}{n} g \tag{1.1}$$

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holds on M, where S and r denote the Ricci tensor and the scalar curvature of (M^n, g) respectively. According to Besse [7, p. 432], (1.1) is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry as well as in general theory of relativity. Also Einstein manifolds form a natural subclass of Riemannian or semi-Riemannian manifolds by a curvature condition imposed on their Ricci tensor ([7, p. 432–433]). For instance, every Einstein manifold belongs to the class of Riemannian or semi-Riemannian manifolds (M^n, g) realizing the following relation:

$$S(X,Y) = ag(X,Y) + bA(X)A(Y), \tag{1.2}$$

where a, b are smooth functions and A is a non-zero 1-form such that

$$g(X,U) = A(X), \tag{1.3}$$

for all vector fields X.

A non-flat Riemannian or a semi-Riemannian manifold (M^n, g) $(n > 2)$ is defined to be a quasi Einstein manifold [8] if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition (1.2). We shall call A the associated 1-form and the unit vector field U is called the generator of the manifold. Such a manifold is denoted by $(QE)_n$.

Quasi Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetime are quasi Einstein manifolds. Also quasi Einstein manifold can be taken as a model of the perfect fluid spacetime in general relativity[14]. So quasi Einstein manifolds have some importance in the general theory of relativity.

The study of quasi Einstein manifolds was continued by M. C. Chaki [8], S. Guha [24], U. C. De and G. C. Ghosh [12, 13], P. Debnath and A. Konar [21], Özgür and Sular [34], Özgür [31], Bejan [5] and many others.

Several authors have generalized the notion of quasi Einstein manifold such as generalized quasi Einstein manifolds ([9, 33, 35, 36]), nearly quasi Einstein manifolds [15], generalized Einstein manifolds [6], super quasi Einstein manifolds [32] and $N(k)$-quasi Einstein manifolds ([28, 34, 31, 42, 20]).

In 2001, Chaki [9] introduced the notion of generalized quasi Einstein manifold. A non-flat Riemannian or a semi-Riemannian manifold (M^n, g) $(n > 2)$ is called a generalized quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)A(Y) + c(A(X)B(Y) + A(Y)B(X)), \tag{1.4}$$

where a, b, c are certian non-zero scalars and A, B are two non-zero 1-form. The unit vector fields U and V corresponding to the 1-forms A and B respectively, defined by

$$g(X,U) = A(X), \qquad g(X,V) = B(X),$$

for every vector field X are orthogonal, that is, $g(U, V) = 0$. Such as n-dimensional manifold is denoted by $G(QE)_n$. The vector fields U and V are called the generators of the manifold and a, b, c are called the associated scalars. If $c = 0$, then the manifold reduces to a quasi Einstein manifold $(QE)_n$. It may be mentioned that De and Ghosh [12] introduced the same notion in another way.

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n \geq 3)$ shall be called a manifold of generalized quasi-constant curvature if its curvature tensor $\tilde{R}$ of type $(0, 4)$ satisfies the condition[9]

$$\begin{aligned}
&\tilde{R}(X,Y,Z,W) = \\
&\quad = p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + q[g(Y,Z)A(X)A(W) \\
&\quad + g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W) - g(Y,W)A(X)A(Z)] \\
&\quad + s[g(Y,Z)\{A(X)B(W) + A(W)B(X)\} + g(X,W)\{A(Y)B(Z) \\
&\quad + B(Y)A(Z)\} - g(X,Z)\{A(Y)B(W) + B(Y)A(W)\} \\
&\quad - g(Y,W)\{A(X)B(Z) + B(X)A(Z)\}]. \qquad (1.5)
\end{aligned}$$

Such an n-dimensional manifold shall be denoted by $G(QC)_n$. If $s = 0$, then the manifold reduces to a manifold of quasi-constant curvature[8].

The notion of quasi-conformal curvature tensor was given by Yano and Sawaki[43] and is defined as follows:

$$\begin{aligned}
C^\star(X,Y)Z &= a_1 R(X,Y)Z \\
&\quad + b_1[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\
&\quad - \frac{r}{n}[\frac{a_1}{n-1} + 2b_1][g(Y,Z)X - g(X,Z)Y], \qquad (1.6)
\end{aligned}$$

where a_1 and b_1 are constants, R is the curvature tensor of type (1,3), S is the Ricci tensor of type (0,2), Q is the Ricci operator and r is the scalar curvature of the manifold.
If $a_1 = 1$ and $b_1 = -\frac{1}{n-2}$, then (1.6) reduces to the conformal curvature tensor C. Thus the conformal curvature tensor C is a particular case of the tensor $C^\star$. For this reason $C^\star$ is called the quasi-conformal curvature tensor. A Riemannian or a semi-Riemannian manifold is called quasi-conformally flat if $C^\star = 0$ for $n > 3$. It is known[4] that a quasi-conformally flat manifold is either conformally flat if $a_1 \neq 0$ or Einstein if $a_1 = 0$ and $b_1 \neq 0$. Since they give no restrictions for manifolds if $a_1 = 0$ and $b_1 = 0$, it is essential for us to consider the case of $a_1 \neq 0$ or $b_1 \neq 0$. The quasi-conformal curvature tensor have been studied by various authors in various ways such as Amur and Maralabhavi [4], De and Sarkar [16], De and Matsuyama [17], De, Jun and Gazi [18], Guha [23], Hosseinzadeh and Taleshian [20], Özgür and Sular [34], Mantica and Suh [26] and many others.

Spacetime of general relativity is regarded as a connected four dimensional semi-Riemannian manifold (M^4, g) with Lorentzian metric g with signature $(-, +, +, +)$. The geometry of the Lorentzian manifold begins with the study of the causal character of vectors of the manifold. It is due to this causality that the Lorentzian manifold becomes a convenient choice for the study of general relativity. Spacetime of general relativity have been studied by different authors in different ways such as Chaki and Ray[10], De and Mallick[19], Mantica and Suh [27], Özen [44], Güler and Demirbağ [25] and many others.

Gray [22] introduced two classes of Riemannian manifolds determined by the covariant differentiation of Ricci tensor. The class A consisting of all Riemannian manifolds whose Ricci tensor S is a Codazzi type tensor, i.e.,

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z).$$

The class B consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, i.e.,

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0.$$

A non-flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$ is called a generalized Ricci recurrent manifold [11] if its Ricci tensor S of type (0,2) satisfies the condition

$$(\nabla_X S)(Y, Z) = \gamma(X)S(Y, Z) + \delta(X)g(Y, Z),$$

where γ and δ are non-zero 1-forms. If $\delta = 0$, then the manifold reduces to a Ricci recurrent manifold[37].

In a smooth manifold (M^n, g) Petrov [38] introduced a tensor $\tilde{P}$ of type $(0, 4)$ and defined by the following:

$$\tilde{P} = \tilde{R} + \frac{\kappa}{2} g \wedge T - \sigma G, \tag{1.7}$$

where $\tilde{R}$ is the curvature tensor of type $(0, 4)$, T is the energy momentum tensor of type $(0, 2)$, κ is the gravitational constant, σ is the energy density, G is a tensor of type (0,4) given by

$$G(X, Y, Z, W) = g(Y, Z)g(X, W) - g(X, Z)g(Y, W), \tag{1.8}$$

for all $X, Y, Z, W \in \chi(M)$ and Kulkarni–Nomizu product $E \wedge F$ of two (0,2) tensors E and F is defined by

$$\begin{aligned}(E \wedge F)(X, Y, Z, W) = &E(Y, Z)F(X, W) + E(X, W)F(Y, Z)\\ &-E(X, Z)F(Y, W) - E(Y, W)F(X, Z),\end{aligned} \tag{1.9}$$

where $X, Y, Z, W \in \chi(M)$. The tensor $\tilde{P}$ is known as the space-matter tensor of type (0,4) of the manifold M. The space-matter tensor have been studied by Ahsan and Siddiqui [1, 2] and many others.

The importance of a $G(QE)_n$ lies in the fact that a four-dimensional semi-Riemannian manifold is relevant to study of a general relativistic fluid spacetime admitting heat flux [39], where U is taken as the velocity vector of the fluid and V is taken as the heat flux vector field.

In the present paper we have studied $G(QE)_n$. The paper is organized as follows:

After introduction in Section 2, we study quasi-conformally flat $G(QE)_n$ and we prove that every quasi-conformally flat $G(QE)_n$ is a $G(QC)_n$. Section 3 is devoted to study $G(QE)_n$ with divergence free quasi-conformal curvature tensor. In Section 4, it is shown that if the generators U and V are Killing vector fields, then the generalized quasi-Einstein manifold satisfies cyclic parallel Ricci tensor. In the next two sections we consider $G(QE)_n$ with generators U and V both as concurrent and recurrent vector fields. In Section 7, we study sectional curvatures at a point of a quasi-conformally flat $G(QE)_n$. $G(QE)_4$ with vanishing space-matter tensor is studied in Section 8. Section 9 is concerned with the study of $G(QE)_4$ with divergence free space-matter tensor. In Section 10, we study perfect fluid $G(QE)_4$ spacetime. Finally, we construct two non-trivial examples of $G(QE)_4$.

2 Quasi-conformally flat $G(QE)_n$ $(n > 3)$

A $G(QE)_n$ $(n > 3)$ is not, in general a $G(QC)_n$. In this section we consider a quasi-conformally flat $G(QE)_n$ $(n > 3)$ and show that such a $G(QE)_n$ is a $G(QC)_n$.

From (1.6) it follows that in a quasi-conformally flat Riemannian or semi-Riemannian manifold (M^n, g) $(n > 3)$, the curvature tensor $\tilde{R}$ of type $(0, 4)$ has the following form:

$$\begin{aligned} a_1\tilde{R}(X,Y,Z,W) = &-b_1[S(Y,Z)g(X,W) - S(X,Z)g(Y,W) \\ &+ g(Y,Z)S(X,W) - g(X,Z)S(Y,W)] \\ &+ \frac{r}{n}\Big(\frac{a_1}{n-1} + 2b_1\Big)[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{2.1}$$

Using (1.4) in (2.1) we obtain

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = &\,\alpha[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + \beta[g(Y,Z)A(X)A(W) \\ &+ g(X,W)A(Y)A(Z) - g(X,Z)A(Y)A(W) - g(Y,W)A(X)A(Z)] \\ &+ \gamma[g(Y,Z)\{A(X)B(W) + A(W)B(X)\} + g(X,W)\{A(Y)B(Z) \\ &+ B(Y)A(Z)\} - g(X,Z)\{A(Y)B(W) + B(Y)A(W)\} \\ &- g(Y,W)\{A(X)B(Z) + B(X)A(Z)\}], \end{aligned} \tag{2.2}$$

where

$$\alpha = \frac{a_1 r + 2(n-1)(r - na)b_1}{n(n-1)a_1}, \quad \beta = -\frac{bb_1}{a_1} \text{ and } \gamma = -\frac{cb_1}{a_1}.$$

In virtue of (1.5) it follows from (2.2) that the manifold under consideration is a $G(QC)_n$. Thus we can state the following:

Theorem 1. *Every quasi-conformally flat $G(QE)_n$ $(n > 3)$ is a $G(QC)_n$.*

3 $G(QE)_n$ $(n > 3)$ with divergence free quasi-conformal curvature tensor

In this section we look for sufficient condition in order that a $G(QE)_n$ $(n > 3)$ may be quasi-conformally conservative.

Quasi-conformal curvature tensor is said to be conservative if the divergence of $C^\star$ vanishes, i.e., $divC^\star = 0$.

In a $G(QE)_n$ if the associated scalars a, b and c are constant, then contracting (1.4) we have

$$r = an + b,$$

which implies that the scalar curvature r is constant, i.e., $dr = 0$.
Using $dr = 0$ we obtain from (1.6) that

$$\begin{aligned}(\nabla_W C^\star)(X, Y, Z) = a_1(\nabla_W R)(X, Y)Z + b_1[(\nabla_W S)(Y, Z)X \\ - (\nabla_W S)(X, Z) + g(Y, Z)(\nabla_W Q)(X) - g(X, Z)(\nabla_W Q)(Y)].\end{aligned} \tag{3.1}$$

We know that $(\operatorname{div} R)(X, Y, Z) = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z)$ and from (1.4) we obtain

$$\begin{aligned}(\nabla_X S)(Y, Z) = b[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z) \\ + c[(\nabla_X A)(Y)B(Z) + A(Y)(\nabla_X B)(Z) \\ + (\nabla_X B)(Y)A(Z) + (\nabla_X A)(Z)B(Y)],\end{aligned} \tag{3.2}$$

since a, b and c are constants.
Contracting (3.1) and using (3.2) we obtain

$$\begin{aligned}(\operatorname{div} C^\star)(X, Y, Z) = (a_1 + b_1)[b\{(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z) \\ - (\nabla_Y A)(X)A(Z) - A(X)(\nabla_Y A)(Z)\} \\ + c\{(\nabla_X A)(Y)B(Z) + A(Y)(\nabla_X B)(Z) + (\nabla_X B)(Y)A(Z) \\ + (\nabla_X A)(Z)B(Y) - (\nabla_Y A)(X)B(Z) - A(X)(\nabla_Y B)(Z) \\ - (\nabla_Y B)(X)A(Z) - (\nabla_Y A)(Z)B(X)\}].\end{aligned} \tag{3.3}$$

Imposing the condition that the generators U and V of the manifold are parallel vector fields gives $\nabla_X U = 0$ and $\nabla_X V = 0$. Hence

$$g(\nabla_X U, Y) = 0, \quad \text{i.e.,} \quad (\nabla_X A)(Y) = 0.$$

and

$$g(\nabla_X V, Y) = 0, \quad \text{i.e.,} \quad (\nabla_X B)(Y) = 0.$$

Therefore from (3.3) it follows that

$$(\operatorname{div} C^\star)(X, Y, Z) = 0.$$

Thus we can state the following:

Theorem 2. *If in a $G(QE)_n$ the associated scalars are constants and the generators U and V of the manifold are parallel vector fields, then the manifold is quasi-conformally conservative.*

4 The generators U and V as Killing vector fields

In this section let us consider the generators U and V of the manifold are Killing vector fields. Then we have

$$(\pounds_U g)(X, Y) = 0 \tag{4.1}$$

and

$$(\pounds_V g)(X, Y) = 0, \tag{4.2}$$

where $\pounds$ denotes the Lie derivative.
From (4.1) and (4.2) it follows that

$$g(\nabla_X U, Y) + g(X, \nabla_Y U) = 0 \tag{4.3}$$

and

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) = 0. \tag{4.4}$$

Since $g(\nabla_X U, Y) = (\nabla_X A)(Y)$ and $g(\nabla_X V, Y) = (\nabla_X B)(Y)$, we obtain from (4.3) and (4.4) that

$$(\nabla_X A)(Y) + (\nabla_Y A)(X) = 0 \tag{4.5}$$

and

$$(\nabla_X B)(Y) + (\nabla_Y B)(X) = 0, \tag{4.6}$$

for all X, Y.
Similarly, we have

$$(\nabla_X A)(Z) + (\nabla_Z A)(X) = 0, \tag{4.7}$$

$$(\nabla_Z A)(Y) + (\nabla_Y A)(Z) = 0, \tag{4.8}$$

$$(\nabla_X B)(Z) + (\nabla_Z B)(X) = 0, \tag{4.9}$$

$$(\nabla_Z B)(Y) + (\nabla_Y B)(Z) = 0, \tag{4.10}$$

for all X, Y, Z.
We also assume that the associated scalars are constants. Then from (1.4) we have

$$\begin{aligned}(\nabla_Z S)(X, Y) = b[(\nabla_Z A)(X)A(Y) + A(X)(\nabla_Z A)(Y)] \\ + c[(\nabla_Z A)(X)B(Y) + A(X)(\nabla_Z B)(Y) \\ + (\nabla_Z B)(X)A(Y) + B(X)(\nabla_Z A)(Y)].\end{aligned} \tag{4.11}$$

Using (4.11) we obtain

$$\begin{aligned}(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = b[\{(\nabla_X A)(Y) \\ + (\nabla_Y A)(X)\}A(Z) + \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}A(Y) \\ + \{(\nabla_Y A)(Z) + (\nabla_Z A)(Y)\}A(X)] + c[\{(\nabla_X B)(Y) \\ + (\nabla_Y B)(X)\}A(Z) + \{(\nabla_X B)(Z) + (\nabla_Z B)(X)\}A(Y) \\ + \{(\nabla_Y B)(Z) + (\nabla_Z B)(Y)\}A(X) + \{(\nabla_X A)(Y) \\ + (\nabla_Y A)(X)\}B(Z) + \{(\nabla_X A)(Z) + (\nabla_Z A)(X)\}B(Y) \\ + \{(\nabla_Y A)(Z) + (\nabla_Z A)(Y)\}B(X)].\end{aligned} \tag{4.12}$$

By virtue of (4.5)–(4.10) we obtain from (4.12) that

$$(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y) = 0.$$

Thus we can state the following theorem:

Theorem 3. *If the generators of a $G(QE)_n$ are Killing vector fields and the associated scalars are constants, then the manifold satisfies cyclic parallel Ricci tensor.*

5 The generators U and V as concurrent vector fields

A vector field ξ is said to be concurrent if [41]

$$\nabla_X \xi = \rho X, \tag{5.1}$$

where ρ is a non-zero constant. If $\rho = 0$, the vector field reduces to a parallel vector field.

In this section we consider the vector fields U and V corresponding to the associated 1-forms A and B respectively are concurrent. Then

$$(\nabla_X A)(Y) = \alpha g(X,Y) \tag{5.2}$$

and

$$(\nabla_X B)(Y) = \beta g(X,Y), \tag{5.3}$$

where α and β are non-zero constants.
Using (5.2) and (5.3) in (4.11) we get

$$\begin{aligned}(\nabla_Z S)(X,Y) = b[\alpha g(X,Z)A(Y) + \alpha g(Y,Z)A(X)] + c[\beta A(X)g(Y,Z) \\ + \alpha B(Y)g(X,Z) + \alpha B(X)g(Y,Z) + \beta A(Y)g(X,Z)].\end{aligned} \tag{5.4}$$

Contracting (5.4) over X and Y we obtain

$$dr(Z) = 2(b\alpha + c\beta)A(Z) + 2c\alpha B(Z), \tag{5.5}$$

where r is the scalar curvature of the manifold.
Again from (1.4) we have

$$r = an + b. \tag{5.6}$$

Since, a, b, $c \in \mathbb{R}$, it follows that $dr(X) = 0$, for all X. Thus equation (5.5) yields

$$(b\alpha + c\beta)A(Z) + c\alpha B(Z) = 0. \tag{5.7}$$

Since α and β are not zero, using (5.7) in (1.4), we finally get

$$S(X,Y) = ag(X,Y) - \frac{(b\alpha + 2c\beta)}{\alpha} A(X)A(Y).$$

Thus the manifold reduces to a quasi-Einstein manifold. Hence we can state the following theorem:

Theorem 4. *If the associated vector fields of a $G(QE)_n$ are concurrent vector fields and the associated scalars are constants, then the manifold reduces to a quasi-Einstein manifold.*

6 The generators U and V as recurrent vector fields

A vector fiels ξ corresponding to the associated 1-form η is said to be recurrent if[41]

$$(\nabla_X \eta)(Y) = \psi(X)\eta(Y), \tag{6.1}$$

where ψ is a non-zero 1-form.

In this section we suppose that the generators U and V corresponding to the associated 1-forms A and B are recurrent. Then we have

$$(\nabla_X A)(Y) = \lambda(X)A(Y) \tag{6.2}$$

and

$$(\nabla_X B)(Y) = \mu(X)B(Y), \tag{6.3}$$

where λ and μ are non-zero 1-forms.
Now, using (6.2) and (6.3) in (4.11) we get

$$\begin{aligned}(\nabla_Z S)(X,Y) = 2b\lambda(Z)A(X)A(Y) + c[\mu(Z)A(X)B(Y) \\ + \lambda(Z)A(X)B(Y) + \lambda(Z)A(Y)B(X) + \mu(Z)B(X)A(Y)].\end{aligned} \tag{6.4}$$

We assume that the 1-forms λ and μ are equal, i.e.,

$$\lambda(Z) = \mu(Z), \tag{6.5}$$

for all Z. Then we obtain from (6.4) and (6.5) that

$$(\nabla_Z S)(X,Y) = 2b\lambda(Z)A(X)A(Y) + 2c\lambda(Z)[A(X)B(Y) + B(X)A(Y)]. \tag{6.6}$$

Using (1.4) and (6.6) we have

$$(\nabla_Z S)(X,Y) = \alpha_1(Z)S(X,Y) + \alpha_2(Z)g(X,Y),$$

where $\alpha_1(Z) = 2\lambda(Z)$ and $\alpha_2(Z) = -2a\lambda(Z)$.
Thus we can state the following:

Theorem 5. *If the generators of a $G(QE)_n$ corresponding to the associated 1-forms are recurrent with the same vector of recurrence and the associated scalars are constants, then the manifold is a generalized Ricci recurrent manifold.*

7 Sectional curvatures at a point of a quasi-conformally flat $G(QE)_n$

Let $U^\perp$ denote the $(n-1)$-dimensional distribution in a quasi-conformally flat $G(QE)_n$ $(n > 3)$ orthogonal to U. Then for any $X \in U^\perp$, $g(X,U) = 0$, that is $A(X) = 0$. In this section we shall determine sectional curvature K at the plane determined by the vectors $X, Y \in U^\perp$ or by $X,\ U$.
Putting $Z = Y$ and $W = X$ in (2.2) we get

$$\tilde{R}(X,Y,Y,X) = \frac{a_1 r + 2(n-1)(r-an)b_1}{a_1 n(n-1)}[g(X,X)g(Y,Y) - \{g(X,Y)\}^2]. \tag{7.1}$$

Putting $Y = Z = U$ and $W = X$ in (2.2) we get

$$\tilde{R}(X,U,U,X) = \left\{ \frac{a_1 r + 2(n-1)(r-an)b_1}{a_1 n(n-1)} - \frac{bb_1}{a_1} \right\} g(X,X). \tag{7.2}$$

Now contracting (1.4) over X and Y we get

$$r = an + b. \tag{7.3}$$

Then using (7.1), (7.3) and (7.2) we obtain

$$K(X,Y) = \frac{\tilde{R}(X,Y,Y,X)}{g(X,X)g(Y,Y) - \{g(X,Y)\}^2} = \frac{a_1(an+b) + 2(n-1)bb_1}{a_1 n(n-1)}, \tag{7.4}$$

and

$$K(X,U) = \frac{\tilde{R}(X,U,U,X)}{g(X,X)g(U,U) - \{g(X,U)\}^2} = \frac{a_1(an+b) - (n-1)(n-2)bb_1}{a_1 n(n-1)}. \tag{7.5}$$

Thus we can state the following theorem:

Theorem 6. *In a quasi-conformally flat $G(QE)_n$ $(n > 3)$ the sectional curvature of the plane determined by two vectors $X, Y \in U^{\perp}$ is*

$$\frac{a_1(an+b) + 2(n-1)(r-an)b_1}{a_1 n(n-1)},$$

while the sectional curvature of the plane determined by two vectors X, U is

$$\frac{a_1(an+b) - (n-1)(n-2)(r-an)b_1}{a_1 n(n-1)}.$$

8 $G(QE)_4$ spacetime with vanishing space-matter tensor

In this section we study $G(QE)_4$ spacetime with vanishing space-matter tensor. Now equation (1.7) can also be written as

$$\begin{aligned} \tilde{P}(X,Y,Z,W) = \tilde{R}(X,Y,Z,W) &+ \frac{\kappa}{2}[g(Y,Z)T(X,W) + g(X,W)T(Y,Z) \\ &- g(X,Z)T(Y,W) - g(Y,W)T(X,Z)] \\ &- \sigma[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{8.1}$$

If $\tilde{P} = 0$, then (8.1) yields

$$\begin{aligned} \tilde{R}(X,Y,Z,W) = &-\frac{\kappa}{2}[g(Y,Z)T(X,W) + g(X,W)T(Y,Z) \\ &- g(X,Z)T(Y,W) - g(Y,W)T(X,Z)] \\ &+ \sigma[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]. \end{aligned} \tag{8.2}$$

Now using (1.4) and (11.2) in (8.2) we obtain

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = {} & (\sigma - a + \frac{r}{2})[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] \\ & - \frac{b}{2}[g(Y,Z)A(X)A(W) + g(X,W)A(Y)A(Z) \\ & - g(X,Z)A(Y)A(W) - g(Y,W)A(X)A(Z)] \\ & - \frac{c}{2}[g(Y,Z)\{A(X)B(W) + A(W)B(X)\} \\ & + g(X,W)\{A(Y)B(Z) + A(Z)B(Y)\} \\ & - g(X,Z)\{A(Y)B(W) + A(W)B(Y)\} \\ & - g(Y,W)\{A(X)B(Z) + A(Z)B(X)\}]. \end{aligned} \tag{8.3}$$

In virtue of (1.5) it follows from (8.3) that the manifold under consideration is a generalized quasi-constant curvature. Thus we can state the following:

Theorem 7. *A $G(QE)_4$ spacetime satisfying Einstein's field equation and with vanishing space-matter tensor is a spacetime of generalized quasi-constant curvature.*

9 $G(QE)_4$ spacetime with divergence free space-matter tensor

In this section we look for sufficient condition in order that a $G(QE)_4$ may be of divergence free space-matter tensor.

In a $G(QE)_n$ if the associated scalars a, b and c are constant, then contracting (1.4) we have

$$r = an + b,$$

which implies that the scalar curvature r is constant, i.e., $dr = 0$.
Using (11.2), we obtain from (8.1) that

$$\begin{aligned}(\operatorname{div} P)(X,Y,Z) = {} & \operatorname{div} R(X,Y)Z + \frac{1}{2}[(\nabla_X S)(Y,Z) - \nabla_Y S)(X,Z)] \\ & - g(Y,Z)[d\sigma(X) + \frac{1}{4}dr(X)] + g(X,Z)[d\sigma(Y) + \frac{1}{4}dr(Y)]. \end{aligned} \tag{9.1}$$

In a semi-Riemannian manifold it is known that

$$(\operatorname{div} R)(X,Y,Z) = (\nabla_X S)(Y,Z) - \nabla_Y S)(X,Z). \tag{9.2}$$

Using (9.1) and (9.2) we obtain

$$\begin{aligned}(\operatorname{div} P)(X,Y,Z) = {} & \frac{3}{2}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] \\ & - g(Y,Z)[d\sigma(X) + \frac{1}{4}dr(X)] + g(X,Z)[d\sigma(Y) + \frac{1}{4}dr(Y)]. \end{aligned} \tag{9.3}$$

Assuming that $(\operatorname{div} P)(X,Y,Z) = 0$ and then contracting (9.3) over Y and Z, we obtain

$$d\sigma(X) = 0.$$

Thus we can state the following:

Theorem 8. *In a $G(QE)_4$ spacetime satisfying Einstein's field equation with divergence free space-matter tensor the energy density is constant.*

Again using (1.4), equation (9.3) can be written as

$$\begin{aligned}(\operatorname{div} P)(X,Y,Z) = {}& \frac{3}{2}[da(X)g(Y,Z) - da(Y)g(X,Z)] + \frac{3}{2}[db(X)A(Y)A(Z) \\ & - db(Y)A(X)A(Z)] + \frac{3}{2}[dc(X)\{A(Y)B(Z) + B(Y)A(Z)\} \\ & - dc(Y)\{A(X)B(Z) + B(X)A(Z)\}] + \frac{3b}{2}[(\nabla_X A)(Y)A(Z) \\ & + A(Y)(\nabla_X A)(Z) - (\nabla_Y A)(X)A(Z) - A(X)(\nabla_Y A)A(Z)] \\ & + \frac{3c}{2}[(\nabla_X A)(Y)B(Z) + A(Y)(\nabla_X B)(Z) + (\nabla_X A)(Z)B(Y) \\ & + A(Z)(\nabla_X B)(Y) - (\nabla_Y A)(X)B(Z) - A(X)(\nabla_Y B)(Z) \\ & - (\nabla_Y A)(Z)B(X) - A(Z)(\nabla_Y B)(X)] - g(Y,Z)[d\sigma(X) \\ & + \frac{1}{4}dr(X)] + g(X,Z)[d\sigma(Y) + \frac{1}{4}dr(Y)]. \end{aligned} \tag{9.4}$$

Imposing the conditions that the associated scalars and the energy density σ are constants and the generators U and V of the manifold are parallel vector fields gives $\nabla_X U = 0$ and $\nabla_X V = 0$. Hence $dr(X) = 0$, $d\sigma(X) = 0$ for all X. Also

$$g(\nabla_X U, Y) = 0, \quad \text{i.e.,} \quad (\nabla_X A)(Y) = 0,$$

and

$$g(\nabla_X V, Y) = 0, \quad \text{i.e.,} \quad (\nabla_X B)(Y) = 0.$$

Therefore from (9.4) it follows that

$$(\operatorname{div} P)(X,Y,Z) = 0.$$

Thus we can state the following:

Theorem 9. *If in a $G(QE)_4$ spacetime satisfying Einstein's field equation the associated scalars and the energy density σ are constants, then the divergence of the space-matter tensor vanishes.*

10 Perfect fluid $G(QE)_4$ spacetime

In a perfect fluid spacetime, the energy momentum tensor T of type $(0,2)$ is of the form:

$$T(X,Y) = pg(X,Y) + (\sigma + p)A(X)A(Y), \tag{10.1}$$

where σ and p are the energy density and the isotropic pressure respectively. Then in the general relativistic spacetime whose matter content is perfect fluid obeying the Einstein's field equation, the Ricci tensor satisfies the following equation

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa T(X,Y). \tag{10.2}$$

In view of (10.1), equation (10.2) can be written as

$$S(X,Y) - \frac{r}{2}g(X,Y) = \kappa[pg(X,Y) + (\sigma + p)A(X)A(Y)]. \tag{10.3}$$

Taking a frame field and contracting (10.3) over X and Y, we obtain

$$r = \kappa(\sigma - 3p). \tag{10.4}$$

We know[40] if the Ricci tensor S of type $(0,2)$ of the spacetime satifies the condition

$$S(X,X) > 0, \tag{10.5}$$

for every time like vector field X, then (10.5) is called the time like convergence condition. Here we consider the general relativistic perfect fluid $G(QE)_4$ spacetime with unit time velocity vector field U, then we have

$$g(U,U) = -1. \tag{10.6}$$

Now putting $X = Y = U$ in (10.3), we get

$$S(U,U) = \frac{\kappa}{2}(\sigma + 3p). \tag{10.7}$$

But we know[3] that a perfect fluid spacetime is filled with radiation if

$$\sigma - 3p = 0. \tag{10.8}$$

Again from (1.4) we have

$$S(U,U) = b - a. \tag{10.9}$$

From equations (10.7) and (10.8) we obtain $S(U,U) = \kappa\sigma > 0$, by (10.5), that is, $\sigma > 0$ which implies that this $G(QE)_4$ spacetime contains pure matter. In this case isotropic pressure p and energy density σ are given by $p = \frac{b-a}{3\kappa}$ and $\sigma = \frac{b-a}{\kappa}$.
Thus we can state the following:

Theorem 10. *If in radiative perfect fluid $G(QE)_4$ spacetime admitting Einstein's equation without cosmological constant, the Ricci tensor obeys the time like convergence condition, then such a spacetime contains pure matter and in this case isotropic pressure is $\frac{b-a}{3\kappa}$ and energy density is $\frac{b-a}{\kappa}$.*

11 Examples of a generalized quasi Einstein spacetime

Example 1. Let (M^4, g) be a viscous fluid spacetime admitting heat flux and satisfying Einstein's equation without cosmological constant. Further, let U be the unit timelike velocity vector field of the fluid and V be the unit heat flux vector field. Then

$$g(U,U) = -1, \quad g(V,V) = 1, \quad g(U,V) = 0.$$

Let $g(X,U) = A(X)$, $g(X,V) = B(X)$, for all X. Further, let T be the $(0,2)$ type energy momentum tensor describing the matter distribution of such a fluid. Then [29]

$$T(X,Y) = (\sigma + p)A(X)A(Y) + pg(X,Y) + [A(X)B(Y) + A(Y)B(X)], \tag{11.1}$$

where σ, p denote the energy density and isotropic pressure of the fluid. It is known [30] that Einstein's equation without cosmological constant can be written as

$$S(X,Y) - \frac{1}{2}rg(X,Y) = \kappa T(X,Y), \tag{11.2}$$

where κ is the gravitational constant and T is the energy momentum tensor of type $(0,2)$. In the present case (11.2) can be written as follows:

$$\begin{aligned} &S(X,Y) - \frac{1}{2}rg(X,Y) \\ &\quad = \kappa[(\sigma + p)A(X)A(Y) + pg(X,Y) + \{A(X)B(Y) + A(Y)B(X)\}]. \end{aligned} \tag{11.3}$$

Hence

$$\begin{aligned} S(X,Y) = &\left(p + \frac{r}{2}\right) g(X,Y) \\ &+ \kappa(\sigma + p)A(X)A(Y) + \kappa[A(X)B(Y) + A(Y)B(X)]. \end{aligned} \tag{11.4}$$

From (11.4) it follows that the spacetime under consideration is a generalized quasi-Einstein manifold with $p+\frac{r}{2}$, $\kappa(\sigma+p)$ and κ as associated scalars, A and B as associated 1-forms. Hence a viscous fluid spacetime admitting heat flux and satisfying Einstein's equation without cosmological constant is a 4-dimensional semi-Riemannian generalized quasi-Einstein manifold.

Example 2. We consider a semi-Riemannian manifold $(\mathbb{R}^4,g)$ endowed with the metric g given by

$$ds^2 = g_{ij}dx^i dx^j = (1 + 2q)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2 \ ],$$

where $q = \frac{e^{x^1}}{k^2}$ and k is a non-zero constant and $i, j = 1, 2, 3, 4$.

The only non-vanishing components of the Christoffel symbols, the curvature tensor and the Ricci tensor are

$$\Gamma^1_{11} = \frac{q}{1+2q}, \quad \Gamma^1_{22} = -\frac{q}{1+2q}, \quad \Gamma^1_{33} = -\frac{q}{1+2q}, \quad \Gamma^1_{44} = \frac{q}{1+2q},$$
$$\Gamma^2_{12} = \frac{q}{1+2q}, \quad \Gamma^3_{13} = \frac{q}{1+2q}, \quad \Gamma^4_{14} = \frac{q}{1+2q},$$
$$R_{1221} = R_{1331} = \frac{q}{1+2q}, \quad R_{1441} = -\frac{q}{1+2q}$$
$$R_{2332} = \frac{q^2}{1+2q}, \quad R_{2442} = R_{3443} = -\frac{q^2}{1+2q},$$
$$R_{11} = \frac{3q}{(1+2q)^2}, \quad R_{22} = R_{33} = \frac{q}{1+2q}, \quad R_{44} = -\frac{q}{1+2q}.$$

The scalar curvature is

$$\frac{6q(1+q)}{(1+2q)^3}$$

which is non-zero and non-constant. We take scalars a, b and c as follows :

$$a = \frac{q}{(1+2q)^2}, \quad b = \frac{3q}{(1+2q)^3} - \frac{q}{(1+2q)^2}, \quad c = \frac{q}{1+2q}.$$

We choose the 1-forms as follows:

$$A_i(x) = \begin{cases} \sqrt{1+2q}, & \text{for } i = 1 \\ 0, & \text{for } i = 2,3,4 \end{cases}$$

and

$$B_i(x) = \begin{cases} \sqrt{\frac{1+2q}{3}}, & \text{for } i = 2,3,4 \\ 0, & \text{for } i = 1 \end{cases}$$

We have,

$$R_{11} = ag_{11} + bA_1A_1 + c(A_1B_1 + A_1B_1), \tag{11.5}$$
$$R_{22} = ag_{22} + bA_2A_2 + c(A_2B_2 + A_2B_2), \tag{11.6}$$
$$R_{33} = ag_{33} + bA_3A_3 + c(A_3B_3 + A_3B_3), \tag{11.7}$$
$$R_{44} = ag_{44} + bA_4A_4 + c(A_4B_4 + A_4B_4). \tag{11.8}$$

Now,

R.H.S. of (11.5) is $\frac{3q}{(1+2q)^2} = R_{11} =$ L.H.S of (11.5).

R.H.S. of (11.6) is $\frac{q}{(1+2q)} = R_{22} =$ L.H.S of (11.6).

Similarly, we can show that the equations (11.7) and (11.8) are also true.

So, the manifold under consideration is a generalized quasi-Einstein space-time.

References

[1] Ahsan, Z.: *A symmetry property of the spacetime of general relativity in terms of the space-matter tensor*. Brazilian J. Phys. **26** (1996), 572–576.

[2] Ahsan, Z., Siddiqui, S. A.: *On the divergence of the space-matter tensor in general relativity*. Adv. Studies Theor. Phys. **4** (2010), 543–556.

[3] Ahsan, Z., Siddiqui, S. A.: *Concircular curvature tensor and fluid spacetimes*. Int J. Theor. Phys. **48**, 11 (2009), 3202–3212.

[4] Amur, K., Maralabhavi, Y. B.: *On quasi-conformal flat spaces*. Tensor (N.S.) **31**, 2 (1977), 194–198.

[5] Bejan, C. L.: *Characterizations of quasi-Einstein manifolds*. An. Stiint. Univ. Al. I. Cuza Iasi. Mat. (N. S.) **53** (2007), 67–72.

[6] Bejan, C. L., Binh, T. Q.: Generalized Einstein manifolds. In: Differential Geometry and its Applications, Proceedings of the 10th International Conference on DGA2007, Olomouc, Czech Republic, 27–31 August 2007, *World Sci. Publ.*, Hackensack, NJ, 2008, 57–64.

[7] Besse, A. L.: Einstein Manifolds. Classics in Mathematics, *Springer-Verlag*, Berlin, Heidelberg, 1987.

[8] Chaki, M. C., Maity, R. K.: *On quasi Einstein manifolds*. Publ. Math. Debrecen **57** (2000), 297–306.

[9] Chaki, M. C.: *On generalized quasi-Einstein manifolds*. Publ. Math. Debrecen **58** (2001), 683–691.

[10] Chaki, M. C., Ray, S.: *Spacetimes with covariant constant energy-momentum tensor*. Int. J. Theor. Phys. **35** (1996), 1027–1032.

[11] De, U. C., Guha, N., Kamilya, D.: *On generalized Ricci-recurrent manifolds*. Tensor (N.S.) **56** (1995), 312–317.

[12] De, U. C., Ghosh, G. C.: *On quasi-Einstein manifolds*. Period. Math. Hungar. **48** (2004), 223–231.

[13] De, U. C., Ghosh, G. C.: *On conformally flat special quasi-Einstein manifolds*. Publ. Math. Debrecen **66** (2005), 129–136.

[14] De, U. C., Ghosh, G. C.: *On quasi Einstein and special quasi Einstein manifolds*. In: Proc. of the Conf. of Mathematics and its Applications, *Kuwait University*, Kuwait, 2004, 178–191.

[15] De, U. C., Gazi, A. K.: *On nearly quasi Einstein manifolds*. Novi Sad J. Math. **38** (2008), 115–121.

[16] De, U. C., Sarkar, A.: *On the quasi-conformal curvature tensor of a (k, μ)-contact metric manifolds*. Math. Rep. **14**, 2 (2012), 115–129.

[17] De, U. C., Matsuyama, Y.: *Quasi-conformally flat manifolds satisfying certain conditions on the Ricci tensor*. SUT J. Math. **42**, 2 (2006), 295–303.

[18] De, U. C., Jun, J. B., Gazi, A. K.: *Sasakian manifolds with quasi-conformal curvature tensor*. Bull. Korean Math. Soc. **45**, 2 (2008), 313–319.

[19] De, U. C., Mallick, S.: *Spacetimes admitting W_2-curvature tensor*. Int. J. Geom. Methods Mod. Phys. **11** (2014), 1–8.

[20] Hosseinzadeh, A., Taleshian, A.: *On conformal and quasi-conformal curvature tensors of an N(k)-quasi-Einstein manifold*. Commun. Korean Math. Soc. **27** (2012), 317–326.

[21] Debnath, P., Konar, A.: *On quasi-Einstein manifolds and quasi-Einstein spacetimes*. Differ. Geom. Dyn. Syst. **12** (2010), 73–82.

[22] Gray, A.: *Einstein-like manifolds which are not Einstein*. Geom. Dedicate **7** (1998), 259–280.

[23] Guha, S.: *On a perfect fluid space-time admitting quasi conformal curvature tensor.* Facta Univ., Ser. Mech. Automat. Control Robot. **3**, 14 (2003), 843–849.

[24] Guha, S. R.: *On quasi-Einstein and generalized quasi-Einstein manifolds.* Facta Univ., Ser. Mech. Automat. Control Robot. **3**, 14 (2003), 821–842.

[25] Güler, S., Demirbağ, S. A.: *A study of generalized quasi-Einstein spacetimes with applications in general relativity.* Int. J. Theor. Phys. **55**, 1 (2016), 548–562.

[26] Mantica, C. A., Suh, Y. J.: *Conformally symmetric manifolds and quasi conformally recurrent Riemannian manifolds.* Balkan J. Geom. Appl. **16**, 1 (2011), 66–77.

[27] Mantica, C. A., Suh, Y. J.: *Pseudo-Z symmetric spacetimes.* J. Math. Phys. **55**, 4 (2014), 1–12.

[28] Nagaraja, H. G.: *On N(k)-mixed quasi-Einstein manifolds.* Eur. J. Pure Appl. Math. **3** (2010), 16–25.

[29] Novello, M., Reboucas, M. J.: *The stability of a rotating universe.* Astrophysical Journal **225** (1978), 719–724.

[30] O'Neill, B.: Semi-Riemannian Geometry. *Academic Press*, New York, 1983.

[31] Özgür, C.: *N(k)-quasi Einstein manifolds satisfying certain conditions.* Chaos, Solutions and Fractals **38** (2008), 1373–1377.

[32] Özgür, C.: *On some classes of super quasi-Einstein manifolds.* Chaos, Solitons and Fractals **40** (2009), 1156–1161.

[33] Özgür, C.: *On a class of generalized quasi-Einstein manifolds.* Applied Sciences, Balkan Society of Geometers **8** (2006), 138–141.

[34] Özgür, C., Sular, S.: *On N(k)-quasi-Einstein manifolds satisfying certain conditions, Balkan J. Geom. Appl.* **13** (2008), 74–79.

[35] Özgür, C., Sular, S.: *On some properties of generalized quasi-Einstein manifolds.* Indian J. Math. **50** (2008), 297–302.

[36] Özgür, C., Sular, S.: *Characterizations of generalized quasi-Einstein manifolds.* An. St. Univ. Ovidius Constanta **20** (2012), 407–416.

[37] Patterson, E. M.: *Some theorems on Ricci-recurrent spaces.* Journal London Math. Soc. **27** (1952), 287–295.

[38] Petrov, A. Z.: Einstein Spaces. *Pergamon Press*, Oxford, 1949.

[39] Ray, D.: *Gödel-like cosmological solutions.* Math. Phys. **21**, 12 (1980), 2797–2798.

[40] Sachs, R. K., Hu, W.: General Relativity for Mathematician. *Springer Verlag*, New York, 1977.

[41] Schouten, J. A.: Ricci-Calculus. *Springer*, Berlin, 1954.

[42] Taleshian, A., Hosseinzadeh, A. A.: *Investigation of some conditions on N(k)-quasi Einstein manifolds.* Bull. Malays. Math. Sci. Soc. **34** (2011), 455–464.

[43] Yano, K., Sawaki, S.: *Riemannian manifolds admitting a conformal transformation group.* J. Differential Geom. **2** (1968), 161–184.

[44] Zengin, F. Ö.: *m-Projectively flat spacetimes.* Math. Reports **4**, 4 (2012), 363–370.