On Uniqueness Theorems for Ricci Tensor*

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Abstract

In Riemannian geometry the prescribed Ricci curvature problem is as follows: given a smooth manifold $M$ and a symmetric 2-tensor $r$, construct a metric on $M$ whose Ricci tensor equals $r$. In particular, DeTurck and Koiso proved the following celebrated result: the Ricci curvature uniquely determines the Levi-Civita connection on any compact Einstein manifold with non-negative section curvature. In the present paper we generalize the result of DeTurck and Koiso for a Riemannian manifold with non-negative section curvature. In addition, we extended our result to complete non-compact Riemannian manifolds with nonnegative sectional curvature and with finite total scalar curvature.

Key words: Uniqueness theorem for Ricci tensor, compact and complete Riemannian manifolds, vanishing theorem.

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1 Introduction

The main point of the papers [1, 2] and the monograph [3, pp. 140–153] is that in certain circumstances the metric (or at last the connection) is uniquely

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determined by the Ricci tensor. In particular, in [1, Corollary 3.3] and [3, Theorem 5.42] anyone can read the following: Let \((M, \bar{g})\) be a compact Einstein manifold with non-negative section curvature and with the Ricci tensor \(\text{Ric}(\bar{g}) = \bar{g}\), then another Riemannian metric \(g\) on \(M\) with \(\text{Ric}(g) = \bar{g}\) has the same Levi-Civita connection as \(\bar{g}\). We note that this proposition is a corollary of the Eells and Sampson vanishing theorem for harmonic maps of compact Riemannian manifolds (see [4, p. 124]).

In the present paper we consider a compact Riemannian manifold \((M, \bar{g})\) with non-negative sectional curvature and with \(\text{Ric}(\bar{g}) \leq \bar{g}\). Under these conditions, we prove that if \(g\) is another Riemannian metric on \(M\) with the Ricci tensor \(\text{Ric}(\bar{g}) = \bar{g}\), then \(g\) and \(\bar{g}\) have the same Levi-Civita connection. Furthermore, if the full holonomy group \(\text{Hol}(\bar{g})\) is reducible then the metric \(g = C\bar{g}\) for some constant \(C > 0\). In turn, it is well known that \(\text{Ric}(\bar{g}) = \text{Ric}(C\bar{g})\). This proposition was announced in report [5] at the 12th International Conference on Geometry and Applications (September 1–5, 2015, Varna, Bulgaria). We extend the above scheme to show that if \((M, \bar{g})\) is a non-compact manifold \((M, \bar{g})\) with non-negative sectional curvature and with the Ricci tensor \(\text{Ric}(\bar{g}) \leq \bar{g}\) then there is no complete Riemannian metric \(g\) such that its Ricci tensor \(\text{Ric}(g) = \bar{g}\) and its total scalar curvature \(s_g(M)\) is finite. This proposition is a corollary of the Schoen and Yau vanishing theorem for harmonic maps of complete non-compact Riemannian manifolds (see [8]).

Our statements generalize and complement the results of the papers [1] and [2], and the monograph [3].

2 Harmonic maps

For the discussion of harmonic maps we will follow Eells and Sampson [4]. Let \((M, g)\) and \((\bar{M}, \bar{g})\) be two Riemannian manifolds with the Levi-Civita connections \(\nabla := \nabla(g)\) and \(\bar{\nabla} := \nabla(\bar{g})\), and \(f: (M, g) \to (\bar{M}, \bar{g})\) be a smooth map. The energy density of \(f\) is defined as the scalar function

\[
e(f) = 2^{-1}||df||^2
\]

where \(||df||^2\) is the squared norm of the differential of \(f\) with respect to metric on the bundle \(T^*M \otimes f^*\bar{T}\bar{M}\). Then the total energy of \(f\) is obtained by integrating the energy density \(e(f)\) over \(M\)

\[
E(f) = \int_M e(f) \, dVol_g
\]

where \(dV_g\) denotes the measure on \((M, g)\) induced by the metric \(g\). If \(f\) is of class \(C^2\) and \(E(f) < +\infty\), and \(f\) is an extremum of the Dirichlet energy functional \(E(f)\), then \(f\) is called a harmonic map and satisfies the Euler–Lagrange equation

\[
\text{trace}_g Ddf = 0
\]

where \(D\) is the connection in the bundle \(T^*M \otimes f^*\bar{T}\bar{M}\) induced from the Levi-Civita connections \(\nabla\) and \(\bar{\nabla}\) of \((M, g)\) and \((\bar{M}, \bar{g})\), respectively.
For any harmonic \( f: (M, g) \to (\bar{M}, \bar{g}) \) we have the Weitzenböck formula (see [4])
\[
\Delta e(f) = Q(f) + \|D df\|^2
\]
where \( \Delta \) is the Laplace–Beltrami operator \( \Delta = \text{div} \nabla \) and
\[
Q(f) = g(\text{Ric}, f^* \bar{g}) - \text{trace}_g(\text{trace}_g(f^* \bar{\text{Riem}}))
\]
where \( \text{Ric} = \text{Ric}(g) \) is the Ricci tensor of \((M, g)\) and \( \bar{\text{Riem}} \) is the Riemannian curvature tensor of \((\bar{M}, \bar{g})\). Let the inequality \( \text{secc} \leq 0 \) be satisfied anywhere on \((\bar{M}, \bar{g})\) and the inequality \( \text{Ric} \geq 0 \) be satisfied anywhere on compact \((M, g)\), then \( Q(f) \) is non-negative everywhere on \( M \). Since our hypothesis implies that the left hand side of (3) is non-negative, then using the Hopf’s lemma (see [6, pp. 30–31]), one can verify that \( e(f) \) is constant. In this case, from (4) we obtain \( D df = 0 \). In this case, \( f \) is totally geodesic map (see [7]). Now we can formulate the following vanishing theorem on harmonic maps. Namely, if \( f: (M, g) \to (\bar{M}, \bar{g}) \) is any harmonic mapping between a compact Riemannian manifold \((M, g)\) with the Ricci tensor \( \text{Ric} \geq 0 \) and a Riemannian manifold \((\bar{M}, \bar{g})\) with the sectional curvature \( \text{secc} \leq 0 \) then \( f \) is totally geodesic and has constant energy density \( e(f) \). Furthermore, if there is at least one point of \( M \) at which its Ricci curvature \( \text{Ric} > 0 \), then every harmonic map \( f: (M, g) \to (\bar{M}, \bar{g}) \) is constant (see [4, p. 124]).

In turn, Schoen and Yau have showed in [8] that \( \sqrt{e(f)} \) is subharmonic function on \((M, g)\) if \( Q(f) \geq 0 \). On other hand, Yau has proved in other his paper [9] that every non-negative \( L^2 \)-integrable subharmonic function on a complete Riemannian manifold must be constant. Applying this to \( \sqrt{e(f)} \), we conclude that \( \sqrt{e(f)} \) is a constant if the total energy \( E(f) < +\infty \) (see also [8]). On the other hand, every complete non-compact Riemannian manifold with nonnegative Ricci curvature has infinite volume (see [9]). In our case, we have \( \text{Ric} \geq 0 \) then the volume of \((M, g)\) is infinite. This forces the constant \( e(f) \) to be zero and \( f \) to be a constant map (see also [8]). Now we can formulate another celebrated vanishing theorem on harmonic maps: If the sectional curvature of \((\bar{M}, \bar{g})\) is non-positive and \((M, g)\) is a complete non-compact manifold with \( \text{Ric} \geq 0 \), then any harmonic map \( f: (M, g) \to (\bar{M}, \bar{g}) \) with the finite energy \( E(f) \) is a constant map (see [8], [10, p. 116]). We remark that in the original paper [8] the manifold \((M, \bar{g})\) was assumed to be compact. However, this assumption is superfluous (see [10, p. 116]).

### 3 The main theorem

If we consider the manifold \( M \) with two Riemannian metrics \( g \) and \( \bar{g} \) then the identity mapping \( \text{Id}: (M, g) \to (M, \bar{g}) \) is harmonic if and only if the deformation tensor \( T = \nabla - \nabla \) is a section of the tensor bundle \( TM \otimes S^2_0 M \), because in this case the Euler–Lagrange equation (3) has the form \( \text{trace}_g T = 0 \) (see [1, 3]). In particular, if \((M, g)\) is a manifold of strictly positive Ricci Ric curvature, then \( \text{Id}: (M, g) \to (M, \text{Ric}) \) is a harmonic map (see [1]). Next we can formulate and prove the following
Theorem 1 Let \((M, \bar{g})\) be a compact Riemannian manifold with the sectional curvature \(\sec \geq 0\) and with the Ricci tensor \(\Ric \leq \bar{g}\). If \(g\) is another Riemannian metric on \(M\) with the Ricci tensor \(\Ric = \bar{g}\), then \(g\) and \(\bar{g}\) have the same Levi-Civita connection. Furthermore, if the full holonomy group \(\text{Hol}(\bar{g})\) of \((M, \bar{g})\) is irreducible then \(\Ric = \bar{Ric}\).

Proof With the above assumptions, we have \(\Ric = \bar{g} > 0\), then the identity map \(\text{Id}: (M, g) \to (M, \bar{g})\) is harmonic. In this case, we have \(e(f) = \frac{1}{2}s\) for the energy density \(e(f)\) of the harmonic identity map \(\text{Id}: (M, g) \to (M, \bar{g})\) and the scalar curvature \(s\) satisfies the Weitzenböck formula (4) which has the following form (see [1]):

\[
\frac{1}{2} \Delta s = Q(f) + \|Df\|^2 \tag{6}
\]

where \(Q(f) = g^{ik}g^{jl}(\bar{g}_{ij}\bar{g}_{kl} - \bar{R}_{ijkl})\) and \(\|Df\|^2 = g^{ij}g^{kl}\bar{g}_{pq}T^p_{ik}T^q_{jl} \geq 0\) for local components \(g_{ij}, \bar{g}_{kl}, \bar{R}_{ijkl}\) and \(T^i_{kl}\) of metric tensors \(g\) and \(\bar{g}\), the Riemannian curvature tensor \(\text{Riem}\) and the deformation tensor \(T\), respectively. On the other hand, we have the identity (see [3, p. 436], [11])

\[
(\bar{g}_{ij}\bar{R}_{kl} - \bar{R}_{ijkl})\varphi^{ik}\varphi^{jl} = \sum_{i<j} \sec(\bar{e}_i, \bar{e}_j)(\bar{\lambda}_i, \bar{\lambda}_j)^2 \tag{7}
\]

where \(\varphi\) is any smooth symmetric tensor field such that \(\varphi(\bar{e}_i, \bar{e}_j) = \bar{\lambda}_i\delta_{ij}\) for the Kronecker delta \(\delta_{ij}\) and some orthonormal basis \(\{\bar{e}_1, \ldots, \bar{e}_n\}\) at any point \(x \in M\). Then equation (6) can be rewritten in the form

\[
\frac{1}{2} \Delta s = \sum_{i<j} \sec(\bar{e}_i, \bar{e}_j)(\bar{\lambda}_i, \bar{\lambda}_j)^2 + g^{ik}g^{jl}(\bar{g}_{ij}\bar{g}_{kl} - \bar{R}_{kl}) + \|T\|^2 \tag{8}
\]

where \(g(\bar{e}_i, \bar{e}_j) = \bar{\lambda}_i\delta_{ij}\). We remark that under the stated assumptions the right side of (8) is non-negative, since then \(\Delta s \geq 0\). Therefore, the scalar curvature \(s\) is a positive subharmonic function on \((M, g)\). If \((M, \bar{g})\) is a compact Riemannian manifold, then using the Hopf’s lemma (see [6, pp. 30–31]), one can verify that \(s = \text{const}\). In this case, from (8) we obtain \(T = 0\). Then \(g\) and \(\bar{g}\) have the same Levi-Civita connection, i.e. \(\bar{\nabla}g = 0\). Furthermore, if the full holonomy group \(\text{Hol}(\bar{g})\) of \((M, \bar{g})\) is irreducible then the metric \(g = C\bar{g}\) for some constant \(C > 0\) (see [3, pp. 282, 285–287]). In this case, we have the identity \(\Ric = \bar{\Ric}\) because \(\text{Ric}(\bar{g}) = \text{Ric}(C\bar{g})\) for some positive constant \(C\) (see [3, pp. 44, 152]).

\[\square\]

4 Two vanishing theorems

In [13] the following non-existence theorem was proved: Let \((M, \bar{g})\) be a compact Riemannian manifold with all sectional curvature less then \((\bar{n} - 1)^{-1}\). Then there is no Riemannian metric \(g\) on \(M\) such that its Ricci tensor \(\Ric = \bar{g}\). In its turn, in [2] the following vanishing theorem was proved: Let \(\bar{g}\) be a metric
on a compact manifold $M$ with the sectional curvature $\overline{\text{sec}} < +1$, then any metric $g$ does not exist on $M$ such that its Ricci tensor $\overline{\text{Ric}} = \bar{g}$. We also get a non-existence result which complements the above propositions. In turn, we can formulate and prove an analogue of these propositions in the following form.

**Theorem 2** Let $(M, \bar{g})$ be a compact Riemannian manifold with nonnegative sectional curvatures and with the Ricci tensor $\overline{\text{Ric}} \leq \bar{g}$. If in addition there is at least one point of $M$ at which the Ricci tensor $\overline{\text{Ric}} < \bar{g}$, then there is no Riemannian metric $g$ on $M$ such that its Ricci tensor $\text{Ric} = \bar{g}$.

**Proof** Let $M$ be a compact manifold. We may assume that $M$ is oriented by taking the twofold covering of $M$ if necessary. Then by Green’s theorem (see [6, pp. 31–33]) we obtain from (6) the following identity

$$
\int_M Q(f) \, d\text{Vol}_g + \int_M \|T\|^2 \, d\text{Vol}_g = 0.
$$

(9)

If the inequalities $\overline{\text{sec}} \geq 0$ and $\overline{\text{Ric}} \leq \bar{g}$ are satisfied and there is a one point $x$ of $M$ in which $\overline{\text{Ric}} < \bar{g}$ then the inequality $\int_M Q(f) \, d\text{Vol}_g > 0$ holds. This inequality contradicts the equation (9). In this case, the harmonic mapping $f$ must be constant. \qed

**Theorem 3** Let $(M, \bar{g})$ be a non-compact Riemannian manifold with the sectional curvature $\overline{\text{sec}} \geq 0$ and with the Ricci tensor $\overline{\text{Ric}} < \bar{g}$. Then there is no complete Riemannian metric $g$ on $(M, \bar{g})$ such that its Ricci tensor $\text{Ric} = \bar{g}$ and its total scalar curvature $s(M)$ is finite.

**Proof** Let $(M, \bar{g})$ be a non-compact Riemannian manifold with the section curvature $\overline{\text{sec}} \geq 0$ and with the Ricci tensor $\overline{\text{Ric}} < \bar{g}$, then $Q(f)$ is non-negative everywhere on $M$. If we assume that there is complete Riemannian metric $g$ on $(M, \bar{g})$ such that its Ricci tensor $\text{Ric} = \bar{g} > 0$, then the volume of $(M, g)$ is infinite (see [9]). Moreover, we have $e(f) = \frac{1}{2} s$ for the energy density $e(f)$ of the harmonic identity map $\text{Id} : (M, g) \rightarrow (M, \bar{g})$ and the scalar curvature $s = \text{trace}_g \text{Ric}$ of the Riemannian manifold $(M, g)$. In this case, $\sqrt{s}$ is a strictly positive subharmonic function on a complete Riemannian manifold $(M, g)$ of infinite volume (see [8]). In addition, if we suppose that the total scalar curvature $\int_M s \, d\text{Vol}_g < +\infty$, then $s$ must be zero (see [8], [12, p. 262]). On the other hand, according to the condition of our theorem the scalar curvature $s = \text{trace}_g \bar{g} > 0$ and hence there is no complete Riemannian metric $g$ on non-compact $(M, \bar{g})$ such that its Ricci tensor $\text{Ric} = \bar{g}$. \qed

**References**


