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Neifeld's Connection Induced on the Grassmann Manifold

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Abstract

The work concerns to investigations in the field of differential geometry. It is realized by a method of continuations and scopes of G. F. Laptev which generalizes a moving frame method and Cartan's exterior forms method and depends on calculation of exterior differential forms. The Grassmann manifold (space of all m-planes) is considered in the n-dimensional projective space P_n. Principal fiber bundle of tangent linear frames is arised above this manifold. Typical fiber of the principal fiber bundle is the linear group working in the tangent space to the Grassmann manifold. Neifeld's connection is given in this fibering. It is proved by Cartan's external forms method, that Bortolotti's clothing of the Grassmann manifold induces this connection.

Key words: Projective space, the Grassmann manifold, principal fiber bundle, Neifeld's connection.

2010 Mathematics Subject Classification: 53A20, 53B25.

1 Introduction

The object of research of this paper is Neifeld's connection on Grassmann manifold. The term "Neifeld's connection" is offered by A. P. Norden [7].

The work concerns to investigations in the field of differential geometry. It is realized by a method of continuations and scopes of G. F. Laptev [4] which generalizes a moving frame method and Cartan's exterior forms method and depends on calculation of exterior differential forms.

Many geometers considered Grassmann manifold as a set of planes of same dimension passing through a fixed point in Euclidean space [2]. Moreover, the derivational formulae of a moving frame containing the forms satisfying the structure equations of linear group and (equi)projective conditions ([9, p. 62]) were usually used in differetial-geometrical researches of Grassmann manifold in the projective space.

We consider Grassmann manifold as manifold of all m-dimensional planes in n-dimensional projective space. The non-classical analytical method is applied in this paper. It has advantage at allocation of subgroups and factor groups of a projective group. The connections in the fiberings associated with Grassmann manifold were investigated in [1] by such method. We show that Neifeld's connection is induced on the Grassmann manifold. By methods of tensor analysis E. G. Neifeld considered two dual linear connections (compare [6]), associated with normalized Grassmann manifold.

We put a projective space P_n to a moving frame $\{A_I\}$ $(I, J, K, \dots = \overline{0, n})$ with derivation formulae

$$dA = \theta_I^J A_J,$$

where the forms θ_J^I satisfy the Cartan structure equations

$$D\theta_J^I = \theta_J^K \wedge \theta_K^I \tag{1}$$

of linear group $GL(n+1)$.

Let's enter new forms (see, eg., [9])

$$\omega_J^I = \theta_J^I - \delta_J^I \theta_0^0. \tag{2}$$

Allocating value 0 indexes $I = \{0, i\}$ we write down more in detail the equality (2)

$$\omega_o^i = \theta_0^i, \quad \omega_j^i = \theta_j^i - \delta_j^i \theta_0^0, \quad \omega_i^0 = \theta_i^0 \quad (\omega_0^0 = 0). \tag{3}$$

The forms (3) are basic forms of projective group $GP(n)$ acting effectively in projective space P_n. From (1) we have

$$D\omega_0^i = \omega_0^j \wedge \omega_j^i, \ D\omega_j^i = \omega_j^0 \wedge \omega_0^i + \omega_j^k \wedge \omega_k^i + \delta_j^i \omega_k^0 \wedge \omega_0^k, \ D\omega_i^0 = \omega_i^j \wedge \omega_j^0.$$

For the further calculations we use compact form of these equations

$$D\omega_J^I = \omega_J^K \wedge \omega_K^I + \delta_J^I \omega_K^0 \wedge \omega_0^K. \tag{4}$$

2 The Grassmann manifold

In P_n we consider the Grassmann manifold $V = Gr(m, n)$ [1], i.e. a manifold of all m-dimensional planes. We make specialization of the moving frame $\{A_a, A_\alpha\}$ $(a, \dots = \overline{0, m};\ \alpha, \dots = \overline{m+1, n})$ putting tops A_a on the plane L_m. From formulae (1) we can see that $\theta_a^\alpha = 0$ are the equations of stationarity of the plane $L_m \in V$, i.e. the forms ω_a^α are the basic forms for the Grassmann manifold V, and $\dim V = (m+1)(n-m)$. These basic forms satisfy the following from (4) structural equations

$$D\omega_a^\alpha = \omega_b^\beta \wedge \omega_{a\beta}^{\alpha b}, \tag{5}$$

where $\omega_{a\beta}^{\alpha b} = \delta_a^b \omega_\beta^\alpha - \delta_\beta^\alpha \omega_a^b$.

We find external differentials of these forms

$$D\omega^{\alpha b}_{a\beta} = \omega^{\gamma b}_{c\beta} \wedge \omega^{\alpha c}_{a\gamma} + \omega^{\gamma}_{c} \wedge \omega^{\alpha bc}_{a\beta\gamma}, \tag{6}$$

where $\omega^{\alpha bc}_{a\beta\gamma} = -\delta^b_a\delta^\alpha_\gamma\omega^c_\beta - \delta^\alpha_\beta\delta^c_a\omega^b_\gamma$.

There is the principal fiber bundle of tangent linear frames $\mathcal{L}(V)$ with structural equations (5), (6) above Grassmann manifold. The typical fiber of the $\mathcal{L}(V)$ is linear group $\mathcal{L} = GL((m+1)(n-m))$, $\dim \mathcal{L} = (m+1)^2(n-m)^2$, acting in the tangent space to the manifold V.

3 Neifeld's connection

In the principal fiber bundle $\mathcal{L}(V)$ we set Neifeld's connection [8, 5].

We enter new forms

$$\tilde{\omega}^{\alpha b}_{a\beta} = \omega^{\alpha b}_{a\beta} - \Gamma^{\alpha bc}_{a\beta\gamma}\omega^{\gamma}_{c} \tag{7}$$

and consider the differentials of these forms (7)

$$D\tilde{\omega}^{\alpha b}_{a\beta} = \tilde{\omega}^{\gamma b}_{c\beta} \wedge \tilde{\omega}^{\alpha c}_{a\gamma} + \omega^{\gamma}_{c} \wedge (\Delta\Gamma^{\alpha bc}_{a\beta\gamma} + \omega^{\alpha bc}_{a\beta\gamma}) + \Gamma^{\gamma be}_{c\beta\mu}\Gamma^{\alpha cd}_{a\gamma\eta}\omega^{\eta}_{d} \wedge \omega^{\mu}_{e}.$$

The connection in the principal fiber bundle $\mathcal{L}(V)$ is set with the help of a field of connection object $\Gamma = \{\Gamma^{\alpha bc}_{a\beta\gamma}\}$ on the base V by equations

$$\Delta\Gamma^{\alpha bc}_{a\beta\gamma} + \omega^{\alpha bc}_{a\beta\gamma} = \Gamma^{\alpha bcd}_{a\beta\gamma\mu}\omega^{\mu}_{d}, \tag{8}$$

where the operator Δ acts:

$$\Delta\Gamma^{\alpha bc}_{a\beta\gamma} = d\Gamma^{\alpha bc}_{a\beta\gamma} + \Gamma^{\alpha bd}_{a\beta\gamma}\omega^{c}_{d} + \Gamma^{\alpha dc}_{a\beta\gamma}\omega^{b}_{d} + \Gamma^{\mu bc}_{a\beta\gamma}\omega^{\alpha}_{\mu} - \Gamma^{\alpha bc}_{a\beta\mu}\omega^{\mu}_{\gamma} - \Gamma^{\alpha bc}_{a\mu\gamma}\omega^{\mu}_{\beta} - \Gamma^{\alpha bc}_{d\beta\gamma}\omega^{d}_{a}.$$

We realize Bortolotti's clothing [3] of the Grassmann manifold which consists in the adjoining to every m-dimensional plane L_m; $(n-m-1)$-dimensional plane P_{n-m-1} not having the common points with the plane L_m.

We define the plane P_{n-m-1} by the points $B_\alpha = A_\alpha + \lambda^a_\alpha A_a$. The differentials of the basic points of clothing plane P_{n-m-1} are

$$dB_\alpha = (\omega^\beta_\alpha + \lambda^b_\alpha\omega^\beta_b)B_\beta + (\Delta\lambda^b_\alpha + \omega^b_\alpha - \lambda^b_\beta\lambda^c_\alpha\omega^\beta_c)A_b.$$

Demanding a relative invariancy of the plane P_{n-m-1} we have

$$\Delta\lambda^a_\alpha + \omega^a_\alpha = \lambda^{ab}_{\alpha\beta}\omega^\beta_b. \tag{9}$$

Bortolotti's clothing allows to cover the components of the connection object Γ

$$\Gamma^{\alpha bc}_{a\beta\gamma} = -\delta^b_a\delta^\alpha_\gamma\lambda^c_\beta - \delta^\alpha_\beta\delta^c_a\lambda^b_\gamma.$$

These functions by virtue of comparisons (9) satisfy to the differential equations (8). Thus, we have

Theorem 1 *Bortolotti's clothing of the Grassmann manifold induces Neifeld's connection in the associated fibering $\mathcal{L}(V)$.*

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