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A Linguistic Fuzzy Approach to the Consensus Reaching in Multiple Criteria Group Decision-making Problems

Vojtěch SUKAČ [1a], Jana TALAŠOVÁ [1b],
Jan STOKLASA [2]

[1] *Department of Mathematical Analysis and Applications of Mathematics, Faculty of Science, Palacký University, 17. listopadu 12, 771 46 Olomouc, Czech Republic*
[a] *e-mail: vojta.sukac@gmail.com*
[b] *e-mail: jana.talasova@upol.cz*

[2] *School of Business and Management, Lappeenranta University of Technology, P.O. Box 20, FI-53851 Lappeenranta, Finland*
e-mail: jan.stoklasa@lut.fi

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Abstract

The paper introduces a new method of reaching a consensus in multiple criteria group decision-making under fuzziness. This model is based on the general definition of the 'soft' consensus introduced by Kacprzyk and Fedrizzi in 1986. The fuzzy evaluations of alternatives express degrees of fulfillment of the given goals by the respective alternatives for each expert. The selection of the best alternative is based on the fuzzy consensus by experts. For this purpose a set of alternatives which are good enough with respect to the most of relevant experts is identified. From this set the alternative with the highest center of gravity (defuzzified fuzzy evaluation) is selected as the most promising one.

Key words: Fuzzy, group decision-making, multicriteria evaluation, fuzzy weighted average, consensus reaching, fuzzy quantifiers.

2010 Mathematics Subject Classification: 90B50

1 Introduction

Making decisions constitutes a large part of our lives. Often the decisions we make are of little importance, e.g. whether to have rolls or bread for breakfast. Sometimes, however, we are faced with decisions whose consequences are not so negligible (e.g. choosing which one of the new cars to buy, choosing the best candidate for a job, etc.). In these situations, an appropriate decision-making model should be used. In the crucial issues it is often advisable to involve more people in decision-making in order to reach a higher objectivity.

In companies the decisions are frequently made by a group of individuals. The biggest problem of group decision-making is that experts can have different preferences and their assessments are often too different to find the option on which everyone agrees. Classic methods based on averaging deal with this issue by providing results that can be characterised by a mean evaluation. These approaches may, however, not be appropriate in situations, when too much disagreement between important evaluators (experts or group of experts) is not welcome (e.g. strategic decisions in companies). Mathematical models that reflect the requirement of unanimity (to some level/extent) seem to be more appropriate in this context. This paper therefore suggests a new multiple-criteria and multi-expert decision making model that enables the decision makers to specify the required level of unanimity (agreement of evaluations provided by the experts/evaluators) and also to specify the borderline evaluation that needs to be achieved by most of the important experts for an alternative to even be considered for selection.

In general the approaches in group decision-making can be divided into two categories. On one hand there are voting systems, which are described in many papers, see e.g. [2, 15, 19, 20, 24]. On the other hand there is consensus reaching, which is also studied quite frequently in the literature, see e.g. [12, 15, 24]. In most of the papers dealing with consensus reaching in group decision-making the consensus reaching is based on the aggregation of fuzzy preference relations.

This paper focuses on the second category of group decision-making models, that is on mathematical models of consensus reaching. It assumes that each expert provides evaluations of each alternative with respect to each criterion considered. For every alternative the overall evaluations by all the experts are calculated using the partial evaluations provided by each expert. These overall evaluations are in the form of fuzzy numbers—such fuzzy evaluations express the degree of fulfillment of the overall goal by each alternative according to each expert. As such these evaluations are of absolute type, that is they are not dependent on the set of alternatives and describe the acceptability of the alternatives. A multiple criteria (fuzzy) evaluation methodology based on the partial goals tree structure that uses this type of evaluations is described in [25].

The evaluation of alternatives proceeds in the following way. At first, each expert evaluates each alternative with respect to each criterion by a fuzzy number. The fuzzy weighted average with fuzzy weights of criteria is then applied for the aggregation of partial evaluations of any expert. Different weights of criteria are admitted for individual experts. The experts are assumed to have various decision competences expressed again by fuzzy weights. The fuzzy weighted average and related issues are described in more details in [21, 22, 23, 26].

Consensus reaching in the model proposed in this paper is based on the idea that the choice of the best alternative is only among the alternatives that are good enough according to the most of relevant experts.

The paper is organized as follows. First, in the next section, some preliminary definitions relating to fuzzy sets, fuzzy numbers and linguistic fuzzy modelling in general are briefly summarized. In the third section the proposed multiple criteria group decision-making model is described in details. In Section 4, an illustrative example is shown. Finally, the last section contains the summary of the model and its discussion.

2 Preliminaries

Let U be a nonempty set called *universe*. A fuzzy set A on U is determined by its *membership function* $\mu_A(x)\colon U \to [0,1]$, where $\mu_A(x)$ expresses the degree of membership of x in the fuzzy set A—from 0 for "x definitely does not belong to A" to 1 for "x definitely belongs to A", through all intermediate values. The family of all fuzzy sets on a universe U is denoted by $\mathcal{F}(U)$.

Let A be a fuzzy set on U and $\alpha \in [0,1]$. A crisp set

$$A_\alpha = \big\{x \in U \mid \mu_A(x) \geq \alpha\big\} \tag{1}$$

is called *the α-cut of a fuzzy set A. The support of a fuzzy set A* is a (crisp) set

$$\mathrm{Supp}(A) = \{x \in U \mid \mu_A(x) > 0\big\}. \tag{2}$$

The kernel of a fuzzy set A is a (crisp) set

$$\mathrm{Ker}(A) = \{x \in U \mid \mu_A(x) = 1\big\}. \tag{3}$$

In cases, when the support of A is a discrete set ($\mathrm{Supp}(A) = \{x_1, \ldots, x_k\}$), then the fuzzy set A can be denoted as $A = \{{}^{\mu_A(x_1)}\!/_{x_1}, \ldots, {}^{\mu_A(x_k)}\!/_{x_k}\}$. A special type of fuzzy sets whose universe is a subset of $\mathbb{R}$, the so called *fuzzy numbers*, can be defined in the following way. Let $U \subset \mathbb{R}$ be an interval. *A fuzzy number* N is a fuzzy set on the universe U which fulfills the following conditions:

(a) $\mathrm{Ker}(N) \neq \emptyset$,

(b) for all $\alpha \in (0,1]$, N_α are closed intervals,

(c) $\mathrm{Supp}(N)$ is bounded.

The family of all fuzzy numbers on U is denoted by $\mathcal{F}_N(U)$.

Each fuzzy number N is determined by

$$N = \Big\{\big[\underline{N}(\alpha), \overline{N}(\alpha)\big]\Big\}_{\alpha\in[0,1]},$$

where $\underline{N}(\alpha)$ and $\overline{N}(\alpha)$ is the lower and upper bound of α-cut of the fuzzy number N respectively, for $0 < \alpha \leq 1$ and $[\underline{N}(0), \overline{N}(0)]$ is the closure of the support of N, i.e. $[\underline{N}(0), \overline{N}(0)] = \mathrm{Cl}(\mathrm{Supp}(N))$.

A trapezoidal fuzzy number N is determined by an ordered quadruple

$$(n^1, n^2, n^3, n^4) \subset U^4$$

of significant values of N satisfying

$$[n^1, n^4] = \mathrm{Cl}(\mathrm{Supp}(N)) \quad \text{and} \quad [n^2, n^3] = \mathrm{Ker}(N).$$

The membership function of a trapezoidal fuzzy number N is

$$\mu_N(x) = \begin{cases} 0 & \text{if } x < n^1, \\ \frac{x-n^1}{n^2-n^1} & \text{if } n^1 \leq x < n^2, \\ 1 & \text{if } n^2 \leq x \leq n^3, \\ \frac{n^4-x}{n^4-n^3} & \text{if } n^3 < x \leq n^4, \\ 0 & \text{if } x > n^4. \end{cases}$$

N is called a triangular fuzzy number if $n^2 = n^3$. Closed real intervals and real numbers can be represented by special cases of trapezoidal fuzzy numbers.

This paper utilizes the linguistic approach to group decision-making problems. *Linguistic variable* [27] is a 5-tuple $\big(\mathcal{V}, \mathcal{T}(\mathcal{V}), U, G, M\big)$, where $\mathcal{V}$ is the name of the linguistic variable, $\mathcal{T}(\mathcal{V})$ is the set of its linguistic terms (the values of $\mathcal{V}$), $U \subset \mathbb{R}$ is the universe on which the fuzzy numbers expressing meanings of these linguistic terms are defined, G is the grammar used to generate the linguistic terms of $\mathcal{V}$ and M is a mapping that assigns to each linguistic term $\mathcal{C} \in \mathcal{T}(\mathcal{V})$ its meaning $C = M(\mathcal{C})$ (a fuzzy number on U).

3 Proposed multiple criteria group decision-making model

In this section the proposed mathematical model will be described in details.

In the proposed model a set $\boldsymbol{X} = \{X_1, \ldots, X_n\}$ of $n \geq 2$ alternatives is considered. Each alternative X_i, $i = 1, \ldots, n$, is evaluated by $p \geq 2$ experts with possibly different competences according to $m \geq 2$ criteria. Competences of experts are given by fuzzy numbers $L^k \in \mathcal{F}_N([0,1])$, $k = 1, \ldots, p$, where 0 means an incompetent expert and 1 means a fully competent expert. Linguistic terms presented in Table 1 (their meanings are illustrated in Figure 1) can be used to assign competences to experts. Obviously, experts' competences are well suited for reflecting the position of the experts in the company or their areas of expertise with respect to the solved problem.

0	definite incompetence
EL	extremely low competence
VL	very low competence
L	low competence
A	average competence
H	high competence
VH	very high competence
EH	extremely high competence
1	full competence

Table 1: Linguistic terms expressing competence of experts

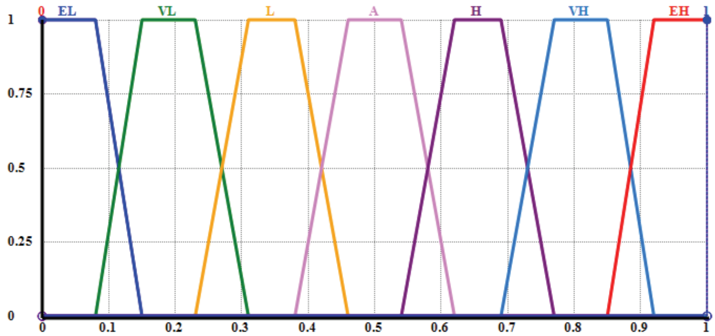

Figure 1: Meanings of linguistic terms expressing competence of experts

As was already mentioned, the experts evaluate each alternative according to m criteria $C_1, \ldots, C_m$. Criteria are set to match the partial goals. Expert E_k, $k = 1, \ldots, p$, evaluates each alternative X_i, $i = 1, \ldots, n$, according to each criterion C_j, $j = 1, \ldots, m$, by a fuzzy number $H_{ij}^k \in \mathcal{F}_N([0,1])$ which expresses the level of fulfillment of the corresponding partial goal (0 means no fulfillment and 1 means complete fulfillment, through all the intermediate values). The following linguistic terms can also be used:

- almost zero fulfillment,
- weak fulfillment,
- average fulfillment,
- substantial fulfillment,
- almost complete fulfillment.

Their meanings are shown in Figure 2.

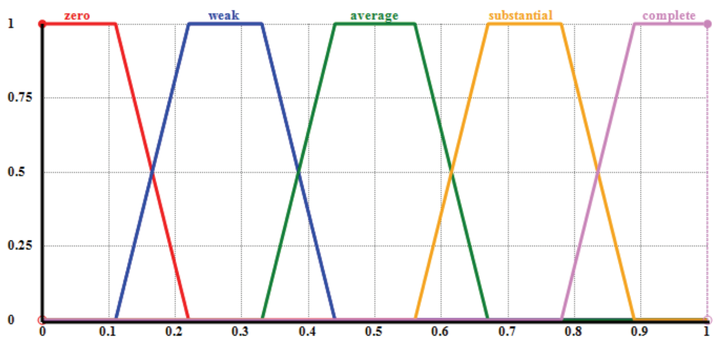

Figure 2: Meanings of linguistic values of the linguistic variable expressing the fulfillment of partial goal

Each expert E_k assigns a weight $W_j^k \in \mathcal{F}_N([0,1])$ to the criterion C_j, $j = 1, \ldots, m$, which expresses the importance of criterion C_j, (0 represents a definitely irrelevant criterion and 1 means definitely necessary criterion). Different weights of criteria are admitted for individual experts. The following linguistic terms can be also used:

- very low importance,
- low importance,
- average importance,
- high importance,
- very high importance.

Their meanings are shown in Figure 3.

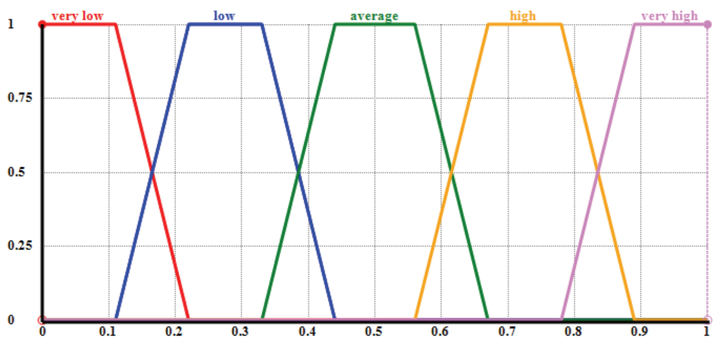

Figure 3: Meanings of linguistic terms expressing the importance of criterion

3.1 Evaluation of alternatives by experts

Fuzzy weighted average operation with fuzzy weights of criteria is applied to aggregate the partial evaluations of alternatives.

The overall evaluation $H_i^k \in \mathcal{F}_N([0,1])$ of alternative X_i, $i = 1, \ldots, n$, according to expert E_k, $k = 1, \ldots, p$, is computed by

$$H_i^k = \left\{ \left[\underline{H}_i^k(\alpha), \overline{H}_i^k(\alpha) \right] \right\}_{\alpha \in [0,1]}, \tag{4}$$

where the lower and upper bounds of the α-cuts

$$\underline{H}_i^k(\alpha) = \min_{w_j^k \in [\underline{W}_j^k(\alpha), \overline{W}_j^k(\alpha)], j=1,\ldots,m} \frac{\sum_{j=1}^m w_j^k \cdot \underline{H}_{ij}^k(\alpha)}{\sum_{j=1}^m w_j^k},$$

$$\overline{H}_i^k(\alpha) = \max_{w_j^k \in [\underline{W}_j^k(\alpha), \overline{W}_j^k(\alpha)], j=1,\ldots,m} \frac{\sum_{j=1}^m w_j^k \cdot \overline{H}_{ij}^k(\alpha)}{\sum_{j=1}^m w_j^k}$$

are computed by a suitable algorithm for nonlinear optimization problems.

In practical cases, calculations are performed for only a few α-cuts and the resulting overall evaluations are replaced by the piecewise linear fuzzy numbers [25] (i.e. fuzzy numbers with a piecewise linear membership function). The overall evaluations $H_i^k \in \mathcal{F}_N([0,1])$ express the level of fulfillment of overall goal by alternative X_i, $i = 1, \ldots, n$, according to expert E_k, $k = 1, \ldots, p$.

Now, one can apply analogously the fuzzy weighted average of the overall evaluations for the given alternative (fuzzy competences of experts L^k, $k = 1, \ldots, p$, are used as weights) to obtain group evaluations of alternatives. The best alternative can then be chosen by comparing the centers of gravity (15) of the group evaluations. But such approach does not use the concept of consensus. Moreover such approach can lead to the selection of an alternative with a low fulfillment level of the overall goal—this can happen e.g. when all the alternatives are evaluated low (have bad evaluations) according to all the experts.

A different perspective is adopted in this paper. The proposed model is based on the idea that the optimal alternative should be chosen among such alternatives which are good enough according to the meaning of a sufficient amount of important experts. Due to this aim a linguistic variable $\widehat{\mathcal{A}}$ with the linguistic term set $\{\widehat{\mathcal{A}}_1, \ldots, \widehat{\mathcal{A}}_5\}$ is introduced to express the level of acceptance of alternatives by experts; Table 2 summarizes the linguistic terms and Figure 4 depicts their respective meanings.

$\widehat{\mathcal{A}}_1$	excellent
$\widehat{\mathcal{A}}_2$	good
$\widehat{\mathcal{A}}_3$	acceptable
$\widehat{\mathcal{A}}_4$	borderline
$\widehat{\mathcal{A}}_5$	unacceptable

Table 2: Linguistic term set expressing acceptance of alternative

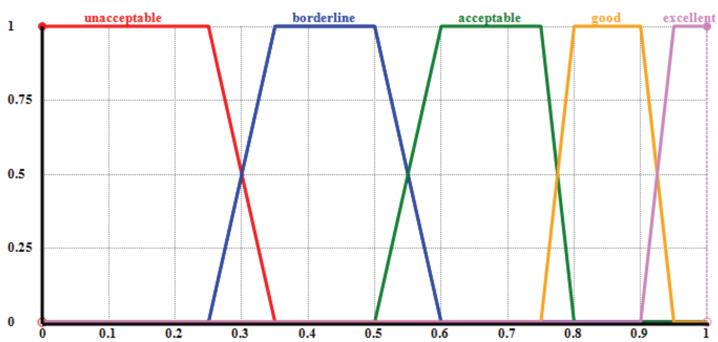

Figure 4: Meanings of the linguistic terms expressing acceptance of alternatives

The membership functions of the trapezoidal fuzzy numbers representing the meanings of the linguistic terms expressing acceptance of alternatives are

$$\mu_{\widehat{A}_r}(x) = \begin{cases} 0 & \text{if } x < \widehat{a}_r^1, \\ \frac{x-\widehat{a}_r^1}{\widehat{a}_r^2-\widehat{a}_r^1} & \text{if } \widehat{a}_r^1 \leq x < \widehat{a}_r^2, \\ 1 & \text{if } \widehat{a}_r^2 \leq x \leq \widehat{a}_r^3, \\ \frac{\widehat{a}_r^4-x}{\widehat{a}_r^4-\widehat{a}_r^3} & \text{if } \widehat{a}_r^3 < x \leq \widehat{a}_r^4, \\ 0 & \text{if } x > \widehat{a}_r^4, \end{cases} \quad r = 1, \ldots, 5.$$

A modified set of linguistic terms $\{\mathcal{A}_1, \ldots, \mathcal{A}_5\}$ expressing "*at least* $\widehat{\mathcal{A}}_r$", $r = 1, \ldots, 5$, whose meanings have the membership functions defined in the following way

$$\mu_{A_r}(x) = \begin{cases} 0 & \text{if } x < \widehat{a}_r^1, \\ \frac{x-\widehat{a}_r^1}{\widehat{a}_r^2-\widehat{a}_r^1} & \text{if } \widehat{a}_r^1 \leq x < \widehat{a}_r^2, \\ 1 & \text{if } x \geq \widehat{a}_r^2, \end{cases} \quad r = 1, \ldots, 5,$$

will also be used in the proposed model.

The numbers θ_{ri}^k, $r = 1, \ldots, 5$, $i = 1, \ldots, n$, $k = 1, \ldots, p$, can be computed in the following way:

$$\theta_{ri}^k = \sup_{x \in [0,1]} \Big\{ \min \left\{ \mu_{A_r}(x), \mu_{H_i^k}(x) \right\} \Big\}. \tag{5}$$

These numbers can be interpreted as truth values of the statement: "the acceptance of alternative X_i by expert E_k is $\mathcal{A}_r$". For example, according to expert E_3, alternative X_2 is *at least acceptable* with the truth value θ_{32}^3, and it is *at least borderline* with the truth value θ_{42}^3. Simultaneously, according to expert E_2, alternative X_3 is *at least excellent* with the truth value θ_{13}^2, and it is *at least good* with the truth value θ_{23}^2.

The fuzzy sets F_{ri} of *experts suggesting* $\mathcal{A}_r$ *for* X_i, $F_{ri} \in \mathcal{F}(\{E_1, \ldots, E_p\})$, $r = 1, \ldots, 5$, $i = 1, \ldots, n$, can now be defined as $F_{ri} = \{{}^{\theta_{ri}^1}\!/_{E_1}, \ldots, {}^{\theta_{ri}^p}\!/_{E_p}\}$.

3.2 Group aggregation

In the proposed model, the selection of the best alternative is based on the idea that the best alternative should be good enough according to the "quantity" of important experts. To represent the desired quantity, the linguistic quantifier set $\{\widehat{\mathcal{Q}}_1, \ldots, \widehat{\mathcal{Q}}_4\}$ is introduced, see Table 3. The meanings $\widehat{Q}_1, \ldots, \widehat{Q}_4$ of these linguistic quantifiers are shown in Figure 5.

$\widehat{Q}_1$	almost all
$\widehat{Q}_2$	more than half
$\widehat{Q}_3$	about half
$\widehat{Q}_4$	minority

Table 3: Linguistic quantifiers

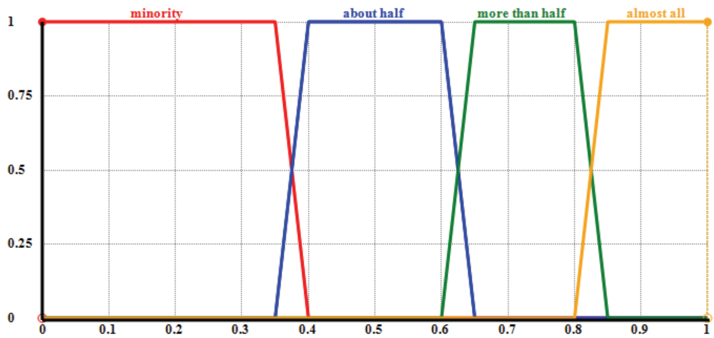

Figure 5: The meanings of the linguistic quantifiers

Their membership functions are

$$\mu_{\widehat{Q}_s}(x) = \begin{cases} 0 & \text{if } x < \widehat{q}_s^1, \\ \frac{x-\widehat{q}_s^1}{\widehat{q}_s^2-\widehat{q}_s^1} & \text{if } \widehat{q}_s^1 \le x < \widehat{q}_s^2, \\ 1 & \text{if } \widehat{q}_s^2 \le x \le \widehat{q}_s^3, \\ \frac{\widehat{q}_s^4-x}{\widehat{q}_s^4-\widehat{q}_s^3} & \text{if } \widehat{q}_s^3 < x \le \widehat{q}_s^4, \\ 0 & \text{if } x > \widehat{q}_s^4, \end{cases} \qquad s = 1, \dots, 4.$$

The meanings $Q_1, \dots, Q_4$ of the linguistic quantifiers expressing "*at least* $\widehat{\mathcal{Q}}_s$", are defined by membership functions

$$\mu_{Q_s}(x) = \begin{cases} 0 & \text{if } x < \widehat{q}_s^1, \\ \frac{x-\widehat{q}_s^1}{\widehat{q}_s^2-\widehat{q}_s^1} & \text{if } \widehat{q}_s^1 \le x < \widehat{q}_s^2, \\ 1 & \text{if } x \ge \widehat{q}_s^2, \end{cases} \qquad s = 1, \dots, 4.$$

To simplify the notation, the linguistic term set {*at least almost all, at least more than half, at least about half, at least minority*} will be substituted by the term set {*almost all, majority, at least half, some*} $= \{Q_1, \dots, Q_4\}$. The linguistic term *important expert* is also required, with its meaning $B \in \mathcal{F}_N([0,1])$ defined by (6).

$$\mu_B(x) = \begin{cases} 0 & \text{if } 0.0 \le x < 0.3 \\ 2x - 0.6 & \text{if } 0.3 \le x < 0.8 \\ 1 & \text{if } 0.8 \le x \le 1.0. \end{cases} \tag{6}$$

Using this linguistic term and its meaning, we can define the fuzzy set of *important experts* $I \in \mathcal{F}(\{E_1, \dots, E_p\})$ as $I = \{{}^{\zeta_1}/_{E_1}, \dots, {}^{\zeta_p}/_{E_p}\}$. For the group aggregation, it is necessary to determine the importance level ζ_k of each expert. If competence of expert E_k is assessed by fuzzy number L^k, then

$$\zeta_k = \sup_{x \in [0,1]} \Big\{ \min \{\mu_B(x), \mu_{L^k}(x)\} \Big\}, \; k = 1, \dots, p.$$

In the proposed model inspired by the concept of 'soft' consensus (see [9]), the group aggregation is based on the *'soft' degrees of consensus* by Fedrizzi and

Kacprzyk described e.g. in [11]. According to [11], a linguistic statement *most experts are convinced* (quantified by “most”) can be symbolically written in the following form:

$$Q \text{ experts are } F, \tag{7}$$

where Q is a linguistic quantifier, “experts” represents the (set of) experts, $E = \{E_1, \ldots, E_p\}$ and F is a property (e.g. convinced). The importance of the experts can be added into the linguistically quantified statement (7) using I:

$$QI \text{ experts are } F, \tag{8}$$

denoting that e.g. *a given quantity (Q) of the important experts (I) are F (e.g. convinced).*

The degree of truth of the linguistically quantified statement (8) now needs to be found—it can be denoted truth(QI experts are F). In accordance with [11] the statement “QI experts are F” can be reformulated as “Q(I and F) experts are I”. The calculations of the truth value proceed in the following two steps. We suppose that at least one of the experts has nonzero importance level and in **step 1** we calculate

$$r' = \frac{\sum \text{count}(I \text{ and } F)}{\sum \text{count}(I)} = \frac{\sum_{k=1}^{p} \left[\mu_I(E_k) \wedge \mu_F(E_k)\right]}{\sum_{k=1}^{p} \mu_I(E_k)}, \tag{9}$$

where $\wedge$ is a minimum t-norm. In **step 2** the actual truth degree of the statement (8) is calculated:

$$\text{truth}(QI \text{ experts are } F) = \mu_Q(r'). \tag{10}$$

With respect to this general approach outlined in [11], in the model proposed in this paper instead of “Q” all the quantifiers $\mathcal{Q}_s$, $s = 1, \ldots, 4$, will be considered in turn; “$\mu_I(E_k)$” can be replaced by “ζ_k”, “F” will be substituted by F_{ri} corresponding to “experts suggesting $\mathcal{A}_r$ for X_i” and hence “$\mu_{F_{ri}}(E_k)$” can be replaced by “θ^k_{ri}”, $r = 1, \ldots, 5$, $i = 1, \ldots, n$.

Thus, the truth value (denoted $\xi_i^{r,s}$) of the statement “the alternative X_i is $\mathcal{A}_r$ (e.g. *at least good*) with respect to the opinion of quantity ($\mathcal{Q}_s$) of important experts (I)” is truth$\big(\mathcal{Q}_s\ I$ experts are experts suggesting $\mathcal{A}_r\big)$. It can be rewritten to truth$\big[\mathcal{Q}_s\big(I$ and suggesting $\mathcal{A}_r\big)$ experts are $I\big]$.

The values $\xi_i^{r,s}$ are computed by

$$\xi_i^{r,s} = \mu_{Q_s}\left(\frac{\sum_{k=1}^{p}\left[\zeta_k \wedge \theta^k_{ri}\right]}{\sum_{k=1}^{p}\zeta_k}\right), \quad r = 1, \ldots, 5,\ s = 1, \ldots, 4. \tag{11}$$

Values $\xi_i^{r,s}$ are computed for all $i = 1, \ldots, n$, $r = 1, \ldots, 5$, $s = 1, \ldots, 4$, that is $20n$ values need to be computed.

For example, $\xi_2^{1,3}$ is the truth value of the statement “alternative X_2 is *excellent* according to the opinions of *at least about half* of important experts” and $\xi_5^{2,2}$ is the truth value of the statement “alternative X_5 is *at least good* according to the opinions of *majority* of important experts", and $\xi_5^{3,1}$ is the truth value of the statement "alternative X_5 is *at least acceptable* according to the opinions of *almost all* important experts".

For all r, s the set $\Upsilon_{r,s}$ is defined, which includes such alternatives which are $\mathcal{A}_r$ according to $\mathcal{Q}_s$ of important experts,

$$\Upsilon_{r,s} = \big\{X_i \in \boldsymbol{X} \,\big|\, \xi_i^{r,s} = 1\big\}. \tag{12}$$

The sets $\Upsilon_{r,s}$ are scanned in the order $\mathcal{K}$ until the first nonempty set is found. The order $\mathcal{K}$ is a given sequence of pairs $\big\{(r,s)_t\big\}_t$ such that the pair $(r,s)_t$ should be checked for nonemptyness before the pair $(r,s)_{t'}$, for $t < t'$.

The order $\mathcal{K}$ can be specified like this:

$$\mathcal{K} = \big\{(1,1),(1,2),(2,1),(2,2),(3,1),(3,2),(1,3),(2,3),\dots\big\}. \tag{13}$$

This order can be rewritten using linguistic terms as

$$\begin{aligned}\mathcal{K} = \big\{&(\textit{excellent} \text{ by } \textit{almost all}),\ (\textit{excellent} \text{ by } \textit{majority}),\\ &(\textit{at least good} \text{ by } \textit{almost all}),\ (\textit{at least good} \text{ by } \textit{majority}),\\ &(\textit{at least acceptable} \text{ by } \textit{almost all}),\ (\textit{at least acceptable} \text{ by } \textit{majority}),\\ &(\textit{excellent} \text{ by } \textit{at least half}),\ (\textit{at least good} \text{ by } \textit{at least half}),\dots\big\}.\end{aligned}$$

The first non-empty set $\Upsilon_{r,s}$ according to $\mathcal{K}$ (denoted Υ^*) includes those alternatives, among which the most promising one is to be chosen.

The sequence $\mathcal{K}$ doesn't have to include all possible combinations of r and s. It is sufficient to consider only relevant combinations (e.g. (*borderline* by *majority*) can be omitted, as such alternatives are definitely not good ones to consider). As $\mathcal{K}$ does not necessarily include all possible combinations of r and s, it can happen that there is no nonempty $\Upsilon_{r,s}$. In this case, no alternative is chosen (none of the alternatives is considered good enough to be adopted in the consensus sense, based on the meanings of the experts).

In the case when all the experts would have zero importance level, which is not very frequent in practice, there would be no relevant information to base our decision on. A reasonable course of action in these cases would be finding experts with nonzero importance levels.

3.3 Choosing of the best alternative

In case of a nonempty Υ^*, the fuzzy weighted average with fuzzy weights L^k, $k = 1, \dots, p$, is applied to aggregate experts' evaluations of alternatives H_i^k, $\forall i\colon X_i \in \Upsilon^*$. The group evaluations of alternatives are expressed by fuzzy numbers

$$H_i = \Big\{\big[\underline{H}_i(\alpha), \overline{H}_i(\alpha)\big]\Big\}_{\alpha\in[0,1]}, \quad i\colon X_i \in \Upsilon^* \tag{14}$$

where the lower and upper bounds of the α-cuts are computed according to the equations

$$\underline{H}_i(\alpha) = \min_{l^k \in [\underline{L}^k(\alpha), \overline{L}^k(\alpha)], k=1,\ldots,p} \frac{\sum_{k=1}^{p} l^k \cdot \underline{H}_i^k(\alpha)}{\sum_{k=1}^{p} l^k},$$

$$\overline{H}_i(\alpha) = \max_{l^k \in [\underline{L}^k(\alpha), \overline{L}^k(\alpha)], k=1,\ldots,p} \frac{\sum_{k=1}^{p} l^k \cdot \overline{H}_i^k(\alpha)}{\sum_{k=1}^{p} l^k}.$$

Alternatives from Υ^* are compared according to their group evaluations by the center of gravity (15).

$$T_{H_i} = \frac{\int_0^1 x\mu_{H_i}(x)\mathrm{d}x}{\int_0^1 \mu_{H_i}(x)\mathrm{d}x} \tag{15}$$

The alternative with the highest center of gravity of its fuzzy group evaluation is chosen as the optimal one.

$$X_{i_0} : T_{H_{i_0}} = \max_{i: X_i \in \Upsilon^*} T_{H_i}. \tag{16}$$

If more than one alternative have the same the highest center of gravity of group evaluation, any of them may be selected as the optimal one.

4 Example—a comparison of the proposed and standard approach

Let us consider a family of four that wants to buy a new car. Each of its four family members evaluates 8 different cars according to a set of relevant criteria including design, safety and fuel consumption. Their overall evaluations of the cars (alternatives) are illustrated in Figure 6.

Competences of all four family members (experts in our terminology) are assigned as follows (the younger child is considered of an extremely low competence, the older one is considered of average competence, one of the parents of very high competence and the other one is considered to be fully competent):

Expert	competence
Expert1	fully competent
Expert2	very high competent
Expert3	average competent
Expert4	extremely low competent

Table 4: Competences of experts.

The experts evaluate each alternative according to all the three above mentioned criteria using linguistic terms shown in Figure 2, which express the fulfillment of the corresponding partial goal. Due to the different relationship of experts to the problem addressed, different weights of criteria were allowed for each expert. The experts determine the importance of criteria using the linguistic terms of the linguistic variable presented in Figure 3.

Each expert evaluates each alternative according to all three criteria and also he/she specifies the weights of all three criteria. The evaluations (including the weights of the criteria) are summarized in the following 4 tables (Tables 5 to 8). The overall evaluations of alternatives according to each expert are illustrated in Figure 6.

criterion	design	fuel consumption	safety
weight of criterion	low	average	very high
car1	complete	weak	average
car2	substantial	average	average
car3	complete	substantial	complete
car4	weak	substantial	zero
car5	weak	weak	average
car6	average	substantial	average
car7	average	complete	complete
car8	substantial	average	substantial

Table 5: Evaluations of alternatives according to each criterion by Expert1 in terms of fulfilment of the three partial goals (criteria)

criterion	design	fuel consumption	safety
weight of criterion	average	very high	high
car1	weak	weak	average
car2	average	average	average
car3	substantial	average	complete
car4	complete	substantial	weak
car5	average	weak	average
car6	average	average	substantial
car7	weak	weak	average
car8	average	substantial	average

Table 6: Evaluations of alternatives according to each criterion by Expert2 in terms of fulfilment of the three partial goals (criteria)

In the *classic approach*, the fuzzy weighted average of overall evaluations by each expert (the competences of experts are used as fuzzy weights) is applied to obtain the group evaluations. The group evaluations of cars are illustrated in Figure 7. The centers of gravity of the group evaluations are also marked in Figure 7. Based on the comparison of the centers of gravity of group evaluations of cars, the family would choose the *car3* as the best one, since its center of gravity is the highest.

criterion weight of criterion	design very low	fuel consumption average	safety very high
car1	weak	weak	average
car2	substantial	substantial	average
car3	average	weak	average
car4	complete	substantial	substantial
car5	weak	average	average
car6	complete	substantial	substantial
car7	substantial	complete	average
car8	complete	substantial	average

Table 7: Evaluations of alternatives according to each criterion by Expert3 in terms of fulfilment of the three partial goals (criteria)

criterion weight of criterion	design very high	fuel consumption average	safety low
car1	average	average	complete
car2	complete	substantial	complete
car3	average	average	complete
car4	weak	complete	substantial
car5	weak	average	average
car6	complete	average	substantial
car7	average	weak	substantial
car8	substantial	complete	substantial

Table 8: Evaluations of alternatives according to each criterion by Expert4 in terms of fulfilment of the three partial goals (criteria)

If we approach the problem with *the multiple criteria group decision-making model proposed in this paper*, we first calculate $\xi_i^{r,s}$ for all $i = 1, \ldots, 8$, $r = 1, \ldots, 5$, $s = 1, \ldots, 4$, according to the equation (11). Then the sets $\Upsilon_{r,s}$ are formed according to the equation (12). This way we obtain the results summarized in Table 9.

According to the order $\mathcal{K}$ specified by (13), the first non-empty set is $\Upsilon_{3,1}$ (see Table 9), that is a set of alternatives evaluated *at least acceptable* by *almost all* of the experts, which contains only three alternatives, namely *car2*, *car6*, *car8*. Finally, these 3 alternatives were compared using method of center of gravity

$$T_{H_2} = 0.5427, \qquad T_{H_6} = 0.6098, \qquad T_{H_8} = 0.6257.$$

and the alternative *car8* was chosen as the best one (all the group evaluations of alternatives and their centers of gravity are illustrated in Figure 7). Note that the set $\Upsilon_{3,1}$ does not contain the alternative *car3*, which was chosen as the best one by the *classic* (previously mentioned) approach. We can observe, that with the *multiple criteria group decision-making model proposed in this paper* the alternative *car8*, which was evaluated *at least acceptable* by *almost* *all* of the experts outperforms the alternative *car3*, which receives a borderline evaluation from Expert 3. This is consistent with the premise of this paper, that an alternative which is good enough with respect to most of the relevant experts should be suggested. The basic difference between the classic approach and the one proposed in this paper is that the approach presented here requires the consensus of the decision makers (e.g. most of the most relevant decision makers need to agree that the alternative is at least acceptable). This way the effect of "averaging" by classic methods is restricted to some extent by discarding alternatives that are not considered "good enough" (in the case of this example "*at least acceptable*") by some important decision makers. The case where a single important expert "disagrees" with a larger group of decision makers of lower importance in the evaluation is also a reason for an alternative to be discarded (not to appear in higher ranking $\Upsilon_{r,s}$ sets).

r	s	alternatives
excellent	almost all	$\emptyset$
excellent	more than half	$\emptyset$
good	almost all	$\emptyset$
good	more than half	$\emptyset$
acceptable	**almost all**	**2,6,8**
acceptable	more than half	2,3,4,6,7,8
excellent	about half	$\emptyset$
good	about half	3,4,7
acceptable	about half	2,3,4,6,7,8
borderline	almost all	1,2,3,4,5,6,7,8
borderline	more than half	1,2,3,4,5,6,7,8
borderline	about half	1,2,3,4,5,6,7,8
excellent	minority	1,2,3,4,5,6,7,8
good	minority	1,2,3,4,5,6,7,8
acceptable	minority	1,2,3,4,5,6,7,8
borderline	minority	1,2,3,4,5,6,7,8
unacceptable	almost all	1,2,3,4,5,6,7,8
unacceptable	more than half	1,2,3,4,5,6,7,8
unacceptable	about half	1,2,3,4,5,6,7,8
unacceptable	minority	1,2,3,4,5,6,7,8

Table 9: Alternatives belonging to $\Upsilon_{r,s}$

5 Conclusion

In this paper a new model for multiple criteria group decision-making in fuzzy environment was presented. This model applies fuzzy evaluations of absolute type for evaluation of alternatives and the definition of the 'soft' consensus for the group aggregation.

In this method, each expert evaluates each alternative with respect to each

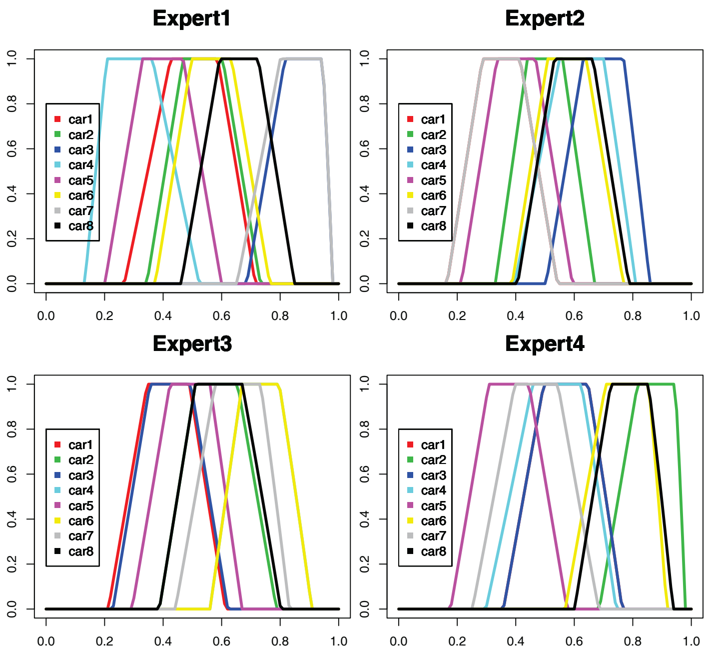

Figure 6: Overall evaluations of cars by the four experts.

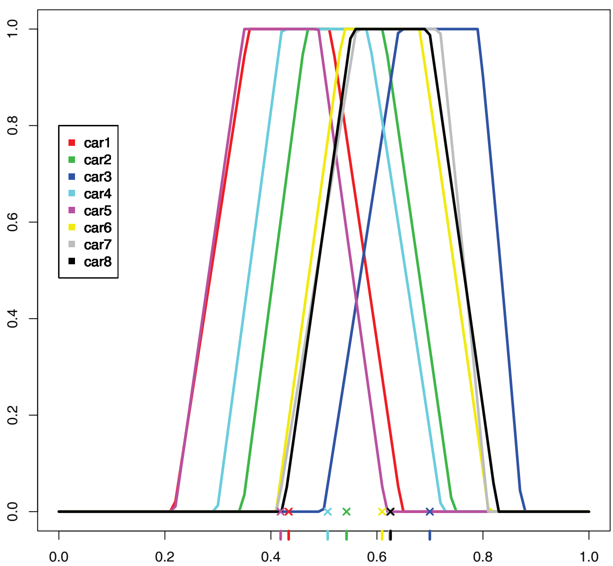

Figure 7: Group evaluations of the eight cars.

criterion by a fuzzy number. The fuzzy weighted average with fuzzy weights of criteria is then applied for the aggregation of partial evaluations of each expert. Different weights of criteria are admitted for individual experts. The experts are assigned various decision competences expressed again by fuzzy weights.

The selection of the best alternative is based on the fuzzy consensus of experts. For this purpose a set of alternatives which are good enough with respect to the most of relevant experts (that is a subset of the available alternatives) is identified. Group evaluations of the alternatives from this set are calculated by fuzzy weighted average where the competences of experts (the meanings of their linguistic labels) are used as fuzzy weights. The alternative which reflects the fuzzy consensus and has the highest group evaluation among such alternatives is chosen as the most promising one.

An example has been provided showcasing that the usage of the proposed model can result in a different recommendation of an "optimal" alternative than standard approaches to group decision-making utilizing only fuzzy weighted average with fuzzy competences. Using the proposed consensus-based model a higher level of agreement of the relevant decision makers (experts) is required for an alternative to be suggested as the best one. The proposed model might thus be an appropriate tool for situations, when disagreement of the evaluators or the diversity of evaluations is not a desired property. Alternatives that are not evaluated above a given threshold set by the definition of the order $\mathcal{K}$ specified by (13) are not considered in the final step among the most promising ones.

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