Lifts of Foliated Linear Connections to the Second Order Transverse Bundles

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Abstract

The second order transverse bundle $T^2_{\text{tr}} M$ of a foliated manifold $M$ carries a natural structure of a smooth manifold over the algebra $\mathbb{D}^2$ of truncated polynomials of degree two in one variable. Prolongations of foliated mappings to second order transverse bundles are a partial case of more general $\mathbb{D}^2$-smooth foliated mappings between second order transverse bundles. We establish necessary and sufficient conditions under which a $\mathbb{D}^2$-smooth foliated diffeomorphism between two second order transverse bundles maps the lift of a foliated linear connection into the lift of a foliated linear connection.

Key words: Foliation, transverse bundle, second order transverse bundle, projectable linear connection, Lie derivative, Weil bundle.

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1 Introduction

Transverse Weil bundle $T^A_{\text{tr}} M$ of a foliated manifold $M$ defined by a Weil algebra $A$ [7, 8] carries a natural structure of a smooth manifold over $A$ [8]. This makes it possible to apply methods of the theory of manifolds over algebras to the study of geometry of $T^A_{\text{tr}} M$. The second order transverse bundle $T^2_{\text{tr}} M$ of a foliated manifold $M$ is naturally equivalent to the Weil bundle $T^{\mathbb{D}^2}_{\text{tr}} M$ defined by the algebra $\mathbb{D}^2$ of truncated polynomials of degree two in one variable. In this paper, we study the behavior of lifts of foliated connections (lifted connections) on second order transverse bundles under $\mathbb{D}^2$-smooth diffeomorphisms preserving the lifted foliations and establish conditions, in terms of transverse
Lie derivatives, under which such a diffeomorphism maps a lifted connection into a lifted one. Another way to obtain conditions under which a $\mathbb{D}^2$-smooth diffeomorphism maps a lifted connection into a lifted one is to generalize the notion of a Lie jet with respect to a field of $A$-velocities [10].

We define the lift of a foliated connection applying to the connection object the functor $T^2_{\tau}$ which is viewed as the functor of $\mathbb{D}^2$-prolongation. Lifts of linear connections to higher order tangent bundles and to Weil bundles were introduced by A. Morimoto [5, 6]. A. P. Shirokov [1] applied theory of manifolds over algebras to the definition and study of these lifts. $\mathbb{D}^2$-smooth linear connections on second order tangent bundles studied in [2]. Applying A. Morimoto’s approach, R. Wolak [12] constructed lifts of linear connections in transverse bundles $T_{\tau}M$ to higher order transverse bundles. V. V. Vishnevskii [11] applied methods used by A. P. Shirokov and A. Morimoto to the study of lifts of projectable linear connections on manifolds fibered by a sequence of submersions.

2 $\mathbb{D}^2$-smooth structure on the second order transverse bundle

The projection $p: \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m \ni \{x^i, y^\alpha\} \mapsto \{x^i\} \in \mathbb{R}^n$, where the indices $i, j, \ldots$ and $\alpha, \beta, \ldots$ run, respectively, through the sets of values $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+m\}$, defines the model codimension $n$ foliation $\mathcal{F}_{n,m}$ on the space $\mathbb{R}^{n+m}$ representing it as a union of $m$-dimensional leaves. A diffeomorphism $f: U \ni \{x^i, y^\alpha\} \mapsto \{f^j(x^i, y^\alpha), f^\beta(x^i, y^\alpha)\} \in U'$ between open subsets $U$ and $U'$ of $\mathbb{R}^{n+m}$ is called a local automorphism of $\mathcal{F}_{n,m}$ if $\partial f^j / \partial y^\alpha = 0$. A codimension $n$ foliation $\mathcal{F}$ on an $(n+m)$-dimensional smooth manifold $M$ is given by an atlas $\mathcal{A}$ whose coordinate changes are local automorphisms of the model foliation $\mathcal{F}_{n,m}$ [4]. Charts from $\mathcal{A}$ are called foliated charts. A manifold $M$ with given foliation $\mathcal{F}$ on it is called a foliated manifold. A foliated manifold is also denoted by $(M, \mathcal{F})$. A connected open subset $U$ of a foliated manifold $M$ is called simple if the induced foliation on $U$ is generated by a submersion with connected leaves. A foliated chart $(U, h)$ is called simple if $U$ is a simple open subset of $M$. The leaf of a foliated manifold $M$ passing through a point $x$ is the maximal connected submanifold $L_x \ni x$ in $M$ defined in terms of simple foliated charts by equations of the form $x^i = x^i_0 = \text{const}$. A smooth mapping $f: M \to M'$ between two foliated manifolds $(M, \mathcal{F})$ and $(M', \mathcal{F}')$ is a foliated mapping (a morphism of foliations) if in terms of any foliated charts $(U, h)$ on $M$ and $(U', h')$ on $M'$ such that $f(U) \subset U'$ it has equations

$$x^i = f^i(x^i, y^\alpha), \quad y^\alpha = f^\alpha(x^i, y^\alpha), \quad \partial_\alpha f^i = 0. \quad (1)$$

Here and in what follows we use the following notation for partial derivatives:

$$\partial_j f^i = \partial f^i / \partial x^j, \quad \partial_\alpha f^i = \partial f^i / \partial y^\alpha, \quad \partial^2_{jk} f^i = \partial^2 f^i / \partial x^j \partial x^k, \quad \partial^2_{j\beta} f^\alpha = \partial^2 f^\alpha / \partial x^j \partial y^\beta,$$

and so on.
A foliated mapping maps leaves of \( M \) into leaves of \( M' \). If \( U \) is a simple open set, equations (1) take the form

\[
x^i = f^i(x^i), \quad y^\alpha = f^\alpha(x^i, y^\alpha).
\]

(2)

In what follows we will assume that equations of foliated mappings in question are written for simple open subsets of their domains.

A transverse 2-velocity on \( M \) at \( x \in M \) is an equivalence class of germs of smooth curves on \( M \) with respect to the following equivalence relation: two germs \( \gamma: (\mathbb{R}, 0) \to (M, x) \) and \( \gamma': (\mathbb{R}, 0) \to (M, x) \) are equivalent if and only if the 2-jets \( j^2(p \circ h \circ \gamma) \) and \( j^2(p \circ h \circ \gamma') \) coincide for any foliated chart \((U, h)\), \( x \in U \). The transverse 2-velocity defined by a germ \( \gamma \) is denoted by \( j^2_{\text{tr}} \gamma \) or \( j^2_{\text{tr}} x \gamma \). The numbers

\[
x^i = (h^i \circ \gamma)(0), \quad y^\alpha = (h^\alpha \circ \gamma)(0),
\]

\[
\dot{x}^i = d(h^i \circ \gamma)/dt|_0, \quad \ddot{x}^i = \frac{1}{2} \frac{d^2(h^i \circ \gamma)}{dt^2}|_0
\]

are the coordinates of the transverse 2-velocity \( j^2_{\text{tr}} x \gamma \) in terms of the chart \((U, h)\).

Let \( T^2_{\text{tr}} M \) denote the set of all transverse 2-velocities at \( x \in M \) and \( T^2_{\text{tr}} M = \bigcup_{x \in M} T^2_{\text{tr}} x M \) the set of all transverse 2-velocities on \( M \). \( T^2_{\text{tr}} M \) carries a structure of a smooth \((3n + m)\)-dimensional manifold fibered over \( M \). This structure is defined as follows. Let

\[
\pi^2_0: T^2_{\text{tr}} M \ni j^2_{\text{tr}} x \gamma \mapsto x \in M
\]

be the canonical projection assigning to a 2-velocity \( j^2_{\text{tr}} x \gamma \in T^2_{\text{tr}} x M \) the point \( x \in M \). A foliated chart \((U, h)\) on \( M \) induces the chart

\[
h^2: (\pi^2_0)^{-1}(U) \ni X = j^2_{\text{tr}} x \gamma \mapsto \{x^i, y^\alpha, \dot{x}^i, \ddot{x}^i\} \in \mathbb{R}^{3n + m}
\]

(4)

on \( T^2_{\text{tr}} M \). If the change of coordinates on a simple open subset of the overlapping of the domains of two charts \((U, h)\) and \((U', h')\) on \( M \) is of the form (2), then the corresponding change of the induced coordinates on \( T^2_{\text{tr}} M \) is of the form

\[
x'^i = f^i(x^i), \quad y'^\alpha = f^\alpha(x^i, y^\alpha), \quad \dot{x}'^i = (\partial_j f^i) \dot{x}^j,
\]

\[
\ddot{x}'^i = \left(\partial_j f^i\right) \ddot{x}^j + \frac{1}{2} \left(\partial^2_{jk} f^i\right) \dot{x}^j \dot{x}^k.
\]

(5)

Thus, the collection \( A^2_{\text{tr}} \) of charts of the form (4), where \( h \) runs through the atlas \( A \), is an atlas defining a structure of a smooth manifold on \( T^2_{\text{tr}} M \).

As it follows from (5), the bundle \( T^2_{\text{tr}} M \) carries a foliation \( \mathcal{F}^2_{\text{tr}} \) with basic coordinates \( x^i, \dot{x}^i, \ddot{x}^i \). We will call \( \mathcal{F}^2_{\text{tr}} \) the lifted foliation [4] and consider \( T^2_{\text{tr}} M \) as a foliated manifold with foliation \( \mathcal{F}^2_{\text{tr}} \). The projection \( \pi^2_0 \) is a morphism of foliations \((T^2_{\text{tr}} M, \mathcal{F}^2_{\text{tr}})\) and \((M, \mathcal{F})\).

The second order transverse bundle \( T^2_{\text{tr}} M \) can be viewed as the bundle \( T^{\mathbb{D}^2} M \) of transverse \( \mathbb{D}^2 \)-velocities on \( M \) [7, 8], where \( \mathbb{D}^2 \) is the algebra of truncated polynomials of degree less or equal to 2 in one variable, i.e. the three-dimensional commutative associative algebra whose elements are of the form \( a + b\varepsilon + c\varepsilon^2 \),
a, b, c ∈ ℝ, with multiplication defined by the relation ε² = 0, and so \( T^2_{tr} M \) carries a natural structure of a smooth manifold over \( \mathbb{D}^2 \). This structure can be described as follows.

On the manifold \( T^2_{tr} \mathbb{R}^{n+m} \), there arises a structure of a \( \mathbb{D}^2 \)-module naturally isomorphic to the \( \mathbb{D}^2 \)-module \( (\mathbb{D}^2)^n \oplus \mathbb{R}^m \) with the action of \( \mathbb{D}^2 \) on \( (\mathbb{D}^2)^n \oplus \mathbb{R}^m \) defined by the relation

\[
\sigma(u \oplus v) = \sigma u \oplus 0
\]

for \( \sigma = be + c\varepsilon^2 \). Coordinate chart (4) defines the mapping

\[
T^2_{tr} h: \pi^{-1} U \ni X = j^2_{\gamma} \gamma \mapsto \{ X^i = x^i + \varepsilon \hat{x}^i + \varepsilon^2 \ddot{x}^i, y^\alpha \} \in T^2_{tr} \mathbb{R}^{n+m} = (\mathbb{D}^2)^n \oplus \mathbb{R}^m .
\]

Let \( U \) be a simple open subset of \( (\mathbb{D}^2)^n \oplus \mathbb{R}^m \). An arbitrary \( \mathbb{D}^2 \)-smooth mapping \( F: U \to (\mathbb{D}^2)^n \oplus \mathbb{R}^m \) is of the form [8]

\[
X^{i'} = f^{i'}(x^i) + \varepsilon(\hat{x}^j \partial_j f^{i'} + g^{i'}(x^i))
+ \varepsilon^2 (\ddot{x}^j \partial_j f^{i'} + \frac{1}{2} \dddot{x}^j \partial^2_j f^{i'} + \hat{x}^j \partial_j g^{i'} + h^{i'}(x^i, y^\alpha)) , \quad y^{\alpha'} = f^{\alpha'}(x^i, y^\alpha) .
\] (6)

Therefore, coordinate changes (5) are \( \mathbb{D}^2 \)-smooth diffeomorphisms between open subsets of the module \( (\mathbb{D}^2)^n \oplus \mathbb{R}^m \), and \( T^2_{tr} M \) carries a structure of a smooth manifold over the algebra \( \mathbb{D}^2 \) modelled by the module \( (\mathbb{D}^2)^n \oplus \mathbb{R}^m \).

Let \( T^2_{tr} \) denote the functor which assigns to a foliated manifold its second order transverse bundle and to a foliated mapping \( f: M \to M' \) the mapping \( T^2_{tr} f: T^2_{tr} M \to T^2_{tr} M' \) defined by the composition of jets: \( T^2_{tr} f: j^2_{\gamma} \gamma \mapsto j^2_{\gamma}(f \circ \gamma) \).

In terms of local coordinates, \( T^2_{tr} f \) is of the form (5). In what follows we assume that the functor \( T^2_{tr} \) assigns to a foliated manifold \( M \) the bundle \( T^2_{tr} M \) endowed with the above described structure of a \( \mathbb{D}^2 \)-smooth manifold.

Let \( i_0: M \to T^2_{tr} M \) denote the zero section which assigns to a point \( x \in M \) the jet \( j^2_{\gamma} \gamma \) of the constant curve \( \gamma(t) = x \). We will identify the image of the zero section \( i_0(M) \subset T^2_{tr} M \) with \( M \). From (6) it follows that an arbitrary \( \mathbb{D}^2 \)-smooth mapping \( F: T^2_{tr} M \to T^2_{tr} M' \) is defined by its restriction \( f = F|_M \to T^2_{tr} M' \) to \( M \). It also follows from (6) that a \( \mathbb{D}^2 \)-smooth mapping \( F: T^2_{tr} M \to T^2_{tr} M' \) is a morphism of foliations (the functions \( h^{i'} \) in (6) do not depend on \( y^\alpha \)) if and only if \( f = F|_M \) is a morphism of foliations. This being the case, we call \( F \) a foliated \( \mathbb{D}^2 \)-smooth mapping. If a \( \mathbb{D}^2 \)-smooth mapping \( F: T^2_{tr} M \to T^2_{tr} M' \) is defined by a morphism of foliations \( f: M \to T^2_{tr} M' \), we denote it by \( f^{T^2_{tr}} \) and say that it is the \( \mathbb{D}^2 \)-prolongation of \( f \). In the case when the image of \( f \) belongs to the zero section of \( T^2_{tr} M' \), the \( \mathbb{D}^2 \)-prolongation of \( f \) coincides with the mapping \( T^2_{tr} f: T^2_{tr} M \to T^2_{tr} M' \). Let in addition \( \tilde{f} = \pi^2_0 \circ f \). The above mentioned mappings form the commutative diagram

\[
\begin{array}{ccc}
T^2_{tr} M & \xrightarrow{F = f^{T^2_{tr}}} & T^2_{tr} M' \\
\pi^2_0 \downarrow & & \downarrow \pi^2_0 \\
M & \xrightarrow{\tilde{f}} & M'.
\end{array}
\] (7)
3 Foliated linear connections and their lifts to the second order transverse bundles

With a foliated manifold \((M, \mathcal{F})\) one can associate the following fiber bundles.

1. The bundle \(P_{\mathcal{F}}^2 M\) of second order foliated frames on \(M\) whose elements are 2-jets of germs of morphisms of foliations

\[
f: (\mathbb{R}^{n+m}, 0) \ni \{u^a, v^\rho\} \mapsto M, \quad a = 1, \ldots, n, \quad \rho = n + 1, \ldots n + m.
\]

A local foliated chart \((x^i, y^\alpha)\) on \(M\) induces the chart

\[
(x^i, y^\alpha; x^i_a, x^i_{ab}; y^\alpha_a, y^\alpha_{ab}, y^\alpha_{ap}, y^\alpha_{p\rho}, y^\alpha_{p\sigma}), \quad (9)
\]

where \(x^i_a = \partial_a x^i = \partial x^i/\partial u^a, x^i_{ab} = \partial^2_{ab} x^i, y^\alpha_a = \partial_a y^\alpha = \partial y^\alpha/\partial u^a, y^\alpha_{ap} = \partial_p y^\alpha = \partial y^\alpha/\partial v^p, y^\alpha_{ab} = \partial_{ab} y^\alpha, y^\alpha_{ap} = \partial_{ap} y^\alpha, y^\alpha_{p\rho} = \partial_{p\rho} y^\alpha\). We will consider \(P_{\mathcal{F}}^2 M\) as a foliated manifold with basic coordinates \((x^i, x^i_a, x^i_{ab})\). \(P_{\mathcal{F}}^2 M\) is a principal fiber bundle over \(M\) with structure group \(G_{n,m}^2\) consisting of 2-jets of germs at zero of automorphisms of the model foliation

\[
g: (\mathbb{R}^{n+m}, 0) \ni \{u^a, v^\rho\} \mapsto \{u'^a, v'^\rho\} \in (\mathbb{R}^{n+m}, 0),
\]

where \(a = 1, \ldots, n, \quad \rho = n + 1, \ldots n + m, \quad a' = 1', \ldots, n', \quad \rho' = (n + 1)', \ldots (n + m)'.

The action \(P_{\mathcal{F}}^2 M \times G_{n,m}^2 \to P_{\mathcal{F}}^2 M\) is defined by the rule of composition of jets: \(j^2_x f \circ j^2_g = j^2_x (f \circ g)\).

2. The principal bundle \(P_{\mathcal{F}}^1 M\) of first order foliated frames on \(M\) whose elements are 1-jets of germs of morphisms of foliations (8).

3. The principal bundles \(P_{\mathcal{F}}^{11} M\) and \(P_{\mathcal{F}}^{12} M\) of first and second order transverse frames on \(M\) defined as bundles whose elements are equivalence classes of germs \(f: (\mathbb{R}^n, 0) \ni \{u^a\} \to M\) such that \(p \circ h \circ f\) is a germ of diffeomorphism for any foliated chart \((U, h)\) with respect to the following equivalence relation: two germs \(f\) and \(f'\) are equivalent if and only if the jets, respectively, of the first and the second order of \(p \circ h \circ f\) and \(p \circ h \circ f'\) coincide. A local foliated chart \((x^i, y^\alpha)\) on \(M\) induces the charts \((x^i, y^\alpha; x^i_a)\) and \((x^i, y^\alpha; x^i_a, x^i_{ab})\) on \(P_{\mathcal{F}}^{11} M\) and \(P_{\mathcal{F}}^{12} M\) respectively. There are natural projections \(P_{\mathcal{F}}^{11}: P_{\mathcal{F}}^2 M \to P_{\mathcal{F}}^1 M\) and \(P_{\mathcal{F}}^{12}: P_{\mathcal{F}}^2 M \to P_{\mathcal{F}}^{11} M\).

4. The transverse bundle (or the first order transverse bundle) \(T_{\mathcal{F}} M\) is defined as the quotient bundle of the tangent bundle \(TM\) by the distribution of tangent spaces to leaves or, equivalently, as the bundle of transverse 1-velocities on \(M\), i.e. the fiber bundle over \(M\) whose elements are equivalence classes \(j^1_{tr,x}\gamma\) of germs of smooth curves on \(M\) with respect to the equivalence relation: two germs \(\gamma: (\mathbb{R}, 0) \to (M, x)\) and \(\gamma': (\mathbb{R}, 0) \to (M, x)\) are equivalent if and only if the 1-jets \(j^1(p \circ h \circ \gamma)\) and \(j^1(p \circ h \circ \gamma')\) coincide. A foliated chart \((U, h)\) on \(M\) induces the chart \(h^2: (\pi_0^1)^{-1}(U) \ni X = j^1_{tr,x}\gamma \mapsto \{x^i, y^\alpha, \dot{x}^i\} \in \mathbb{R}^{2n+m}\) on \(T_{\mathcal{F}} M\), where \(\pi_0^1: T_{\mathcal{F}} M \ni j^1_{tr,x}\gamma \mapsto x \in M\) and the numbers \(\dot{x}^i\) are the same as in (3). The bundle \(T_{\mathcal{F}} M\) can also be obtained as the base of the projection \(\pi_0^2: T_{\mathcal{F}}^2 M \to T_{\mathcal{F}} M\) induced by the algebra epimorphism \(\pi_0^2: \mathbb{D}^2 \to \mathbb{D}\), where \(\mathbb{D}\)
is the algebra of Study dual numbers. \( T_1M \) carries a natural structure of a smooth manifold over the algebra \( \mathbb{D} \) modeled by the \( \mathbb{D} \)-module \( \mathbb{D}^n \oplus \mathbb{R}^m \).

A linear connection on \( M \) is a right invariant horizontal distribution on the first order frame bundle \( P^1M \) [1, 3] and can be viewed as a field \( \Gamma : P^2M \to \mathbb{R}^{(n+m)^3} \) of second order geometric objects on \( M \) corresponding to the representation \( G^2_{n+m} \times \mathbb{R}^{(n+m)^3} \to \mathbb{R}^{(n+m)^3} \) of the second order differential group \( G^2_{n+m} \) [1, 3] on the space \( \mathbb{R}^{(n+m)^3} \) defined as follows:

\[
\Gamma^A_{BC} = z^A_B z^A_C, \quad A, B, C = 1, \ldots n + m, \quad A', B', C' = 1', \ldots, (n + m)',
\]

where \( \Gamma^A_{BC} \) and \( \Gamma^A_{B'C'} \) are the coordinates of elements of \( \mathbb{R}^{(n+m)^3} \) and \( z^A_{_A} = \partial_A z^A_{,} \). \( z^A_{AB} = \partial_{AB} z^A_{,} \) are the coordinates of an element from \( G^2_{n+m} \) defined by a germ of diffeomorphism at zero given by equations \( z^A_{,} = z^A_{(z^A_{,})} \).

The subgroup \( G^2_{n,m} \subset G^2_{n+m} \) of 2-jets of germs of automorphisms of the model foliation leaves invariant the submanifold \( F \subset \mathbb{R}^{(n+m)^3} \) defined by the equations

\[
\Gamma^a_{\hat{b} \hat{c}} = \Gamma^a_{b c} = \Gamma^a_{\hat{b} \hat{c}} = 0
\]

and acts on \( F \) as follows:

\[
\Gamma^a_{bc} = u^a_{b c} u^a_{b c} + \Gamma^a_{b c} u^a_{b c} u^a_{a c},
\]

\[
\Gamma^a_{bc} = v^a_{b c} v^a_{b c} + \Gamma^a_{b c} v^a_{b c} v^a_{a c} + v^a_{b c},
\]

\[
\Gamma^a_{bc} = v^a_{b c} v^a_{b c} + \Gamma^a_{b c} v^a_{b c} v^a_{a c},
\]

\[
\Gamma^a_{bc} = v^a_{b c} v^a_{b c} + \Gamma^a_{b c} v^a_{b c} v^a_{a c} + v^a_{b c},
\]

\[
\Gamma^a_{bc} = v^a_{b c} v^a_{b c} + \Gamma^a_{b c} v^a_{b c} v^a_{a c},
\]

\[
\Gamma^a_{bc} = v^a_{b c} v^a_{b c} + \Gamma^a_{b c} v^a_{b c} v^a_{a c} + v^a_{b c},
\]

The manifold \( F \) is fibered over \( \mathbb{R}^3 \) with coordinates \( \Gamma^a_{bc} \), and action (10) defines the action of the differential group \( G^2_{n} \) on \( \mathbb{R}^{n^3} \) given by the first relation of (10).

Denote by \( \hat{E}(M) \) the bundle associated to \( P^2_{\text{fol}} M \) corresponding to action (10). A local foliated chart \( (x^i, y^i) \) on \( M \) induces the chart \( (x^i, y^i, \Gamma^i_{jk}, \Gamma^i_{jk}, \Gamma^i_{jk}, \Gamma^i_{jk}, \Gamma^i_{jk}) \) on \( \hat{E}(M) \). By a foliated linear connection on \( M \) we will mean a foliated section

\[
\nabla : M \to \hat{E}(M)
\]

with respect to the foliation on \( \hat{E}(M) \) with basic coordinates \( x^i, \Gamma^i_{jk} \). In terms of a simple foliated chart, such a section is given by equations

\[
\Gamma^\alpha_{jk} = \Gamma^\alpha_{jk}(x^i),
\]

\[
\Gamma^\alpha_{jk} = \Gamma^\alpha_{jk}(x^i, y^i), \quad \Gamma^\alpha_{jk} = \Gamma^\alpha_{jk}(x^i, y^i), \quad \Gamma^\alpha_{jk} = \Gamma^\alpha_{jk}(x^i, y^i).
\]
A foliated connection $\nabla$ defines a projectable connection in the transverse frame bundle $P^1_{tr}M$ with coefficients (12) in terms of simple foliated charts. A projectable connection in $P^1_{tr}M$ exists if and only if the Atiah class $a(M)$ of $M$ is zero [4]. Therefore, vanishing of the Atiah class $a(M)$ is necessary condition for existence of a foliated linear connection on $M$. This condition is also sufficient. In fact, let $\mathfrak{g}_n^1$ be the Lie algebra of the Lie group $G_n^1 \cong GL(n, \mathbb{R})$, $\mathfrak{g}_{n,m}^1$ the Lie algebra of the Lie group $G_{n,m}^1$, and let $\omega_{tr}$ be the $\mathfrak{g}_{n,m}^1$-valued connection form of a projectable connection in $P^1_{tr}M$. A local trivialization of the bundle $P^1_{fol}M$ over a domain of a foliated chart $U \subset M$ defines a local trivialization of $P^1_{tr}M$ over $U$. Along a section of $P^1_{fol}M$ over $U$ one can choose a $\mathfrak{g}_{n,m}^1$-valued connection form $\omega_U$ which projects into $\omega_{tr}$ and then extend it by right translations on $P^1_{fol}M$ over $U$. Then, using a partition of zero for $M$ over a covering $\{U_\lambda\}$ consisting of domains of foliated charts over which $P^1_{fol}M$ is trivial, one can glue such local connection forms and obtain a connection form which defines a foliated linear connection on $M$. In what follows we assume that the Atiah classes of foliated manifolds under consideration are zero. This takes place, e.g., for foliations defined by submersions.

Applying the functor $T^2_{tr}$ to the bundle $P^2_{fol}M$ with structure group $G^2_{n,m}$, we arrive at the $\mathbb{D}^2$-smooth principal bundle $T^2_{tr}P^2_{fol}M$ over $T^2_{tr}M$ with structure group $T^2_{tr}G^2_{n,m}$. A local trivialization of the transverse frame bundle $T^2_{tr}P^2_{fol}M$ gives the expressions for the action of $T^2_{tr}G^2_{n,m}$ on $T^2_{tr}F$. To write down these expressions, one should replace the first relation in (10) by

$$\overline{\Gamma}^{a}_{bc} = U^a_{ab}U^{a'}_{bc} + \overline{\Gamma}^{a'}_{b'c'}U_b^{b'}U_c^{c'}U_{a'},$$  

(15)

where all components in (15) belong to $\mathbb{D}^2$. This action leads in turn to the associated bundle $T^2_{tr}E(T^2_{tr}M)$. A local foliated chart $(x^i, y^\alpha)$ on $M$ induces the chart $(X^i, y^\alpha, \Gamma_{jk}^i, \Gamma_{\beta k}^i, \Gamma_{\gamma j}^i, \Gamma_{\gamma \beta j}^i, \Gamma_{\gamma \beta \gamma j}^i)$ on $T^2_{tr}E(T^2_{tr}M)$ with $\Gamma_{jk}^i \in \mathbb{D}^2$. Finally, the application of $T^2_{tr}$ to (11) defines a $\mathbb{D}^2$-smooth $\mathbb{D}^2$-linear connection $\nabla_{tr}$ on $T^2_{tr}M$, which will be called the lift of a foliated connection (11), or a lifted connection. If a foliated connection $\nabla$ on $M$ is given in terms of a simple foliated chart by equations (12) and (13), then to get the equation of its lift in terms of the unduced chart on $T^2_{tr}E(T^2_{tr}M)$, one should take all equations (13) and replace equations (12) by the equations

$$\overline{\Gamma}^{i}_{jk}(X^\ell) = \Gamma^{i}_{jk}(x^{\ell}) + \varepsilon \dot{x}^{\ell} \partial^i \Gamma^{\ell}_{jk} + \varepsilon^2 \left( \ddot{x}^{\ell} \partial^i \Gamma^{\ell}_{jk} + \frac{1}{2} \dot{x}^{\ell} \partial^i \partial^\ell \Gamma^{\ell}_{jk} \right).$$  

(16)

Let now $M$ and $M'$ be two isomorphic foliated manifolds and $F: T^2_{tr}M \to T^2_{tr}M'$ a foliated $\mathbb{D}^2$-smooth diffeomorphism. Our aim is to find conditions under which a foliated $\mathbb{D}^2$-smooth diffeomorphism $F$ maps the lift of a given foliated connection on $T^2_{tr}M$ into a lifted connection on $T^2_{tr}M'$.

Consider diagram (7) for a foliated $\mathbb{D}^2$-smooth diffeomorphism $F$. It is obvious that the prolongation $T^2_{tr}\tilde{f}$ of an isomorphism of foliations $\tilde{f}: M \to M'$
maps the lift $T^2_1\nabla$ of any foliated connection $\nabla$ into the lift of the image of $\nabla$ under $\tilde{f}$. Hence $F$ maps the lift $T^2_1\nabla$ of a foliated connection $\nabla$ into a lifted connection $T^2_1\nabla'$ if and only if the composition $T^2_1(\tilde{f}^{-1}) \circ F$ maps $T^2_1\nabla$ into itself. This composition is the $D^2$-prolongation of the section $\varphi = T^2(\tilde{f}^{-1}) \circ F|_M : M \to T^2_1M$. In terms of local charts, the section $\varphi$ and the $D^2$-diffeomorphism $T^2(\tilde{f}^{-1}) \circ F = \varphi_{D^2}$ are given, respectively, by equations of the form $X'^i = x^i + \varepsilon g'(x^k) + \varepsilon^2 h'(x^k)$, $y'\alpha = y\alpha$ and

$$X'^i = x^i + \varepsilon'(x^i + g^i(x^k)) + \varepsilon^2 (x^i + \dot{x}^i + \partial_j g^i + h^i(x^k)), \quad y'\alpha = y\alpha. \quad (17)$$

**Note 1** As was mentioned above, the first order transverse bundle $T_{tr}M$ is the base of the projection $\pi^2_1 : T^2_1M \to T_{tr}M$ corresponding to the algebra epimorphism $\pi^2_1 : D^2 \to D$, where the algebra of dual numbers is viewed as the quotient algebra of $D^2$ by the ideal $\varepsilon^2 D^2$. Applying this epimorphism to the relations in the above discussion, we obtain the respective formulas for the bundle $T_{tr}M$. To write down these formulas, one should reject in formulas for $T^2_1M$ the terms containing $\varepsilon^2$.

In accordance with Note 1 made above, we apply first the $D$-prolongation $g^D : T_{tr}M \to T_{tr}M$ of the section $g = \pi^2_1 \circ \varphi : M \to T_{tr}M$ to the connection object

$$\Gamma^1_{jk}(x^\ell) = \Gamma^i_{jk}(x^\ell) + \varepsilon \dot{x}^\ell \partial_\ell \Gamma^i_{jk}. \quad (15)$$

Using formulas similar to (15) in which $U^a_\alpha$ are replaced by $\partial X'^i / \partial X^k = \partial X'^i / \partial x^k = \delta^i_k + \varepsilon \partial_k g^i$ and $U^a_\alpha$ by $\partial^2 X'^i / \partial X^k \partial X^j = \varepsilon g^i_{jk}$, we obtain the following formulas for this image:

$$\Gamma^j_{jk}(x^\ell) + \varepsilon (\dot{x}^\ell \partial_\ell \Gamma^i_{jk} + \partial^2_{jk} g^i + g^i \partial_\ell \Gamma^j_{jk} - \Gamma^i_{jk} \partial_\ell g^i + \Gamma^i_{jk} \partial_j g^\ell + \Gamma^i_{jk} \partial_k g^\ell). \quad (18)$$

The formulas

$$\partial^2_{jk} g^i + g^i \partial_\ell \Gamma^j_{jk} - \Gamma^i_{jk} \partial_\ell g^i + \Gamma^i_{jk} \partial_j g^\ell + \Gamma^i_{jk} \partial_k g^\ell \quad (19)$$

are the coordinate expression for a projectable section of the tensor bundle $T^2_{tr}M$ of type $(1, 2)$ associated to the vector bundle $T_{tr}M$. We will denote it by $\mathcal{L}_g \Gamma$ and call the Lie derivative of the connection object of the foliated connection $\nabla$ on $T_{tr}M$ with respect to a projectable section $g$ of $T_{tr}M$. The Lie derivative (19) can be defined pointwise as the inverse image of the Lie derivative with respect to the vector field $g^i(x^\ell)$ of the connection object $\Gamma^i_{jk}(x^\ell)$ of the linear connection given on a local quotient manifold of $M$ [4] relative to the foliation (within a simple foliated domain). Thus, vanishing of the Lie derivative $\mathcal{L}_g \Gamma$ is a necessary condition for the image of $T^2_{tr} \nabla$ to be a lifted connection.

It is a matter of direct verification that a projectable section $\varphi : M \to T^2_{tr}M$ given locally by equations (17) defines in addition a projectable section $u : M \to T_{tr}M$ given locally by the equations $\dot{x}^i = h^i - \frac{1}{2} g^i_{kj} \partial_k g^j$. We will call the two sections $g$ and $u$ of $T_{tr}M$ the sections associated to the diffeomorphism $F : T^2_{tr}M \to T^2_{tr}M'$ in question.
**Theorem 1** Let $M$ and $M'$ be two isomorphic foliated manifolds and $\nabla$ a foliated linear connection on $M$ with connection object $\Gamma$ (12), (13). A foliated $\mathbb{D}^2$-smooth diffeomorphism $F: T^2_{tr}M \rightarrow T^2_{tr}M'$ maps the lift $T^2_{tr}\nabla$ of $\nabla$ to $T^2_{tr}M$ into a lifted connection on $T^2_{tr}M'$ if and only if

$$L_g\Gamma = L_u\Gamma = 0,$$

where $g$ and $u$ are the two projectable sections of $T_{tr}M$ associated to $F$.

**Proof** A direct verification shows that a projectable section $g: M \rightarrow T_{tr}M$ with local coordinate expression $x^i = g^i(x^k)$ defines a projectable section $\tilde{g}: M \rightarrow T^2_{tr}M$ with local coordinate expression $\tilde{x}^i = g^i(x^k), \dot{\tilde{x}}^i = \frac{1}{2}\hat{g}^k \partial_k g^i$. We show next that if $L_g\Gamma = 0$, then the prolongation $\tilde{g}^{\mathbb{D}^2}: T^2_{tr}M \rightarrow T^2_{tr}M$ defined by diagram (7) maps the connection $T^2_{tr}\nabla$ into itself. Using a partition of zero for $M$ over a covering $\{U_\lambda\}$ of $M$ consisting of domains of simple foliated charts, one can glue vector fields $\tilde{g}_\lambda$ which are defined on $U_\lambda$ and are projected under the mapping $\pi: TM \rightarrow T_{tr}M$ into the restrictions $g|U_\lambda$ of the section $g: M \rightarrow T_{tr}M$ to $U_\lambda$ and obtain, as a result, a vector field $\tilde{g}$ on $M$ which is projected by $\pi$ into the section $g$. In terms of a local foliated chart, the vector field $\tilde{g}$ is given by equations $\{g^i(x^k), g^\alpha(x^k, y^\beta)\}$. Applying the functor $T_{tr}$ to the vector field $\tilde{g}$ and the section $g$, one obtains a $\mathbb{D}^2$-smooth vector field $\tilde{G} = T_{tr}\tilde{g}$ on $T^2_{tr}M$ and a projectable section $G$ of the transverse bundle of $T^2_{tr}M$ with respect to the lifted foliation. The functor $T_{tr}$ applied to the relation $L_g\Gamma = 0$ gives $L_G T_{tr}\Gamma = 0$, and the vector field $\tilde{G}$ generates a local $\mathbb{D}^2$-smooth one-parameter group $\Psi = \{\Psi_T(X)\}$, $T = t + \dot{t} + \ddot{t}^2$, $X \in T^2_{tr}M$ of transformations of $T^2_{tr}M$ which transforms the connection $T^2_{tr}\nabla$ into lifted connections. We also have $\Psi = T_{tr}\psi$, where $\psi = \{\psi_t(x)\}$ is the local one-parameter group of transformations of $M$ generated by the vector field $\tilde{g}$. If, in terms of a simple foliated chart, $\psi$ is given by equations $\psi^i(x^k, t), \psi^\alpha(x^k, y^\beta, t)$, then $\Psi$ has equations

$$\begin{align*}
\Psi^i(X^k, T) &= \psi^i(x^k, t) + \varepsilon \left( \dot{x}^k \partial_k \psi^i + i \partial_t \psi^i \right) \\
+ \varepsilon^2 \left( \dot{x}^k \partial_k \psi^i + i \partial_t \psi^i + \frac{1}{2} \dot{x}^k \dot{x}^j \partial_{kj}^2 \psi^i + \frac{1}{2} \left( i \right)^2 \partial_{ltt}^2 \psi^i + \dot{x}^k i \partial_{kt}^2 \psi^i \right), \psi^\alpha(x^k, y^\beta, t).
\end{align*}$$

(20)

The $\mathbb{D}^2$-valued parameter $T$ is equivalent to the three independent $\mathbb{R}$-valued parameters $t, \dot{t}, \ddot{t}$. If a transformation $\psi_{t_0}(x)$ is defined for some $t_0 \in M$, then the transformation $\Psi_T(X)$ is defined for all $T = t_0 + \dot{t} + \ddot{t}^2$ and $X \in (\pi^2_0)^{-1}(x)$. Letting $t = \dot{t} = 0, \ddot{t} = 1$ in (20), we obtain the transformation $\tilde{g}^{\mathbb{D}^2}: T^2_{tr}M \rightarrow T^2_{tr}M$.

Let $\iota^2_1: T_{tr}M \rightarrow T^2_{tr}M$ denote the canonical embedding given in terms of foliated charts by equations $\{x^i, y^\alpha, \dot{x}^i\} \mapsto \{x^i, y^\alpha, 0, \dot{x}^i\}$. The composition $\iota^2_1 \circ u$ is a section of $T^2_{tr}M$, and the $\mathbb{D}^2$-diffeomorphism $\varphi^{\mathbb{D}^2}$ can be represented as the composition $\varphi^{\mathbb{D}^2} = (\iota^2_1 \circ u)^{\mathbb{D}^2} \circ \tilde{g}^{\mathbb{D}^2}$. It remains to apply $(\iota^2_1 \circ u)^{\mathbb{D}^2}$ to the connection object (16). Using again formulas similar to (15) where $U_{\alpha}^{\alpha'}$ are replaced by $\partial X^i / \partial X^k = \partial X^i / \partial x^k = \delta_k^i + \varepsilon^2 \partial_k u^i$ and $U_{\alpha}^{\alpha'}$ by $\partial^2 X^i / \partial X^k \partial X^j = \varepsilon^2 \partial_{jk}^2 u^i$, where
we obtain the following formulas for the image:

\[ \tilde{\Gamma}^i_{jk}(X^\ell) + \varepsilon^2 \left( \partial^2_{jk} u^i + u^\ell \partial_\ell \Gamma^i_{jk} - \Gamma^\ell_{jk} \partial_\ell u^i + \Gamma^i_{\ell k} \partial_j u^\ell + \Gamma^i_{j \ell} \partial_k u^\ell \right), \]

which proves the theorem. □

References


