Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **54**, 2 (2015) 71–82

Some Applications of new Modified q-Szász–Mirakyan Operators

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(Received May 19, 2015)

Abstract

This paper we introducing a new sequence of positive q-integral new Modified q-Szász-Mirakyan Operators. We show that it is a weighted approximation process in the polynomial space of continuous functions defined on $[0, \infty)$. Weighted statistical approximation theorem, Korovkin-type theorems for fuzzy continuous functions, an estimate for the rate of convergence and some properties are also obtained for these operators.

Key words: q-analogue Baskakov operators, q-Durrmeyer operators, rate of convergence, weighted approximation.

2010 Mathematics Subject Classification: 41A25, 41A35

1 Introduction

The approximation of functions by linear positive operators is an important research topic in general mathematics and it also provides powerful tools to application areas such as computer-aided geometric design, numerical analysis, and solutions of differential equations. q-Calculus is a generalization of any subjects, such as hyper geometric series, complex analysis and particle physics. Currently it continues being an important subject of study. It has been shown that linear positive operators constructed by q-numbers are quite effective as far as the rate of convergence is concerned and we can have some unexpected results, which are not observed for classical case. This type of construction was first used to generate Bernstein operators. In 1987, Lupas defined a q-analogue of Bernstein operators and studied some approximation properties of them. In 1997, Phillips introduced another generalization of Bernstein operators based on the q-integers called q-Bernstein operators. Research results show that q-Bernstein operators possess good convergence and approximation properties in $C[0,1]$. Aral [1] introduced the q-Szász–Mirakyan operators. Aral and Gupta [1], [14] extended the study and established some approximation properties for q-Szász–Mirakyan operators. In the last decade some new generalizations of well known positive linear operators, based on q-integers were introduced and studied by several authors. For instance q-Meyer–König and Zeller operators studied by Trif., Dogru and Duman [12] and Gupta [2] etc. In 20011, Aral and Gupta [1], [14] introduced a q-generalization of the classical Baskakov operators. In 2012, Honey Sharma [4],[5] introduced the q-Durrmeyer type operators. In this paper motivated by Honey Sharma we introduced a q-analogue of the q-Durrmeyer operators and we study better rate of convergence.

Definition 1 For any fixed real number $q > 0$ and $k \in N$, the q-integers is defined by

$$[k]_q = \begin{cases} k, & \text{if } q = 1, \\ 1 + q + q^1 + q^2 + \ldots + q^{k-1}, & \text{if } q \neq 1. \end{cases}$$

In this way for a real number n we may write $[n]_q = \frac{1-q^n}{1-q}$; $q \neq 1$.

Definition 2 The q-factorial is defined by

$$[k]_q! = \begin{cases} 1, & \text{if } k = 0, \\ [1]_q \cdot [2]_q \cdot \ldots \cdot [k]_q, & \text{if } k = 1, 2, \ldots \end{cases}$$

Definition 3 For any number $k \in (0, n)$, the q-binomial coefficient is defined by

$$\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Aral and Gupta [1] introduced a q-generalization of the classical Baskakov operators. For $f \in C[0,\infty)$, $q \in (0,1)$ and each positive integer n, the operators is defined as

$$(B_{n,q}f)(x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k}_q q^{\frac{k(k-1)}{2}} \frac{(kx)^k}{(1+qx)_q^{n+k}} f\left(\frac{[k]_q}{q^{q-1}[n]_q}\right). \qquad (1.1)$$

For $q = 1$ above operators becomes classical Baskakov operators.

N. Deo et. al. [9] introduced new version of Bernstein–Durrmeyer-type operators defined as: for $f \in CI_n$ where $I_n = [0, \frac{n}{n+1}]$

$$(M_{n,q}f)(x) = n\left(1 + \frac{1}{n}\right)^2 \sum_{k=0}^{n} p_{n,k}(x) \int_0^{\frac{n}{n+1}} p_{n,k}(t) f(t) dt \qquad (1.2)$$

where,

$$p_{n,k}(x) = \left(1 + \frac{1}{n}\right)^n \binom{n}{k} x^k \left(\frac{n}{n+1} - x\right)^{n-k}$$

and established some approximation results on it.

H. Sharma [4] introduced the following q-Durrmeyer type operators defined as: for $f \in CI_{n,q}$ where $I_{n,q} = \left[0, \frac{[n]_q}{[n+1]_q}\right]$

$$(M^{\star}_{n,q}f)(x) = \frac{[n+1]_q^2}{[n]_q} \sum_{k=0}^{n} q^{-k} p^{\star}_{n,k}(q;x) \int_0^{\frac{[n]_q}{[n+1]_q}} p^{\star}_{n,k}(q;qt)f(t)d_qt \tag{1.3}$$

where

$$p^{\star}_{n,k}(q;x) = \binom{n}{k}_q \left(\frac{[n+1]_q}{[n]_q}x\right)^k \left(1 - \frac{[n+1]_q}{[n]_q}x\right)_q^{n-k}$$

and established some approximation results on it.

In this paper motivated by H. Sharma [4], [5], and N. Deo [8] we introduce a q-analogue of the q-Szász–Mirakyan type operators defined as: for $f \in CI_{n,q}$

$$(S_{n,q}f)(x) = \frac{[n+1]_q^2}{[n]_q E_q([n]_q x)} \sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}} \frac{([n]_q x)^k}{[k]_q!} \int_0^{\frac{[n]_q}{[n+1]_q}} p_{n,k}(q;qt)f(t)d_qt. \tag{1.4}$$

Again we modified above equations for $p \geqq 0$ so, we get

$$(S_{n,q,p}f)(x) = $$
$$= \frac{[n+1]_q^2}{[n]_q E_q([n+p]_q x)} \sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}} \frac{([n+p]_q x)^k}{[k]_q!} \int_0^{\frac{[n]_q}{[n+1]_q}} p_{n,k}(q;qt)f(t)\, d_qt. \tag{1.5}$$

H. S. Kasana et. el. [3] obtained a sequence of modified Szász operators for integrable function on $[0,\infty)$ defined as:

$$(M_{n,x}f)(x) \equiv M_{n,x}(f(y);t) = n \sum_{k=0}^{\infty} b_{n,k}(t) \int_0^{\infty} b_{n,k}(y) f(x+y)\, dy \tag{1.6}$$

where, x and t belong to $[0,\infty)$ and x is fixed.

In this paper motivated by H. S. Kasana and H. Sharma, we introduce a q-analogue of the q-Szász–Mirakyan type operators defined as: for $f \in CI_{n,q}$;

$$(S^{\star}_{n,q,x,p}f)(t) =$$
$$= \frac{[n+1]_q^2}{[n]_q E_q([n+p]_q t)} \sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}} \frac{([n+p]_q t)^k}{[k]_q!} \int_0^{\frac{[n]_q}{[n+1]_q}} p_{n,k}(q;qy) f(x+y)\, d_qy \tag{1.7}$$

where, x and t belong to $I_{n,q}$ and x is fixed.

The aim of this paper is to study the approximation properties of a new generalization of the q-Szász-Mirakyan operators based on q-integers. We estimate moments for these operators. Also, we study asymptotic formula for these operators. Finally, we give better error estimations for operators (1.5) and (1.7).

2 Estimation of moments

Theorem 1 *Let the sequence of positive linear operators* $(S_{n,q,p}f)(x)$ *defined by* (1.5). *For all* $n \in N$; $q \in (0,1)$, $p \geqq 0$; $f \in CI_{n,q}$; $x \in I_{n,q}$, *we get*

$$(S_{n,q,p}1)(x) = 1 \tag{2.1}$$

$$(S_{n,q,p}t)(x) = \frac{[n]_q([n+p]_q x+1)}{[n+2]_q[n+1]_q} \tag{2.2}$$

$$(S_{n,q,p}t^2)(x) = \frac{(1+q)[n]_q^2 + q(1+q)^2 x[n+p]_q[n]_q^2 + q^3x^2[n+p]_q^2[n]_q^2}{[n+3]_q[n+2]_q[n+1]_q^2}. \tag{2.3}$$

Proof We put $f(t) = 1$ in the operators $S_{n,q,p}$, we get

$$\begin{aligned}
&(S_{n,q,p}1)(x) = \\
&= \frac{[n+1]_q^2}{[n]_q E_q([n+p]_q x)} \sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}} \frac{([n+p]_q x)^k}{[k]_q!} \int_0^{\frac{[n]_q}{[n+1]_q}} p_{n,k}(q;qt) 1 d_q t \\
&= \frac{[n+1]_q^2}{[n]_q E_q([n+p]_q x)} \sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}} \frac{([n+p]_q x)^k}{[k]_q!} \frac{[n]_q}{[n+1]_q} q^k \frac{[n]_q!}{[n+1]_q!} \\
&= \frac{1}{E_q([n+p]_q x)} \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{([n+p]_q x)^k}{[k]_q!} = 1.
\end{aligned}$$

Again we put $f(t) = t$ in the operators $S_{n,q,p}$, we get

$$\begin{aligned}
&(S_{n,q,p}t)(x) = \\
&= \frac{[n+1]_q^2}{[n]_q E_q([n+p]_q x)} \sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}} \frac{([n+p]_q x)^k}{[k]_q!} \int_0^{\frac{[n]_q}{[n+1]_q}} p_{n,k}(q;qt) t d_q t \\
&= \frac{[n+1]_q^2}{[n]_q E_q([n+p]_q x)} \sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}} \frac{([n+p]_q x)^k}{[k]_q!} \frac{[n]_q^2}{[n+1]_q^2} q^k \frac{[n]_q![k+1]_q!}{[k]_q![n+2]_q!} \\
&= \frac{[n]_q}{[n+2]_q[n+1]_q E_q([n+p]_q x)} \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{([n+p]_q x)^k}{[k]_q!} [k+1]_q \\
&= \frac{[n]_q([n+p]_q x+1)}{[n+2]_q[n+1]_q}.
\end{aligned}$$

Similarly, we put $f(t) = t^2$ in the operators $S_{n,q,p}$, we get

$$\begin{aligned}
&(S_{n,q,p}t^2)(x) = \\
&= \frac{[n+1]_q^2}{[n]_q E_q([n+p]_q x)} \sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}} \frac{([n+p]_q x)^k}{[k]_q!} \int_0^{\frac{[n]_q}{[n+1]_q}} p_{n,k}(q;qt) t^2 d_q t
\end{aligned}$$

$$= \frac{[n+1]_q^2}{[n]_q E_q([n+p]_q x)} \sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}} \frac{([n+p]_q x)^k}{[k]_q!} \frac{[n]_q^3}{[n+1]_q^3} q^k \frac{[n]_q! [k+2]_q!}{[k]_q! [n+3]_q!}$$
$$= \frac{[n]_q^2}{[n+3]_q [n+2]_q [n+1]_q^2 E_q([n+p]_q x)} \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{([n+p]_q x)^k}{[k]_q!} [k+1]_q [k+2]_q$$
$$= \frac{[n]_q^2 \Big[1 + q + q(1+q)^2 [n+p]_q x}{[n+3]_q [n+2]_q [n+1]_q^2}$$
$$+ \frac{q^4 \left(\left(\frac{[n+1]_q [n+p]_q x^2}{q} + [n+p]_q x \right) - [n+p]_q x \right) \Big]}{[n+3]_q [n+2]_q [n+1]_q^2}$$
$$= \frac{(1+q)[n]_q^2 + q(1+q)^2 x [n+p]_q [n]_q^2 + q^3 x^2 [n+p]_q^2 [n]_q^2}{[n+3]_q [n+2]_q [n+1]_q^2}.$$

This completes the proof of the theorem. □

Lemma 1 *For the special case $q = 1$ we have*

$$(S_{n,1,p} 1)(x) = 1;$$
$$(S_{n,1,p} t)(x) = \frac{n(n+p)x + n}{(n+2)(n+1)};$$
$$(S_{n,1,p} t^2)(x) = \frac{n^2[(n+p)^2 x^2 + 4(n+p)x + 2]}{(n+3)(n+2)(n+1)^2}.$$

Lemma 2 *The sequence of positive linear operators $S_{n,q,p}$, we get following central moments: let $\phi^i = (t-x)^i$, $i = 1, 2, \dots$*

$$(S_{n,q,p} \phi^1)(x) = (S_{n,q,p} t)(x) - x (S_{n,q,p} 1)(x)$$
$$= \frac{[n]_q([n+p]_q x + 1)}{[n+2]_q [n+1]_q} - x \cdot 1 = \frac{[n]_q(1 + (p-3)x) - 2x}{[n+2]_q [n+1]_q};$$

$$(S_{n,q,p} \phi^2)(x) = (S_{n,q,p} t^2)(x) - 2x (S_{n,q,p} t)(x) + x^2 (S_{n,q,p} 1)(x)$$
$$= \frac{(1+q)[n]_q^2 + q(1+q)^2 x [n+p]_q [n]_q^2 + q^3 x^2 [n+p]_q^2 [n]_q^2}{[n+3]_q [n+2]_q [n+1]_q^2}$$
$$- 2x \frac{[n]_q([n+p]_q x + 1)}{[n+2]_q [n+1]_q} + x^2 \cdot 1$$
$$= x^2 \left(1 - \frac{2[n]_q [n+p]_q}{[n+2]_q [n+1]_q} + \frac{q^3 [n]_q^2 [n+p]_q^2}{[n+3]_q [n+2]_q [n+1]_q} \right)$$
$$+ x \left(\frac{q(1+q)^2 [n]_q^2 [n+p]_q}{[n+3]_q [n+2]_q [n+1]_q^2} - \frac{2[n]_q}{[n+2]_q [n+1]_q} \right) + \frac{(1+q)[n]_q^2}{[n+3]_q [n+2]_q [n+1]_q^2}.$$

Lemma 3 *For the special case $q = 1$ we have the following central moment*

$$(S_{n,q,p}\phi^1)(x) = \frac{n(1 + (p - 3)x) - 2x}{(n + 2)(n + 1)}$$

$$(S_{n,q,p}\phi^2)(x) =$$

$$= \frac{n^3[2x - x^2] + n^2[(p^2 + 11)x^2 + (4p - 8)x + 2] + n[(17 - 12p)x^2 - 6x] + 6x^2}{(n + 3)(n + 2)(n + 1)^2}.$$

Lemma 4 *For the special case $q = 1; p = 0$ we have*

$$\begin{aligned} (S_{n,1,p}1)(x) &= 1; \\ (S_{n,1,p}t)(x) &= \frac{n^2x + n}{(n + 2)(n + 1)}; \\ (S_{n,1,p}t^2)(x) &= \frac{[n^4x^2 + 4n^3x + 2n^2]}{(n + 3)(n + 2)(n + 1)^2}. \end{aligned}$$

3 Weighted statistical approximation theorem

The aim of this section is to use statistical convergence to study Korovkin-type approximation of a function f by means of sequence of positive linear operators from a weighted space into a weighted subspace.

Theorem 2 *Let a sequence $(q_n)_n$; $q_n \in (0, 1)$ such that $st - \lim_{n\to\infty} q_n = 1$ and let the sequence of positive linear operators $S_{n,q_n,p}$; $n \in N$ be defined by* (1.5)*. Then for any compact set $x \in I_n$ and for non-decreasing function $f \in C_{\rho_0} I_n$, we get*

$$st - \lim_{n\to\infty} \|(S_{n,q_n,p}f)(x) - f(x)\|_{\rho_\alpha} = 0; \ \alpha > 0. \tag{3.1}$$

Proof The weight functions $\rho_0(x)$ and weighted subspace $C_{\rho_0} I_n$ defined by; for $x \in I_n$; $\alpha > 0$, $\rho_0(x) = 1 + x^2$; $\rho_\alpha(x) = 1 + x^{2+\alpha}$ and $f \in C_{\rho_0} I_n = f \in B_\rho I_n$ such that f continuous on I_n with norm $\|f\|_\rho = \sup_{x\in I_n} \frac{\|f(x)\|}{\rho(x)}$; here $B_\rho(I_n)$; $C_\rho(I_n)$ are Banach Space. By using Theorem 1, we get

$$st - \lim_{n\to\infty} \|(S_{n,q_n,p}1)(x) - 1\|_{\rho_0} = 0 \tag{3.2}$$

Since,

$$\frac{|(S_{n,q_n,p}t)(x) - x|}{1 + x^2} = \frac{|\frac{[n]_q([n+p]_q x+1)}{[n+2]_q[n+1]_q} - x|}{1 + x^2} \leq \frac{1}{[n]_{q_n}}$$

and $st - \lim_{n\to\infty} q_n = 1$ this implies $st - \lim_{n\to\infty} \frac{1}{[n]_{q_n}} = 0$, we get

$$st - \lim_{n\to\infty} \|(S_{n,q_n,p}t)(x) - x\|_{\rho_0} = 0 \tag{3.3}$$

Again since,

$$\frac{|(S_{n,q_n,p}t^2)(x) - x^2|}{1 + x^2} \leq \frac{1 + q_n}{[n]^2_{q_n}} + \frac{q_n(1 + q_n)^2}{[n]_{q_n}} + \frac{q^3}{[n]_{q_n}},$$

we get

$$st-\lim_{n\to\infty}\|(S_{n,q_n,p}t^2)(x)-x^2\|_{\rho_0}=0. \tag{3.4}$$

By using A-statistical convergence theorem given by Duman and Orhan [11], here we let $A=C_1$ equation (3.2), (3.3) and (3.4), we get

$$st-\lim_{n\to\infty}\|(S_{n,q_n,p}t^k)(x)-x^k\|_{\rho_0}=0$$

for $k=0,1,2$ if and only if

$$st-\lim_{n\to\infty}\|(S_{n,q_n,p}f)(x)-f(x)\|_{\rho_\alpha}=0;\ \alpha>0.$$

This completes the proof of the theorem. □

Theorem 3 *Let a sequence $(q_n)_n; q_n\in(0,1)$ such that $st-\lim_{n\to\infty}q_n=1$ and let the sequence of positive linear operators $S^\star_{n,q_n,p}$; $n\in N$ be defined by (1.7). Then for any compact set x; $t\in I_n$ and for non-decreasing function $f\in C_0 I_n$, we get*

$$st-\lim_{n\to\infty}\|(S^\star_{n,q_n,p}f)(t)-f(t)\|_{\rho_\alpha}=0;\alpha>0.$$

Proof The proof of the theorem is analogous as Theorem 2. □

Theorem 4 *Let a sequence $(q_n)_n; q_n\in(0,1)$ such that $st-\lim_{n\to\infty}q_n=1$ and let the sequence of positive linear operators $S_{n,q_n,p}$; $n\in N$ be defined by (1.5). Then for any compact set $x\in I_n$ and for non-decreasing function $f\in C_{x^2}I_n$, we get*

$$\lim_{n\to\infty}\|(S_{n,q_n,p}f)(x)-f(x)\|_{x^2}=0.$$

Proof To prove the theorem we use modulus of continuity of f on closed interval I_n is given by

$$\omega(f,\delta)=\sup_{|t-x|\le\delta}\sup_{t\in I_n}|f(t)-f(x)|.$$

We see that $f\in C_{x^2}I_n$, the modulus of continuity $\omega(f,\delta)$ tends to zero.

$$\|(S_{n,q_n,p}t)(x)-x\|_{x^2}\le$$

$$\le\frac{[n+p]_{q_n}-q[n+p-1]_{q_n}}{[n+2]_{q_n}[n+1]_{q_n}}\sup_{t\in I_n}\frac{x}{1+x^2}+\frac{[n]_{q_n}}{[n+2]_{q_n}[n+1]_{q_n}}\sup_{t\in I_n}\frac{1}{1+x^2}$$

we get,

$$\lim_{n\to\infty}\|(S_{n,q_n,p}t)(x)-x\|_{x^2}=0. \tag{3.5}$$

Again

$$\lim_{n\to\infty}\|(S_{n,q_n,p}t^2)(x)-x^2\|_{x^2}\le\left(\frac{q^3[n]_q^2[n+p]_q^2}{[n+3]_q[n+2]_q[n+1]_q}-1\right)\sup_{t\in I_n}\frac{x^2}{1+x^2}+...$$

$$\frac{q(1+q)^2x[n]_{q_n}^2[n+p]_{q_n}}{[n+3]_{q_n}[n+2]_{q_n}[n+1]_{q_n}^2}\sup_{t\in I_n}\frac{x}{1+x^2}+\frac{(1+q)[n]_{q_n}^2}{[n+3]_{q_n}[n+2]_{q_n}[n+1]_{q_n}^2}\sup_{t\in I_n}\frac{1}{1+x^2}$$

we get,

$$\lim_{n\to\infty}\|(S_{n,q_n,p}t^2)(x)-x^2\|_{x^2}=0. \tag{3.6}$$

By equation, (3.5) and (3.6) , we get

$$\lim_{n\to\infty}\|(S_{n,q_n,p}t^k)(x)-x^k\|_{\rho_0}=0$$

for $k=0,1,2$ if and only if $\lim_{n\to\infty}\|(S_{n,q_n,p}f)(x)-f(x)\|_{x^2}=0$. This completes the proof of the theorem. □

Theorem 5 *Let a sequence $(q_n)_n; q_n\in(0,1)$ such that $st-\lim_{n\to\infty}q_n=1$ and let the sequence of positive linear operators $S^\star_{n,q_n,p}; n\in N$ be defined by (1.7). Then for any compact set $x; t\in I_n$ and for non-decreasing function $f\in C_{t^2}I_n$, we get*

$$\lim_{n\to\infty}\|(S^\star_{n,q_n,p}f)(t)-f(t)\|_{t^2}=0.$$

Proof The proof of the theorem is analogous as theorem 4. □

4 Korovkin-type theorems for fuzzy continuous functions

In this section we mention some important definitions given by M. Burgin [6].

Definition 4 A number a is called an *r-limit* of a sequence S (it is denoted by $a=r-limS$) if for any $\epsilon\in R$, the inequality $|a-a_i|<r+\epsilon$ is valid for almost all a_i, i.e. there is such n that for any $i>n$, we have $|a-a_i|<r+\epsilon$.

Definition 5 A sequence S that has an r-limit is called *r-convergent* and it is said that S *r-converges* to its r-limit a. It is denoted by $S\to ra$.

Definition 6 A function $f\colon R\to R$ is called *r-continuous* in $X\subset R$ if $\gamma(f,X)\le r$ and is called fuzzy continuous in X if $\gamma(f,X)\le\infty$ where $\gamma(f,X)$ defined as,

$$\gamma(f,X)\ge\inf\{\sup\{|f(x)-g(x)|\colon x\in X\}\colon g(x)\in C(X)\}.$$

For example the functions $f(x)=x^n$ when $x\in[n,n+1)$, $n\in Z$ and $g(x)=[x]^n$ are fuzzy continuous in each finite interval of the real line R, but they are not continuous in any interval with the length larger than 1. To define the Riemann integral for a continuous function $f(x)$, step functions are utilized. If the integral of $f(x)$ exists, then any such step function is fuzzy continuous.

Theorem 6 *Let a sequence $(q_n)_n$; $q_n\in(0,1)$ such that $r-\lim_{n\to\infty}q_n=1$ and let the sequence of positive linear operators $S_{n,q_n,p}$; $n\in N$ be defined by (1.5). If*

$$r_i-\lim_{n\to\infty}\|(S_{n,q_n,p}e_i)(x)-e_i\|=0$$

for $i = 0, 1, 2$. *Then for non-decreasing function* $f \in C(I_n)$, *we get*

$$r - \lim_{n\to\infty} \|(S_{n,q_n,p}f)(x) - f\| = 0$$

where, r *is any real number such that* $r \geq K_3(r_0 + r_1 + r_2)$ *for some* $K_3 > 0$.

Proof Let the functions e_i defined as; $e_i(x) = t^i$ far all $x \in I_n$. Now, for each $\epsilon > 0$, there corresponds $\delta > 0$ such that $|\lambda(t-x)| \leq \epsilon$ whenever $|t - x| \leq \delta$. Again for $|t - x| > \delta$, then there exist a positive number M such that $|\lambda(t-x)| \leq M \leq M\frac{(t-x)^2}{\delta^2}$. Thus for all t and $x \in I_n$, we get

$$|\lambda(t-x)| \leq \epsilon + M\frac{(t-x)^2}{\delta^2}. \tag{4.1}$$

Applying S_{n,q_n} on (4.1), we get

$$|(S_{n,q_n,p}f)(x) - f(x)| \leq \epsilon(S_{n,q_n,p}e_0)(x) + \frac{M}{\delta^2}(S_{n,q_n,p}(t-x)^2)(x)$$

$$\|(S_{n,q_n,p}f)(x) - f(x)\| \leq \epsilon + \epsilon\|(S_{n,q_n,p}e_0)(x) - e_0(x)\|$$

$$+ K_3 \sum_{i=0}^{2} \|(S_{n,q_n,p}e_i)(x) - e_i(x)\|$$

where

$$K_3 = \max\left\{\frac{M}{\delta^2}, \frac{2Mx}{\delta^2}, \frac{Mx^2}{\delta^2}\right\}.$$

Then for every $\varepsilon > 0$ there exist $N = N(\epsilon) > 0$ such that for all $n \in N$, we get

$$\|(S_{n,q_n,p}f)(x) - f(x)\| \leq \epsilon + \epsilon(r_0 + \epsilon) + K_3(3\epsilon + r_0 + r_1 + r_2) \leq r + \varepsilon_1$$

here, $\varepsilon_1 = \epsilon(1 + r_0 + \epsilon + 3K_3)$. Since ϵ is arbitrary and small, $r - \lim_{n\to\infty} q_n = 1$, we get $r - \lim_{n\to\infty} \|(S_{n,q_n,p}f)(x) - f\| = 0$. This completes the proof of the theorem. □

Theorem 7 *Let a sequence* $(q_n)_n$; $q_n \in (0, 1)$ *such that* $r - \lim_{n\to\infty} q_n = 1$ *and let the sequence of positive linear operators* $S^{\star}_{n,q_n,p}$; $n \in N$ *be defined by* (1.7). *If*

$$r_i - \lim_{n\to\infty} \|(S^{\star}_{n,q_n,p}e_i)(x) - e_i\| = 0$$

for $i = 0, 1, 2$. *Then for non-decreasing function* $f \in C(I_n)$, *we get*

$$r - \lim_{n\to\infty} \|(S^{\star}_{n,q_n,p}f)(x) - f\| = 0$$

where, r *is any real number such that* $r \geq K_4(r_0 + r_1 + r_2)$ *for some* $K_4 > 0$.

Proof The proof of the theorem is analogous as Theorem 6. □

Theorem 8 *Let f be the integrable and bounded in the interval I_n and let if f'' exists at a point $x \in I_n$. Let a sequence $(q_n)_n; q_n \in (0,1)$ such that $\lim_{n\to\infty} q_n = 1$ and let the sequence of positive linear operators $S_{n,q_n,p}$; $n \in N$ be defined by* (1.5)*. Then, one gets that*

$$\lim_{n\to\infty}[n]_{q_n}|(S_{n,q_n,p}f)(x) - f(x)| = (1 + (p - 3x))f'(x) + \frac{2x - x^2}{2}f''(x)$$

Proof Let if f'' exists at a point $x \in I_n$, then by using Taylor's expansion, we write

$$f(t) = f(x) + (t - x)f'(x) + \frac{(t - x)^2}{2}f''(x) + (t - x)^2\lambda(t - x) \tag{4.2}$$

where, $\lambda(t - x) \to 0$ as $t \to x$. Applying $S_{n,q_n,p}$, we get

$$(S_{n,q_n,p}f)(x) = f(x)(S_{n,q_n,p}1)(x) + f'(x)(S_{n,q_n,p}(t - x))(x)$$
$$+ \frac{f''(x)}{2}(S_{n,q_n,p}(t - x)^2)(x) + (S_{n,q_n,p}(t - x)^2\lambda(t - x))(x).$$

By using Lemma 1 and multiplying $[n]_{q_n}$ both sides, we get

$$[n]_{q_n}[(S_{n,q_n,p}f)(x) - f(x)] = f'(x)[n]_{q_n}\left(\frac{[n]_q([n + p]_q x + 1)}{[n + 2]_q[n + 1]_q}\right)\dots$$
$$\dots + \frac{f''(x)[n]_{q_n}}{2}S_{n,q_n,p}\phi^2(x) + [n]_{q_n}R_{[n]_{q_n}}(t, x). \tag{4.3}$$

Here we write,

$$[n]_{q_n}R_{[n]_{q_n}}(t, x) =$$
$$= \frac{[n + 1]_q^2}{[n]_q E_q([n + p]_q x)}\sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}}\frac{([n + p]_q x)^k}{[k]_q!}\int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} p_{n,k}(q_n; q_n t)\phi^2\lambda\phi d_q t$$

$$|[n]_{q_n}R_{[n]_{q_n}}(t, x)| \le$$
$$\le \frac{[n + 1]_q^2}{[n]_q E_q([n + p]_q x)}\sum_{k=0}^{\infty} q^{\frac{k^2-k-2}{2}}\frac{([n + p]_q x)^k}{[k]_q!}\int_0^{\frac{[n]_{q_n}}{[n+1]_{q_n}}} p_{n,k}(q_n; q_n t)|\phi^2\lambda\phi| d_q t$$
$$\le [n]_{q_n}\epsilon(S_{n,q_n,p}(t - x)^2)(x) + \frac{[n]_{q_n}M}{\delta^2}(S_{n,q_n,p}(t - x)^4)(x)$$
$$\le [n]_{q_n}\epsilon o\left(\frac{1}{[n]_{q_n}}\right) + \frac{[n]_{q_n}M}{\delta^2}o\left(\frac{1}{[n]_{q_n}^2}\right)$$
$$\le \epsilon + \frac{M}{([n]_{q_n})^{\frac{-1}{2}}}o\left(\frac{1}{[n]_{q_n}}\right) \le \epsilon + Mo\left(\frac{1}{\sqrt{[n]_{q_n}}}\right).$$

Here we choose $\delta = ([n]_{q_n})^{\frac{-1}{4}}$.

Since ϵ is arbitrary and small, $\lim_{n\to\infty} q_n = 1$ and whenever $n \to \infty$, we get

$$|[n]_{q_n} R_{[n]_{q_n}}(t,x)| \to 0. \tag{4.4}$$

By using (4.3) in equation (4.4), we get

$$\lim_{n\to\infty} [n]_{q_n} |(S_{n,q_n,p} f)(x) - f(x)| = (1 + (p - 3x))f'(x) + \frac{2x - x^2}{2} f''(x)$$

This completes the proof of the theorem. □

Theorem 9 *Let f be the integrable and bounded in the interval I_n and let if f'' exists at a point $x; t \in I_n$. Let a sequence $(q_n)_n$; $q_n \in (0,1)$ such that* $\lim_{n\to\infty} q_n = 1$ *and let the sequence of positive linear operators $S^{\star}_{n,q_n,p}$; $n \in N$ be defined by* (1.7)*. Then, one gets that*

$$\lim_{n\to\infty} [n]_{q_n} |(S^{\star}_{n,q_n,p} f)(t) - f(t)| = (1 + (p - 3t))f'(x + t) + \frac{2t - t^2}{2} f''(x + t)$$

Proof The proof of the theorem is analogous as Theorem 8. □

5 Conclusion

We conclude that q-Szász–Mirakyan modified operators (1.5) and (1.7) improve the approximation process when the value of n is very large i.e. when n tends to infinity. Although some theorems written in similar way but proofs are different.

Acknowledgement This work was supported and grant funded by the Chhattisgarh Council of Science Technology, Raipur, 492001, India. The authors are thankful to Director of National Institute of Technology, Raipur (C.G.) for encouragement. We are also thankful to Director of CCOST Chhattisgarh, India for time to time eminent support and encouragement.

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