Some Properties of Lorentzian $\alpha$-Sasakian Manifolds with Respect to Quarter-symmetric Metric Connection

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Abstract

The aim of this paper is to study generalized recurrent, generalized Ricci-recurrent, weakly symmetric and weakly Ricci-symmetric, semi-generalized recurrent, semi-generalized Ricci-recurrent Lorentzian $\alpha$-Sasakian manifold with respect to quarter-symmetric metric connection. Finally, we give an example of 3-dimensional Lorentzian $\alpha$-Sasakian manifold with respect to quarter-symmetric metric connection.

Key words: Quarter-symmetric metric connection, Lorentzian $\alpha$-Sasakian manifold, generalized recurrent manifold, generalized Ricci-recurrent manifold, weakly symmetric manifold, weakly Ricci-symmetric manifold, semi-generalized recurrent manifold, Einstein manifold.

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1 Introduction

The idea of a semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten [5]. Further, Hayden [7], introduced the idea of metric connection with torsion on a Riemannian manifold. In [32], Yano studied some curvature conditions for semi-symmetric connections in Riemannian manifolds.

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In 1975, Golab [6] defined and studied a quarter-symmetric connection in a differentiable manifold. A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold $(M^n, g)$ is said to be a quarter-symmetric connection [6] if its torsion tensor $\tilde{T}$ defined by

$$\tilde{T}(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y],$$  

(1.1)

is of the form

$$\tilde{T}(X, Y) = \eta(Y)\phi X - \eta(X)\phi Y,$$  

(1.2)

where $\eta$ is a non-zero 1-form and $\phi$ is a tensor field of type $(1, 1)$. In addition, if a quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$((\tilde{\nabla} g)(Y, Z) = 0$$  

(1.3)

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on $M$, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection. In particular, if $\phi X = X$ and $\phi Y = Y$ for all $X, Y \in \chi(M)$, then the quarter-symmetric connection reduces to a semi-symmetric connection [5].

M. M. Tripathi [29] studied semi-symmetric metric connections in a Kenmotsu manifolds. In [31], the semi-symmetric non-metric connection in a Kenmotsu manifold was studied by M. M. Tripathi and N. Nakkar. Also in [30], M. M. Tripathi proved the existence of a new connection and showed that in particular cases, this connection reduces to semi-symmetric connections; even some of them are not introduced so far.

In 2005, Yildiz and Murathan [36] studied Lorentzian $\alpha$-Sasakian manifolds and proved that conformally flat and quasi conformally flat Lorentzian $\alpha$-Sasakian manifolds are locally isometric with a sphere. In 2012, Yadav and Suthar [34] studied Lorentzian $\alpha$-Sasakian manifolds.


On the other hand, De and Guha introduced generalized recurrent manifold with the non-zero 1-form $\alpha_1$ and another non-zero associated 1-form $\beta_1$. Such a manifold has been denoted by $GK_n$. If the associated 1-form becomes zero, then the manifold $GK_n$ reduces to a recurrent manifold introduced by Ruse [24] which is denoted by $K_n$. The idea of Ricci-recurrent manifold was introduced by Patterson [17]. He denoted such a manifold by $R^n$. Ricci-recurrent manifolds have been studied by many authors ([3], [18], [35], [9], [10], [11], [12]).
A non-flat $n$-dimensional differentiable manifold $M$, $n > 3$, is called generalized recurrent if its curvature tensor $R$ satisfies the condition
\[
(\nabla_X R)(Y, Z)W = \alpha_1(X)R(Y, Z)W + \beta_1(X)[g(Z, W)Y - g(Y, W)Z], \tag{1.4}
\]
where $\nabla$ is the Levi-Civita connection and $\alpha_1$ and $\beta_1$ are two 1-forms ($\beta_1 \neq 0$) defined by
\[
\alpha_1(X) = g(X, A), \quad \beta_1(X) = g(X, B), \tag{1.5}
\]
and $A, B$ are vector fields related with 1-forms $\alpha_1$ and $\beta_1$ respectively. A non-flat $n$-dimensional differentiable manifold $M$, $n > 3$, is called generalized Ricci-recurrent if its Ricci tensor $S$ satisfies the condition
\[
(\nabla_X S)(Y, Z)W = \alpha_1(X)S(Y, Z)W + (n - 1)\beta_1(X)g(Y, Z), \tag{1.6}
\]
where $\alpha_1$ and $\beta_1$ defined as (1.5).

The notions of weakly symmetric and weakly Ricci-symmetric manifolds were introduced by L. Tamassy and T. Q. Binh in ([27], [28]).

A non-flat $n$-dimensional differentiable manifold $M$, $n > 3$, is called pseudosymmetric if there is a 1-form $\alpha_1$ on $M$ such that
\[
(\nabla_X R)(Y, Z)V = 2\alpha_1(X)R(Y, Z)V + \alpha_1(Y)R(X, Z)V + \alpha_1(Z)R(Y, X)V
+ \alpha_1(V)R(Y, Z)X + g(R(Y, Z)V, X)A, \tag{1.7}
\]
where $\nabla$ is the Levi-Civita connection and $X, Y, Z, V$ are vector fields on $M$. $A \in \chi(M)$ is the vector field associated with 1-form $\alpha_1$ which is defined by $g(X, A) = \alpha_1(X)$ in [1]. Later R. Deszcz [4] started to use “pseudosymmetric” term in different sense, see([11], [12] [13]).

A non-flat $n$-dimensional differentiable manifold $M$, $n > 3$, is called weakly symmetric ([27], [28]) if there are 1-forms $\alpha_1, \beta_1, \gamma_1, \sigma_1$ such that
\[
(\nabla_X R)(Y, Z)V = \alpha_1(X)R(Y, Z)V + \beta_1(Y)R(X, Z)V + \gamma_1(Z)R(Y, X)V
+ \sigma_1(V)R(Y, Z)X + g(R(Y, Z)V, X)A \tag{1.8}
\]
for all vector fields $X, Y, Z, V$ on $M$. A weakly symmetric manifold $M$ is pseudosymmetric if $\beta_1 = \gamma_1 = \sigma_1 = \frac{1}{2}\alpha_1$ and $P = A$, locally symmetric if $\alpha_1 = \beta_1 = \gamma_1 = \sigma_1 = 0$ and $P = 0$. A weakly symmetric manifold is said to be proper if at least one of the 1-forms $\alpha_1, \beta_1, \gamma_1$ and $\sigma_1$ is not zero or $P \neq 0$.

A non-flat $n$-dimensional differentiable manifold $M$, $n > 3$, is called weakly Ricci-symmetric ([27], [28]) if there are 1-forms $\rho, \mu, \nu$ such that
\[
(\nabla_X S)(Y, Z) = \rho(X)S(Y, Z) + \mu(Y)S(Y, Z) + \nu(Z)S(X, Y) \tag{1.9}
\]
for all vector fields $X, Y, Z, V$ on $M$. If $\rho = \mu = \nu$, then $M$ is called pseudo Ricci-symmetric (see [2]).

If $M$ is weakly symmetric, from (1.8), we have

$$
(\nabla_X S)(Y, Z) = \alpha_1(X)S(Z, V) + \beta_1(R(X, Z)V) + \gamma_1(Z)S(X, V) \\
+ \sigma_1(V)S(X, Z) + p(R(X, V)Z),
$$

(1.10)

where $p$ is defined by $p(X) = g(X, P)$ for any $X \in \chi(M)$ in [28].

Generalizing the notion of recurrency, the author Khan [21] introduced the notion of generalized recurrent Sasakian manifolds. In the paper B. Prasad [19] introduced the notion of semi-generalized recurrent manifold and obtained few interesting results. L. Rachůnek and J. Mikeš studied the similar problems in ([14], [15], [25]).

A Riemannian manifold is called a semi-generalized recurrent manifold if its curvature tensor $R$ satisfies the condition

$$
(\nabla_X R)(Y, Z)W = \alpha_1(X)R(Y, Z)W + \beta_1(X)g(Z, W)Y,
$$

(1.11)

where $\alpha_1$ and $\beta_1$ defined as (1.5).

A Riemannian manifold is called a semi-generalized Ricci-recurrent manifold if its curvature tensor $R$ satisfies the condition

$$
(\nabla_X S)(Y, Z) = \alpha_1(X)S(Y, Z) + n\beta_1(X)g(Y, Z),
$$

(1.12)

where $\alpha_1$ and $\beta_1$ defined as (1.5).

Motivated by the above studies, in the present paper we have proved that $\beta_1 = (\alpha - \alpha^2)\alpha_1$ holds on both generalized recurrent and generalized Ricci-recurrent Lorentzian $\alpha$-Sasakian manifold with respect to the quarter-symmetric metric connection. We also show that there is no weakly symmetric or weakly Ricci-symmetric Lorentzian $\alpha$-Sasakian manifold with respect to the quarter-symmetric metric connection, $n > 3$, unless $\alpha_1 + \sigma_1 + \gamma_1$ or $\rho + \mu + \nu$ is everywhere zero, respectively. We have also studied semi-generalized recurrent Lorentzian $\alpha$-Sasakian manifold with respect to the quarter-symmetric metric connection.

## 2 Preliminaries

A $n(=2m+1)$-dimensional differentiable manifold $M$ is said to be a Lorentzian $\alpha$-Sasakian manifold if it admits a $(1, 1)$ tensor field $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and Lorentzian metric $g$ which satisfy the following conditions

$$
\phi^2 X = X + \eta(X)\xi,
$$

(2.1)
Some properties of Lorentzian $\alpha$-Sasakian manifolds...

\begin{align*}
\eta(\xi) &= -1, \phi \xi = 0, \eta(\phi X) = 0, \quad (2.2) \\
g(\phi X, \phi Y) &= g(X, Y) + \eta(X)\eta(Y), \quad (2.3) \\
g(X, \xi) &= \eta(X), \quad (2.4) \\
(\nabla_X \phi)(Y) &= \alpha\{g(X, Y)\xi + \eta(Y)X\} \quad (2.5)
\end{align*}

$\forall X, Y \in \chi(M)$ and for non-zero smooth functions $\alpha$ on $M$, $\nabla$ denotes the covariant differentiation with respect to Lorentzian metric $g$ ([20], [37]).

For a Lorentzian $\alpha$-Sasakian manifold, it can be shown that ([20], [37]):

\begin{align*}
\nabla_X \xi &= \alpha \phi X, \quad (2.6) \\
(\nabla_X \eta)(Y) &= \alpha g(\phi X, Y) \quad (2.7)
\end{align*}

for all $X, Y \in \chi(M)$.

Further on a Lorentzian $\alpha$-Sasakian manifold, the following relations hold [20]

\begin{align*}
g(R(X, Y)Z, \xi) &= \eta(R(X, Y)Z) = \alpha^2[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \quad (2.8) \\
R(\xi, X)Y &= \alpha^2[g(X, Y)\xi - \eta(Y)X], \quad (2.9) \\
R(X, Y)\xi &= \alpha^2[\eta(Y)X - \eta(X)Y], \quad (2.10) \\
R(\xi, X)\xi &= \alpha^2[X + \eta(X)\xi], \quad (2.11) \\
S(X, \xi) &= S(\xi, X) = (n - 1)\alpha^2\eta(X), \quad (2.12) \\
S(\xi, \xi) &= -(n - 1)\alpha^2, \quad (2.13) \\
Q\xi &= (n - 1)\alpha^2\xi, \quad (2.14)
\end{align*}

where $Q$ is the Ricci operator, i.e.,

\begin{align*}
g(Q X, Y) &= S(X, Y). \quad (2.15)
\end{align*}

If $\nabla$ is the Levi-Civita connection manifold $M$, then quarter-symmetric metric connection $\tilde{\nabla}$ in $M$ is denoted by

\begin{align*}
\tilde{\nabla}_X Y &= \nabla_X Y + \eta(Y)\phi(X). \quad (2.16)
\end{align*}
3 Curvature tensor and Ricci tensor of Lorentzian $\alpha$-Sasakian manifold with respect to quarter-symmetric metric connection

Let $\tilde{R}(X,Y)Z$ and $R(X,Y)Z$ be the curvature tensors with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and with respect to the Riemannian connection $\nabla$ respectively on a Lorentzian $\alpha$-Sasakian manifold $M$. A relation between the curvature tensors $\tilde{R}(X,Y)Z$ and $R(X,Y)Z$ on $M$ is given by

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \alpha [g(\phi X, Z)\phi Y - g(\phi Y, Z)\phi X] + \alpha \eta(Z)[\eta(Y)X - \eta(X)Y].$$  

(3.1)

Also from (3.1), we obtain

$$\tilde{S}(X,Y) = S(X,Y) + \alpha [g(X,Y) + n\eta(X)\eta(Y)],$$

(3.2)

where $\tilde{S}$ and $S$ are the Ricci tensor with respect to $\tilde{\nabla}$ and $\nabla$ respectively.

Contracting (3.2), we obtain,

$$\tilde{r} = r,$$

(3.3)

where $\tilde{r}$ and $r$ are the scalar curvature tensor with respect to $\tilde{\nabla}$ and $\nabla$ respectively.

Also we have

$$\tilde{R}(\xi,X)Y = -\tilde{R}(X,\xi)Y = \alpha^2[g(X,Y))\xi - \eta(Y)X] + \alpha \eta(Y)[X + \eta(X)\xi],$$

(3.4)

$$\eta(\tilde{R}(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)],$$

(3.5)

$$\tilde{R}(X,Y)\xi = (\alpha^2 - \alpha)[\eta(Y)X - \eta(X)Y],$$

(3.6)

$$\tilde{S}(X,\xi) = \tilde{S}(\xi,X) = (n - 1)(\alpha^2 - \alpha)\eta(X),$$

(3.7)

$$\tilde{S}(\xi,\xi) = -(n - 1)(\alpha^2 - \alpha),$$

(3.8)

$$\tilde{Q}X = QX - \alpha(n - 1)X,$$

(3.9)

$$\tilde{Q}\xi = (n - 1)(\alpha^2 - \alpha)\xi,$$

(3.10)

$$\tilde{R}(\xi,X)\xi = (\alpha^2 - \alpha)[X + \eta(X)\xi].$$

(3.11)
4 Generalized recurrent Lorentzian $\alpha$-Sasakian manifold with respect to quarter-symmetric metric connection

A non-flat $n$-dimensional differentiable manifold $M$, $n > 3$, is called generalized recurrent with respect to the quarter-symmetric metric connection if its curvature tensor $\tilde{R}$ satisfies the condition

$$ (\tilde{\nabla}_X \tilde{R})(Y, Z)W = \alpha_1(X)\tilde{R}(Y, Z)W + \beta_1(X)[g(Z, W)Y - g(Y, W)Z] $$

(4.1)

for all $X, Y, Z, W \in \chi(M)$, where $\tilde{\nabla}$ is the quarter-symmetric metric connection and $\tilde{R}$ is the curvature tensor of $\tilde{\nabla}$.

A non-flat $n$-dimensional differentiable manifold $M$, $n > 3$, is called generalized Ricci-recurrent with respect to the quarter-symmetric metric connection if its Ricci tensor $\tilde{S}$ satisfies the condition

$$ (\tilde{\nabla}_X \tilde{S})(Y, Z) = \alpha_1(X)\tilde{S}(Y, Z) + (n - 1)\beta_1(X)g(Y, Z) $$

(4.2)

for all $X, Y, Z \in \chi(M)$.

In [26] Sular studied that if $M$ be a generalized recurrent Kenmotsu manifold and generalized Ricci recurrent Kenmotsu manifold respect to semi-symmetric metric connection, then $\beta_1 = 2\alpha_1$ holds on $M$.

Now we consider generalized recurrent and generalized Ricci-recurrent Lorentzian $\alpha$-Sasakian manifold with respect to quarter-symmetric metric connection.

We start with the following theorem:

**Theorem 4.1.** If a generalized recurrent Lorentzian $\alpha$-Sasakian manifold $M$ admits quarter-symmetric metric connection, then $\beta_1 = (\alpha - \alpha^2)\alpha_1$ holds on $M$.

**Proof.** Suppose that $M$ is a generalized recurrent Lorentzian $\alpha$-Sasakian manifold admitting a quarter-symmetric metric connection. Taking $Y = W = \xi$ in (4.1), we get

$$ (\tilde{\nabla}_X \tilde{R})(\xi, Z)\xi = \alpha_1(X)\tilde{R}(\xi, Z)\xi + \beta_1(X)[g(Z, \xi)\xi + Z]. $$

(4.3)

By using the equation (2.4), (2.10) and (3.6) in (4.3), we have

$$ (\tilde{\nabla}_X \tilde{R})(\xi, Z)\xi = [\alpha_1(X)(\alpha^2 - \alpha) + \beta_1(X)]\{\eta(Z)\xi + Z\}. $$

(4.4)

On the other hand, it is clear that

$$ (\tilde{\nabla}_X \tilde{R})(\xi, Z)\xi = \tilde{\nabla}_X \tilde{R}(\xi, Z)\xi - \tilde{R}(\tilde{\nabla}_X \xi, Z) - \tilde{R}(\xi, \tilde{\nabla}_X Z) - \tilde{R}(\xi, Z)\tilde{\nabla}_X \xi. $$

(4.5)

Now using the equation (2.10) and (3.6) in (4.5), we have

$$ (\tilde{\nabla}_X \tilde{R})(\xi, Z)\xi = 0. $$

(4.6)
Hence comparing the right hand sides of the equations (4.4) and (4.6) we obtain

\[
[\alpha_1(X)(\alpha^2 - \alpha) + \beta_1(X)]\{\eta(Z)\xi + Z\} = 0, \tag{4.7}
\]

which imply

\[
\beta_1(X) = (\alpha - \alpha^2)\alpha_1(X) \tag{4.8}
\]

for any vector field \(X \in M\). So our theorem is proved.

**Theorem 4.2.** Let \(M\) be a generalized Ricci-recurrent Lorentzian \(\alpha\)-Sasakian manifold admitting quarter-symmetric metric connection, then \(\beta_1 = (\alpha - \alpha^2)\alpha_1\) holds on \(M\).

**Proof.** Suppose that \(M\) is a generalized Ricci-recurrent Lorentzian \(\alpha\)-Sasakian Manifold \(M\) with respect to quarter-symmetric metric connection. Now putting \(Z = \xi\) in (4.2), we get

\[
(\tilde{\nabla}_X \tilde{S})(Y, \xi) = \alpha_1(X)\tilde{S}(Y, \xi) + (n-1)\beta_1(X)g(Y, \xi). \tag{4.9}
\]

Then by using the equation (2.4), (2.12) and (3.7) in (4.9), we have

\[
(\tilde{\nabla}_X \tilde{S})(Y, \xi) = \alpha_1(X)[(n-1)(\alpha^2 - \alpha)\eta(Y) + (n-1)\beta_1(X)\eta(Y)]. \tag{4.10}
\]

On the other hand, by using the definition of covariant derivative of \(\tilde{S}\) with respect to the quarter-symmetric metric connection, it is well-known that

\[
(\tilde{\nabla}_X \tilde{S})(Y, \xi) = \tilde{\nabla}_X \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_XY, \xi) - \tilde{S}(Y, \tilde{\nabla}_X\xi) \tag{4.11}
\]

Now using the equation (2.6), (2.7), (2.12), (2.16), (3.2) and (3.7) in (4.11), we obtain

\[
(n-1)(\alpha^2 - \alpha)\alpha g(Y, \phi X) - (\alpha - 1)[S(Y, \phi X) + \alpha g(Y, \phi X)]. \tag{4.12}
\]

Hence comparing the right hand sides of the equations (4.10) and (4.12) we obtain

\[
\alpha_1(X)[(n-1)(\alpha^2 - \alpha)\eta(Y) + (n-1)\beta_1(X)\eta(Y)
= (n-1)(\alpha^2 - \alpha)\alpha g(Y, \phi X)
- (\alpha - 1)[S(Y, \phi X) + \alpha g(Y, \phi X)]. \tag{4.13}
\]

Now putting \(Y = \xi\) in (4.13), we get

\[
\beta_1(X) = (\alpha - \alpha^2)\alpha_1(X) \tag{4.14}
\]

for any vector field \(X \in M\). So this completes the proof.
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5 Weakly symmetric Lorentzian $\alpha$-Sasakian manifold with respect to quarter-symmetric metric connection

A non-flat $n$-dimensional differentiable manifold $M$, $n > 3$, is called weakly symmetric with respect to quarter-symmetric metric connection if there are 1-forms $\alpha_1, \beta_1, \gamma_1, \sigma_1$ such that

$$\nabla_X \tilde{R}(Y, Z)V = \alpha_1(X) \tilde{R}(Y, Z)V + \beta_1(Y) \tilde{R}(X, Z)V + \gamma_1(Z) \tilde{R}(Y, X)V + \sigma_1(V) \tilde{R}(Y, Z)X + g(\tilde{R}(Y, Z)V, X)A \tag{5.1}$$

for all vector fields $X, Y, Z, V$ on $M$.

A non-flat $n$-dimensional differentiable manifold $M$, $n > 3$, is called weakly Ricci-symmetric with respect to quarter-symmetric metric connection if there are 1-forms $\rho, \mu, \upsilon$ such that

$$\nabla_X \tilde{S}(Y, Z) = \rho(X) \tilde{S}(Y, Z) + \mu(Y) \tilde{S}(Y, Z) + \upsilon(Z) \tilde{S}(X, Y) \tag{5.2}$$

for all vector fields $X, Y, Z, V$ on $M$. If $M$ is weakly symmetric with respect to the quarter-symmetric metric connection, by a contraction from (1.8), we have

$$\nabla_X \tilde{S}(Z, V) = \alpha_1(X) \tilde{S}(Z, V) + \beta_1(\tilde{R}(X, Z)V) + \gamma_1(Z) \tilde{S}(X, V) + \sigma_1(V) \tilde{S}(X, Z) + p(\tilde{R}(X, V)Z) \tag{5.3}$$

In [26], Sular studied weakly symmetric and weakly Ricci-symmetric Kenmotsu manifold with respect to semi-symmetric metric connection and obtained some results.

i) If $M$ be a weakly symmetric Kenmotsu manifold with respect to quarter-symmetric metric connection then there is no weakly symmetric $n > 3$, unless $\alpha_1 + \sigma_1 + \gamma_1$ is everywhere zero.

ii) If $M$ be a weakly Ricci-symmetric Kenmotsu manifold with respect to semi-symmetric metric connection then there is no weakly Ricci-symmetric $n > 3$, unless $\rho + \mu + \upsilon$ is everywhere zero.

Now we consider weakly symmetric and weakly Ricci-symmetric Lorentzian $\alpha$-Sasakian manifold with respect to quarter-symmetric metric connection.

We start with the following theorem:

**Theorem 5.1.** There is no weakly symmetric Lorentzian $\alpha$-Sasakian manifold with respect to quarter-symmetric metric connection $n > 3$, unless $\alpha_1 + \sigma_1 + \gamma_1$ is everywhere zero, provided $\alpha \neq 0, 1$.

**Proof.** Let $M$ be a weakly symmetric Lorentzian $\alpha$-Sasakian manifold with respect to quarter-symmetric metric connection $\nabla$. By the covariant differentiation of the Ricci tensor $\tilde{S}$ of the quarter-symmetric metric connection with respect to $X$, we have

$$\nabla_X \tilde{S}(Z, V) = \tilde{\nabla}_X \tilde{S}(Z, V) - \tilde{S}(\tilde{\nabla}_X Z, V) - \tilde{S}(Z, \tilde{\nabla}_X V). \tag{5.4}$$
Putting $V = \xi$ in (5.4) and using (2.6), (2.7), (2.12), (2.16) and (3.7), it follows that
\[(\tilde{\nabla}_X \tilde{S})(Z, \xi) = (n - 1)(\alpha^2 - \alpha)(\nabla_X \eta)Z - (\alpha - 1)\tilde{S}(Z, \phi X). \tag{5.5}\]
Replacing $V = \xi$ in (5.3), we get
\[(\tilde{\nabla}_X \tilde{S})(Z, \xi) = \alpha_1(X)\tilde{S}(Z, \xi) + \beta_1(\tilde{R}(X, Z)\xi) + \gamma_1(Z)\tilde{S}(X, \xi) + \sigma_1(\xi)\tilde{S}(X, Z) + p(\tilde{R}(X, \xi)Z). \tag{5.6}\]
Now using (2.6), (2.7), (2.12), (2.16), (3.6) and (3.7) in (5.6), we obtain
\[(\tilde{\nabla}_X \tilde{S})(Z, \xi) = \alpha_1(X)(n - 1)(\alpha^2 - \alpha)\eta(Z)
+ (\alpha^2 - \alpha)[\eta(Z)\beta_1(X) - \eta(X)\beta_1(Z)]
+ \gamma_1(Z)(n - 1)(\alpha^2 - \alpha)\eta(X) + \sigma_1(\xi)\tilde{S}(X, Z)
- \alpha^2[g(X, Z)p(\xi) - \eta(Z)p(X)] - \alpha_1\eta(Z)[p(X)
+ \eta(X)p(\xi)]. \tag{5.7}\]
Thus, comparing the right hand sides of the equations (5.5) and (5.7) we obtain
\[(n - 1)(\alpha^2 - \alpha)(\nabla_X \eta)Z - (\alpha - 1)\tilde{S}(Z, \phi X) = \alpha_1(X)(n - 1)(\alpha^2 - \alpha)\eta(Z)
+ (\alpha^2 - \alpha)[\eta(Z)\beta_1(X) - \eta(X)\beta_1(Z)]
+ \gamma_1(Z)(n - 1)(\alpha^2 - \alpha)\eta(X) + \sigma_1(\xi)\tilde{S}(X, Z)
- \alpha^2[g(X, Z)p(\xi) - \eta(Z)p(X)] - \alpha_1\eta(Z)[p(X)
+ \eta(X)p(\xi)]. \tag{5.8}\]
Then taking $X = Z = \xi$ in (5.8) and using (2.1), (2.2), (2.4), (2.12) and (3.8), we get
\[(n - 1)(\alpha^2 - \alpha)[\alpha_1(\xi) + \gamma_1(\xi) + \sigma_1(\xi)] = 0. \tag{5.9}\]
Now as $n > 3$ and $\alpha \neq 0, 1,$ So,
\[\alpha_1(\xi) + \gamma_1(\xi) + \sigma_1(\xi) = 0. \tag{5.10}\]
Now putting $Z = \xi$ in (5.3), we get
\[(\tilde{\nabla}_X \tilde{S})(\xi, V) = \alpha_1(X)\tilde{S}(\xi, V) + \beta_1(\tilde{R}(X, \xi)V) + \gamma_1(\xi)\tilde{S}(X, V)
+ \sigma_1(\xi)\tilde{S}(X, \xi) + p(\tilde{R}(X, \xi)V). \tag{5.11}\]
Also putting $Z = \xi$ in (5.4) and using (2.6), (2.7), (2.12), (2.16) and (3.7), it follows that
\[(\tilde{\nabla}_X \tilde{S})(\xi, V) = (n - 1)(\alpha^2 - \alpha)(\nabla_X \eta)V - (\alpha - 1)\tilde{S}(V, \phi X). \tag{5.12}\]
Similarly using (2.6), (2.7), (2.12), (2.16), (3.6) and (3.7) in (5.11), we obtain
\[(\tilde{\nabla}_X \tilde{S})(\xi, V) = \alpha_1(X)(n - 1)(\alpha^2 - \alpha)\eta(V) - \alpha^2[g(X, V)\beta_1(\xi)
- \eta(V)\beta_1(X)] - \alpha\eta(V)[\beta_1(X) + \eta(X)\beta_1(\xi)]
+ \gamma_1(\xi)\tilde{S}(X, V) + \sigma_1(V)(n - 1)(\alpha^2 - \alpha)\eta(X)
+ (\alpha^2 - \alpha)[\eta(V)p(X) - \eta(V)p(X)]. \tag{5.13}\]
Thus, comparing the right hand sides of the equations (5.12) and (5.13), we obtain

\[(n - 1)(\alpha^2 - \alpha)(\nabla_X \eta)V - (\alpha - 1)\tilde{S}(V, \phi X) = \alpha_1(X)(n - 1)(\alpha^2 - \alpha)\eta(V)\]
\[\quad - \alpha^2[g(X, V)\beta_1(\xi) - \eta(V)\beta_1(X)] - \alpha\eta(V)[\beta_1(X)\]
\[\quad + \eta(X)\beta_1(\xi)] + \gamma_1(\xi)\tilde{S}(X, V) + \sigma_1(V)(n - 1)(\alpha^2\]
\[\quad - \alpha)\eta(X) + (\alpha^2 - \alpha)[\eta(V)p(X)\]
\[\quad - \eta(V)p(X)]. \tag{5.14}\]

Now putting \(V = \xi\) in (5.14), we obtain

\[-\alpha_1(X)(n - 1)(\alpha^2 - \alpha) - (\alpha^2 - \alpha)[\eta(X)\beta_1(\xi) + \beta_1(X)]\]
\[\quad + (\sigma_1(\xi) + \gamma_1(\xi))(n - 1)(\alpha^2 - \alpha)\eta(X)\]
\[\quad - (\alpha^2 - \alpha)[p(X) + \eta(X)p(\xi)] = 0. \tag{5.15}\]

Taking \(X = \xi\) in (5.14), we obtain

\[\alpha_1(\xi)(n - 1)(\alpha^2 - \alpha)\eta(V) + \gamma_1(\xi)(n - 1)(\alpha^2 - \alpha)\eta(V)\]
\[\quad - \sigma_1(V)(n - 1)(\alpha^2 - \alpha) + (\alpha^2\]
\[\quad - \alpha)[p(V) + \eta(V)p(\xi)] = 0. \tag{5.16}\]

In (5.16) taking \(V = X\) and summing with (5.15), by virtue of (5.10) we find

\[-(n - 1)(\alpha^2 - \alpha)[\alpha_1(X) + \sigma_1(X)] - (\alpha^2 - \alpha)[\eta(X)\beta_1(\xi) + \beta_1(X)]\]
\[\quad + (n - 1)(\alpha^2 - \alpha)\eta(X)\gamma_1(\xi) = 0. \tag{5.17}\]

Again putting \(X = \xi\) in (5.8), we obtain

\[\alpha_1(\xi)(n - 1)(\alpha^2 - \alpha)\eta(Z) + (\alpha^2 - \alpha)[\eta(Z)\beta_1(\xi) + \beta_1(Z)]\]
\[\quad - \gamma_1(Z)(n - 1)(\alpha^2 - \alpha)\]
\[\quad + \sigma_1(\xi)(n - 1)(\alpha^2 - \alpha)\eta(Z) = 0. \tag{5.18}\]

Now in the equation (5.18) taking \(Z = X\), we obtain

\[\alpha_1(\xi)(n - 1)(\alpha^2 - \alpha)\eta(X) + (\alpha^2 - \alpha)[\eta(X)\beta_1(\xi) + \beta_1(X)]\]
\[\quad - \gamma_1(X)(n - 1)(\alpha^2 - \alpha)\]
\[\quad + \sigma_1(\xi)(n - 1)(\alpha^2 - \alpha)\eta(X) = 0. \tag{5.19}\]

Then adding (5.17) and (5.19), we find

\[(n - 1)(\alpha^2 - \alpha)\eta(X)[\alpha_1(\xi) + \gamma_1(\xi) + \sigma_1(\xi)] - (n - 1)(\alpha^2 - \alpha)[\alpha_1(X)\]
\[\quad + \gamma_1(X) + \sigma_1(X)] = 0. \tag{5.20}\]

Since \(n > 3, \alpha \neq 0, 1, \) and

\[\alpha_1(\xi) + \gamma_1(\xi) + \sigma_1(\xi) = 0,
\]
so we get
\[ \alpha_1(X) + \gamma_1(X) + \sigma_1(X) = 0 \]
for all \( X \in M \).

So our proof is completed.

**Theorem 5.2.** There is no weakly Ricci-symmetric Lorentzian \( \alpha \)-Sasakian manifold with respect to quarter-symmetric metric connection \( n > 3 \), unless \( \rho + \mu + \upsilon \) is everywhere zero, provided \( \alpha \neq 0, 1 \).

**Proof.** Assume that \( M \) is a weakly Ricci-symmetric Lorentzian \( \alpha \)-Sasakian manifold with respect to quarter-symmetric metric connection \( \tilde{\nabla} \). Now taking \( Z = \xi \) in (5.2) and using (3.2) and (3.7), we obtain
\[
(\tilde{\nabla}_X \tilde{S})(Y, \xi) = \rho(X)(n - 1)(\alpha^2 - \alpha)\eta(Y) + \mu(X)(n - 1)(\alpha^2 - \alpha)\eta(X) + v(\xi)[S(X, Y) + \alpha\{g(X, Y) + n\eta(X)\eta(Y)\}].
\] (5.21)
Also we have
\[
(\tilde{\nabla}_X \tilde{S})(Y, \xi) = (n - 1)(\alpha^2 - \alpha)(\nabla_X \eta)(Y) - (\alpha - 1)[S(Y, \phi X) + \alpha g(X, \phi Y)].
\] (5.22)
Now equating (5.21) and (5.22), we obtain
\[
\rho(X)(n - 1)(\alpha^2 - \alpha)\eta(Y) + \mu(X)(n - 1)(\alpha^2 - \alpha)\eta(X) + v(\xi)[S(X, Y) + \alpha\{g(X, Y) + n\eta(X)\eta(Y)\}] = (n - 1)(\alpha^2 - \alpha)(\nabla_X \eta)(Y) - (\alpha - 1)[S(Y, \phi X) + \alpha g(X, \phi Y)].
\] (5.23)
Now putting \( X = Y = \xi \) in (5.23), we find
\[
(n - 1)(\alpha^2 - \alpha)[\rho(\xi) + \mu(\xi) + v(\xi)] = 0.
\] (5.24)
As \( n > 3 \) and \( \alpha \neq 0, 1 \), So
\[
\rho(\xi) + \mu(\xi) + v(\xi) = 0.
\] (5.25)
Taking \( X = \xi \) in (5.23), we find
\[
(n - 1)(\alpha^2 - \alpha)\eta(Y)[\rho(\xi) + v(\xi)] + \mu(Y)(n - 1)(\alpha^2 - \alpha) = 0.
\] (5.26)
So in view of (5.25), the above equation turns into
\[
-\eta(Y)\mu(\xi) = \mu(Y).
\] (5.27)
Similarly in (5.23), taking \( Y = \xi \), we find
\[
-\rho(X)(n - 1)(\alpha^2 - \alpha) + (\alpha^2 - \alpha)\eta(X)[\mu(\xi)(n - 1) + v(\xi)] = 0.
\] (5.28)
So in view of (5.25), we get finally

$$\rho(X) = -\rho(\xi)\eta(X). \quad (5.29)$$

Since \((\nabla_\xi \tilde{S})(Y, \xi) = 0\), then from (5.2), we get

$$[\rho(\xi) + \mu(\xi)]\eta(X) = v(X), \quad (5.30)$$

that is

$$-v(\xi)\eta(X) = v(X). \quad (5.31)$$

Thus replacing \(Y\) with \(X\) in (5.27) and then summing of the equations (5.27), (5.29) and (5.31) we get

$$\rho(X) + \mu(X) + v(X) = -\eta(X)[\rho(\xi) + \mu(\xi) + v(\xi)]. \quad (5.32)$$

From the equation (5.25), it is clear that

$$\rho(X) + \mu(X) + v(X) = 0 \quad (5.33)$$

for any vector field \(X\) holds on \(M\), which means that

$$\rho + \mu + v = 0.$$

Hence our proof is completed.

6 On semi-generalized recurrent Lorentzian \(\alpha\)-Sasakian manifold with respect to quarter-symmetric metric connection

A Lorentzian \(\alpha\)-Sasakian manifold is called a semi-generalized recurrent manifold with respect to quarter-symmetric metric connection if its curvature tensor \(\tilde{R}\) satisfies the condition

$$(\tilde{\nabla}_X \tilde{R})(Y, Z)W = \alpha_1(X)\tilde{R}(Y, Z)W + \beta_1(X)g(Z, W)Y, \quad (6.1)$$

where \(\alpha_1\) and \(\beta_1\) defined as (1.5) for any vector field and \(\tilde{\nabla}\) denotes the operator of covariant differentiation with respect to the metric.

Taking \(Y = W = \xi\) in (6.1), we have

$$(\tilde{\nabla}_X \tilde{R})(\xi, Z)\xi = \alpha_1(X)\tilde{R}(\xi, Z)\xi + \beta_1(X)g(Z, \xi)\xi. \quad (6.2)$$

From (4.5), the left hand side of (6.2) can be written in the form

$$(\tilde{\nabla}_X \tilde{R})(\xi, Z)\xi = X\tilde{R}(\xi, Z)\xi - \tilde{R}(\tilde{\nabla}_X \xi, Z) - \tilde{R}(\xi, \tilde{\nabla}_X Z) - \tilde{R}(\xi, Z)\tilde{\nabla}_X \xi. \quad (6.3)$$
Now using (2.6), (2.16), (3.4), (3.6) and (3.11), the right hand site of the equation (6.3) becomes
\[
(\tilde{\nabla}_X \tilde{R})(\xi, Z)\xi = -(\alpha^2 - \alpha)(\alpha - 1)\eta(Z)\phi X - (\alpha^2 - \alpha)\eta(Z)\phi X. \tag{6.4}
\]
Now using (3.11), the right hand side of (6.2) can be written in the form
\[
(\tilde{\nabla}_X \tilde{R})(\xi, Z)\xi = \alpha_1(X)(\alpha^2 - \alpha)[Z + \eta(Z)\xi] + \beta_1(X)\eta(Z)\xi. \tag{6.5}
\]
Now from (6.4) and (6.5), we have
\[
-(\alpha^2 - \alpha)(\alpha - 1)\eta(Z)\phi X - (\alpha^2 - \alpha)\eta(Z)\phi X = \alpha_1(X)(\alpha^2 - \alpha)[Z + \eta(Z)\xi] + \beta_1(X)\eta(Z)\xi. \tag{6.6}
\]
Now putting \(Z = \xi\) in (6.6), we obtain
\[
(\alpha^2 - \alpha)\tilde{\nabla}_X \xi + \alpha \tilde{\nabla}_X \xi = -\beta_1(X)\xi, \tag{6.7}
\]
that is
\[
\alpha^2 \tilde{\nabla}_X \xi = -\beta_1(X)\xi. \tag{6.8}
\]
Hence we can state the following theorem:

**Theorem 6.1.** If a semi-generalized recurrent Lorentzian \(\alpha\)-Sasakian manifold admits quarter-symmetric metric connection, the associated vector field \(\xi\) is not constant and \(\tilde{\nabla}_X \xi\) is parallel to \(\xi\), provided \(\alpha \neq 0\).

Permutting equation (6.1) with respect to \(X, Y, Z\) and adding the three equations and using Bianchi identity, we have
\[
\alpha_1(X)\tilde{R}(Y, Z)W + \beta_1(X)g(Z, W)Y + \alpha_1(Y)\tilde{R}(Z, X)W + \beta_1(Y)g(X, W)Z + \alpha_1(Z)\tilde{R}(X, Y)W + \beta_1(Z)g(Y, W)X = 0. \tag{6.9}
\]
Contracting (6.9) with respect to \(Y\), we get
\[
\alpha_1(X)\tilde{S}(Z, W) + n\beta_1(X)g(Z, W) + \tilde{R}'(Z, X, W, A) + \beta_1(Z)g(X, W) - \alpha_1(Z)\tilde{S}(X, W) + \beta_1(Z)g(X, W) = 0. \tag{6.10}
\]
In view of \(\tilde{S}(Z, W) = g(\tilde{Q}Z, W)\), the equation (6.10) becomes
\[
\alpha_1(X)g(\tilde{Q}Z, W) + n\beta_1(X)g(Z, W) - g(\tilde{R}(Z, X)A, W) + \beta_1(Z)g(X, W) - \alpha_1(Z)g(\tilde{Q}X, W) + \beta_1(Z)g(X, W) = 0. \tag{6.11}
\]
From (6.11), we have
\[
\alpha_1(X)\tilde{Q}Z + n\beta_1(X)Z - \tilde{R}(Z, X)A + \beta_1(Z)X - \alpha_1(Z)\tilde{Q}X + \beta_1(Z)X = 0. \tag{6.12}
\]
Contracting (6.12) with respect to $Z$, we obtain
\[ \alpha_1(X) \tilde{r} + (n^2 + 2) \beta_1(X) - 2 \tilde{S}(X, A) = 0. \]  
(6.13)

Putting $X = \xi$ in (6.13), we get
\[ \eta(A) \tilde{r} + (n^2 + 2) \eta(B) - 2(n - 1)(\alpha^2 - \alpha) \eta(A) = 0, \]  
(6.14)

that is
\[ \tilde{r} = \frac{1}{\eta(A)} [2(n - 1)(\alpha^2 - \alpha) \eta(A) - (n^2 + 2) \eta(B)], \]  
(6.15)

where $\tilde{r}$ is the scalar curvature with respect to quarter-symmetric metric connection.

Hence we can state the following theorem:

**Theorem 6.2.** The scalar curvature of a semi-generalized recurrent Lorentzian $\alpha$-Sasakian manifold admitting a quarter-symmetric metric connection is related in terms of contact forms $\eta(A)$ and $\eta(B)$ as given by (6.15).

### 7 On semi-generalized Ricci-recurrent Lorentzian $\alpha$-Sasakian manifold with respect to quarter-symmetric metric connection

A Lorentzian $\alpha$-Sasakian manifold is called a semi-generalized Ricci-recurrent manifold with respect to quarter-symmetric metric connection if its Ricci tensor $S$ satisfies the condition
\[ (\tilde{\nabla}_X \tilde{S})(Y, Z) = \alpha_1(X) \tilde{S}(Y, Z) + n \beta_1(X) g(Y, Z), \]  
(7.1)

where $\alpha_1$ and $\beta_1$ defined as (1.5).

Taking $Z = \xi$ in (7.1), we have
\[ (\tilde{\nabla}_X \tilde{S})(Y, \xi) = \alpha_1(X) \tilde{S}(Y, \xi) + n \beta_1(X) g(Y, \xi). \]  
(7.2)

The left hand side of (7.2), clearly can be written in the form
\[ (\tilde{\nabla}_X \tilde{S})(Y, \xi) = X \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_X Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_X \xi). \]  
(7.3)

Using (3.2) and (3.7), the right hand site of the equation (7.3) becomes
\[ - \tilde{S}(Y, \tilde{\nabla}_X \xi) + (n - 1) \alpha(\alpha^2 - \alpha) g(\phi X, Y). \]  
(7.4)

The right hand site of (7.2) can be written as using (3.7)
\[ \alpha_1(X)(n - 1)(\alpha^2 - \alpha) \eta(Y) + n \beta_1(X) \eta(Y). \]  
(7.5)
From (7.4) and (7.5), we get
\[ \tilde{S}(Y, \nabla_X \xi) + (n - 1)\alpha (\alpha^2 - \alpha)g(\phi X, Y) = \alpha_1(X)\eta(Y). \]
(7.6)

Now putting \( Y = \xi \) in (7.6), we obtain
\[ \alpha_1(X)(n - 1)(\alpha^2 - \alpha) + n\beta_1(X) = 0, \]
(7.7)

that is
\[ \alpha_1(X) = -\frac{n}{(n - 1)(\alpha^2 - \alpha)}\beta_1(X). \]
(7.8)

This leads to the following theorem:

**Theorem 7.1.** If a semi-generalized Ricci-Recurent Lorentzian \( \alpha \)-Sasakian manifold admits a quarter-symmetric metric connection, then
\[ \alpha_1(X) = -\frac{n}{(n - 1)(\alpha^2 - \alpha)}\beta_1(X) \]
holds, that is, the 1-form \( \alpha_1 \) and \( \beta_1 \) are in opposite direction.

A Lorentzian \( \alpha \)-Sasakian manifold \((M^n, g)\) with respect to quarter-symmetric metric connection is said to be an Einstein manifold if its Ricci tensor \( \tilde{S} \) is of the form
\[ \tilde{S}(X, Y) = kg(X, Y), \]
(7.9)

where \( k \) is constant. For an Einstein manifold,
\[ (\nabla_U \tilde{S}) = 0 \]
\( \forall \ U \in \chi(M) \). From (7.1), we have
\[ [k\alpha_1(X) + n\beta_1(X)]g(Y, Z) + [k\alpha_1(y) + n\beta_1(y)]g(Z, X) + [k\alpha_1(Z) + n\beta_1(Z)]g(X, Y) = 0. \]
(7.10)

Putting \( Y = \xi \) in (7.10) and using (1.5) and (2.4), we obtain
\[ [k\alpha_1(X) + n\beta_1(X)]\eta(Y) + [k\alpha_1(y) + n\beta_1(y)]\eta(X) + [k\alpha_1(Z) + n\beta_1(Z)]g(X, Y) = 0. \]
(7.11)

Now putting \( X = Y = \xi \) in (7.11) and using (1.5), (2.2) and (2.4), we obtain
\[ k\eta(A) + n\eta(B) = 0, \]
(7.12)

that is
\[ \eta(A) = -\frac{n}{k}\eta(B). \]
(7.13)
Using (1.5) and (2.4) in the above relation, we have
\[ \alpha_1(\xi) = -\frac{n}{k}\beta_1(\xi). \] (7.14)

So, we have the following theorem:

**Theorem 7.2.** If a semi-generalized Ricci-recurrent Lorentzian \( \alpha \)-Sasakian manifold \( M \) admitting a quarter-symmetric metric connection is an Einstein manifold, then the contact form \( \eta(A) \) and \( \eta(B) \) and the 1-form \( \alpha_1 \) and \( \beta_1 \) are both in opposite direction.

8 Example of 3-dimensional Lorentzian \( \alpha \)-Sasakian manifold with respect to quarter-symmetric metric connection

We consider a 3-dimensional manifold \( M = \{(x, y, u) \in \mathbb{R}^3\} \), where \( (x, y, u) \) are the standard coordinates of \( \mathbb{R}^3 \). Let \( e_1, e_2, e_3 \) be the vector fields on \( M \) given by
\[ e_1 = e^{-u}\frac{\partial}{\partial x}, \quad e_2 = e^{-u}\frac{\partial}{\partial y}, \quad e_3 = e^{-u}\frac{\partial}{\partial u}. \]

Clearly, \( \{e_1, e_2, e_3\} \) is a set of linearly independent vectors for each point of \( M \) and hence a basis of \( \chi(M) \). The Lorentzian metric \( g \) is defined by
\[ g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0, \]
\[ g(e_1, e_1) = 1, \quad g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1. \]

Let \( \eta \) be the 1-form defined by \( \eta(Z) = g(Z, e_3) \) for any \( Z \in \chi(M) \) and the \((1,1)\) tensor field \( \phi \) is defined by
\[ \phi e_1 = e_1, \quad \phi e_2 = e_2, \quad \phi e_3 = 0. \]

From the linearity of \( \phi \) and \( g \), we have
\[ \eta(e_3) = -1, \]
\[ \phi^2 X = X + \eta(X)e_3 \]
and
\[ g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y) \]
for any \( X \in \chi(M) \). Then for \( e_3 = \xi \), the structure \((\phi, \xi, \eta, g)\) defines a Lorentzian paracontact structure on \( M \).

Let \( \nabla \) be the Levi-Civita connection with respect to the Lorentzian metric \( g \). Then we have
\[ [e_1, e_2] = 0, \quad [e_1, e_3] = e_1e^{-u}, \quad [e_2, e_3] = e_2e^{-u}. \]
Koszul’s formula is defined by

\[ 2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \]

\[ -g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \]

Then from above formula we can calculate the followings:

\[ \nabla_{e_1} e_1 = e_3 e^{-u}, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = e_1 e^{-u}, \]

\[ \nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = e_3 e^{-u}, \quad \nabla_{e_2} e_3 = e_2 e^{-u}, \]

\[ \nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0. \]

From the above calculations, we see that the manifold under consideration satisfies \( \eta(\xi) = -1 \) and \( \nabla_X \xi = \alpha \phi X \) for \( \alpha = e^{-u} \).

Hence the structure \((\phi, \xi, \eta, g)\) is a Lorentzian \(\alpha\)-Sasakian manifold.

Using (2.16), we find \(\tilde{\nabla}\), the quarter-symmetric metric connection on \(M\) following:

\[ \tilde{\nabla}_{e_1} e_1 = e_3 e^{-u}, \quad \tilde{\nabla}_{e_1} e_2 = 0, \quad \tilde{\nabla}_{e_1} e_3 = e_1 (e^{-u} - 1), \]

\[ \tilde{\nabla}_{e_2} e_1 = 0, \quad \tilde{\nabla}_{e_2} e_2 = e_3 e^{-u}, \quad \tilde{\nabla}_{e_2} e_3 = e_2 (e^{-u} - 1), \]

\[ \tilde{\nabla}_{e_3} e_1 = 0, \quad \tilde{\nabla}_{e_3} e_2 = 0, \quad \tilde{\nabla}_{e_3} e_3 = 0. \]

Using (1.2), the torson tensor \(T\), with respect to quarter-symmetric metric connection \(\tilde{\nabla}\) as follows:

\[ \tilde{T}(e_i, e_i) = 0, \quad \forall i = 1, 2, 3, \]

\[ \tilde{T}(e_1, e_2) = 0, \quad \tilde{T}(e_1, e_3) = -e_1, \quad \tilde{T}(e_2, e_3) = -e_2. \]

Also,

\[ (\tilde{\nabla}_{e_1} g)(e_2, e_3) = 0, \quad (\tilde{\nabla}_{e_2} g)(e_3, e_1) = 0, \quad (\tilde{\nabla}_{e_3} g)(e_1, e_2) = 0. \]

Thus \(M\) is Lorentzian \(\alpha\)-Sasakian manifold with quarter-symmetric metric connection \(\tilde{\nabla}\).

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**References**


Some properties of Lorentzian \( \alpha \)-Sasakian manifolds


