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Existence and Stability of Periodic Solutions for Nonlinear Neutral Differential Equations with Variable Delay Using Fixed Point Technique

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Abstract

Our paper deals with the following nonlinear neutral differential equation with variable delay

$$\frac{d}{dt}Du_t(t) = p(t) - a(t)u(t) - a(t)g(u(t-\tau(t))) - h(u(t), u(t-\tau(t))).$$

By using Krasnoselskii's fixed point theorem we obtain the existence of periodic solution and by contraction mapping principle we obtain the uniqueness. A sufficient condition is established for the positivity of the above equation. Stability results of this equation are analyzed. Our results extend and complement some results obtained in the work [13].

Key words: Fixed point theorem, contraction, compactness, neutral differential equation, integral equation, periodic solution, positive solution, stability.

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1 Introduction

Delay differential equations have attracted a rapidly growing attention in the field of nonlinear dynamics and have become a powerful tool for investigating the complexities of the real-world problems such as infectious diseases, biotic population, neuronal networks, and even economics and finance. When employing delay differential equations to solve practical problems, it is very crucial to be able to completely characterize the dynamical properties of the delay differential equations.

Lyapunov functions and functionals have been successfully used to obtain boundedness, stability and the existence of periodic solutions of differential and functional differential equations with functional delays. In the study of differential equations with functional delays by using Lyapunov functionals, many difficulties arise if the delay is unbounded or if the differential equation in question has unbounded terms, see [3, 4, 6, 8]. In recent years, several investigators have tried stability by using a new technique. Particularly, Burton, Furumochi, Zhang and others began a study in which they noticed that some of these difficulties vanish or might be overcome by means of fixed point theory (see [1, 2, 5, 6, 7, 11, 13, 14]). The most striking object is that the fixed point method does not only solve the problem but has a significant advantage over Liapunov's direct method. While it remains an art to construct a Liapunov's functional when it exists, a fixed point method, in one step, yields existence, uniqueness and stability. All we need, to use the fixed point method, is a complete metric space, a suitable fixed point theorem and an elementary integral methods to solve problems that have frustrated investigators for decades.

Y. Yuan and Z. Guo in [13], discussed the existence of periodic solutions and stability for the following neutral functional differential equation

$$\frac{d}{dt}Du_t(t) = p(t) - au(t) - qau(t-\tau) - h(u(t), u(t-\tau)), \tag{1.1}$$

where $Du_t(t) = u(t) - qu(t-\tau)$, $|q| < 1$, $a > 0$, $\tau > 0$, $h \in C(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ and $p \in C(\mathbb{R}, \mathbb{R})$. Such a kind of NFDE has been used for the study of distributed networks containing a transmission line [9, 10].

In this paper, we are interested on the existence of positive periodic solutions and stability of the following nonlinear neutral differential equation

$$\frac{d}{dt}Du_t(t) = p(t) - a(t)u(t) - a(t)g(u(t-\tau(t))) - h(u(t), u(t-\tau(t))). \tag{1.2}$$

where $Du_t(t) = u(t) - g(u(t-\tau(t)))$, a, p, τ are real valued continuous functions with a and τ are positive functions. The functions $h\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$ are continuous in their respective arguments. It is easy to see that the equation (1.2) reduce to the equation (1.1) when, $\tau(t) = \tau$ is a constant, $a(t) = a$ is a strictly positive constant and $g(u(t-\tau(t))) = qu(t-\tau)$ with $|q| < 1$.

The outline of this work is as follows. In Section 2, we introduce the functional setting of the problem and fix the different notations and facts needed in the sequel. Section 3 is devoted to the existene and uniqueness of periodic solution of the equation (1.2). In Section 4, we give a sufficient conditions to ensure the positivity solution of (1.2). The stability of the periodic solution is the topic of Section 5.

2 Preliminaries

For $T > 0$, define $\mathcal{C}_T = \{\varphi\colon \varphi \in C(\mathbb{R},\mathbb{R}), \varphi(t+T) = \varphi(t)\}$, where $C(\mathbb{R},\mathbb{R})$ is the space of all real valued continuous functions. Then $\mathcal{C}_T$ is a Banach space when it is endowed with the supremum norm

$$\|\varphi\| = \max_{t\in[0,T]} |\varphi(t)|.$$

Since we are searching for the existence of periodic solutions for the equation (1.2), it is natural to assume that

$$a(t+T) = a(t),\ p(t+T) = p(t), \quad \tau(t+T) = \tau(t), \tag{2.1}$$

with $\tau(t) \geq \tau^* > 0$ and

$$\int_0^T a(r)dr > 0. \tag{2.2}$$

The functions $g(\cdot)$, $h(\cdot,\cdot)$ are also globally Lipschitz continuous in x and in x and y, respectively. That, there are a positive constants k_1, k_2 and k_3, such that

$$|g(x) - g(y)| \leq k_1\|x-y\| \text{ and } k_1 < 1, \tag{2.3}$$

$$|h(x,y) - h(z,w)| \leq k_2\|x-z\| + k_3\|y-w\|. \tag{2.4}$$

Lemma 1 *Suppose (2.1) and (2.2) hold. If $u \in \mathcal{C}_T$, then u is a solution of the equation (1.2) if and only if*

$$\begin{aligned} u(t) &= g(u(t-\tau(t))) \\ &\quad + \gamma\int_t^{t+T} [p(s) - 2a(s)g(u(s-\tau(s))) - h(u(s), u(s-\tau(s)))]e^{-\int_s^t a(r)dr}ds, \end{aligned} \tag{2.5}$$

where

$$\gamma = (e^{\int_0^T a(r)dr} - 1)^{-1}.$$

Proof Let $u \in \mathcal{C}_T$ be a solution of (1.2). Multiply both sides of the equation (1.2) by $e^{\int_0^t a(r)dr}$ and then integrate from t to $t+T$, to obtain

$$\begin{aligned} &\int_t^{t+T} [Du_s(s)]' e^{\int_0^s a(r)dr} ds \\ &= \int_t^{t+T} [p(s) - a(s)u(s) - a(s)g(u(s-\tau(s))) - h(u(s), u(s-\tau(s)))]e^{\int_0^s a(r)dr}ds. \end{aligned}$$

Performing an integration by part, we obtain

$$\begin{aligned} &Du_t(t)e^{\int_0^t a(r)dr}(e^{\int_0^T a(r)dr}-1)-\int_t^{t+T} a(s)Du_s(s)e^{\int_0^s a(r)dr}ds \\ &=-\int_t^{t+T} a(s)[u(s)-g(u(s-\tau(s)))]e^{\int_0^s a(r)dr}ds \\ &+\int_t^{t+T}[p(s)-2a(s)g(u(s-\tau(s)))-h(u(s),u(s-\tau(s)))]e^{\int_0^s a(r)dr}ds. \end{aligned}$$

By dividing both sides of the above equation by $e^{\int_0^t a(r)dr}(e^{\int_0^T a(r)dr}-1)$, we arrive at

$$\begin{aligned} u(t) &= g(u(t-\tau(t)))+(e^{\int_0^T a(r)dr}-1)^{-1} \\ &\times\int_t^{t+T}[p(s)-2a(s)g(u(s-\tau(s)))-h(u(s),u(s-\tau(s)))]e^{-\int_s^t a(r)dr}ds. \end{aligned}$$

The converse implication is easily obtained and the proof is complete. □

We end this section by stating the fixed point theorems that we employ to help us show the existence and stability of solutions to equation (1.2); see [6, 12].

Theorem 1 (Contraction Mapping Principle) *Let $(\mathcal{X},\rho)$ a complete metric space and let $P\colon \mathcal{X}\to\mathcal{X}$. If there is a constant $\alpha<1$ such that for any $x,y\in\mathcal{X}$ we have*

$$\rho(Px,Py)\le\alpha\rho(x,y),$$

then there is one and only one point $z\in\mathcal{X}$ with $Pz=z$.

Theorem 2 (Krasnoselskii) *Let $\mathbb{M}$ be a closed bounded convex nonempty subset of a Banach space $(\mathcal{X},\|\cdot\|)$. Suppose that A and B map $\mathbb{M}$ into $\mathcal{X}$ such that*

(i) A is compact and continuous,

(ii) B is a contraction mapping,

(iii) $x,y\in\mathbb{M}$, implies $Ax+By\in\mathbb{M}$,

Then there exists $z\in\mathbb{M}$ with $z=Az+Bz$.

3 Existence and uniqueness of periodic solution

By applying Theorems 1 and 2, we obtain in this Section the existence and the uniqueness of periodic solution of (1.2). So, let a Banach space $(\mathcal{C}_T,\|\cdot\|)$, a closed bounded convex subset of $\mathcal{C}_T$,

$$\mathcal{M}=\{\varphi\in\mathcal{C}_T,\ \|\varphi\|\le L\},\tag{3.1}$$

with $L>0$, and by the Lemma 1, we define the mapping $\mathcal{P}$ given by

$$\begin{aligned} (\mathcal{P}\varphi)(t) &= g(\varphi(t-\tau(t))) \\ &+\gamma\int_t^{t+T}[p(s)-2a(s)g(\varphi(s-\tau(s)))-h(\varphi(s),\varphi(s-\tau(s)))]e^{-\int_s^t a(r)dr}ds. \end{aligned}\tag{3.2}$$

Therefore, we express equation (3.2) as

$$\mathcal{P}\varphi = \mathcal{A}\varphi + \mathcal{B}\varphi,$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by

$$(\mathcal{A}\varphi)(t) = \gamma \int_t^{t+T} [p(s) - 2a(s)g(\varphi(s-\tau(s))) - h(\varphi(s), \varphi(s-\tau(s)))] e^{-\int_s^t a(r)dr} ds, \tag{3.3}$$

and

$$(\mathcal{B}\varphi)(t) = g(\varphi(t-\tau(t))). \tag{3.4}$$

Since $\varphi \in \mathcal{C}_T$ and (2.1) holds, we have for any $\varphi \in \mathcal{M}$

$$\begin{aligned}(\mathcal{A}\varphi)(t+T) &= \gamma \int_{t+T}^{t+T+T} [p(s) - 2a(s)g(\varphi(s-\tau(s))) \\ &- h(\varphi(s), \varphi(s-\tau(s)))] e^{-\int_s^{t+T} a(r)dr} ds \\ &= \gamma \int_t^{t+T} [p(s+T) - 2a(s+T)g(\varphi(s+T-\tau(s+T))) \\ &- h(\varphi(s+T), \varphi(s+T-\tau(s+T)))] e^{-\int_{s+T}^{t+T} a(r)dr} ds \\ &= (\mathcal{A}\varphi)(t),\end{aligned}$$

and

$$(\mathcal{B}\varphi)(t+T) = g(\varphi(t+T-\tau(t+T))) = g(\varphi(t-\tau(t))) = (\mathcal{B}\varphi)(t).$$

Then

$$\mathcal{A}\mathcal{M}, \mathcal{B}\mathcal{M} \subset \mathcal{C}_T. \tag{3.5}$$

Theorem 3 *Assume that (2.1)–(2.4) hold. Let a constant $L > 0$ defined in $\mathcal{M}$ such that*

$$k_1 L + |g(0)| + \gamma\beta T(\mu + 2\lambda k_1 L + |g(0)| + k_2 L + k_3 L + |h(0,0)|) \leq L \tag{3.6}$$

where

$$\beta = e^{\int_0^T a(r)dr}, \qquad \lambda = \sup_{t\in[0,T]} \{a(t)\}, \qquad \mu = \sup_{t\in[0,T]} |p(t)|.$$

Then (1.2) has a T-periodic solution.

Proof First, let $\mathcal{A}$ defined by (3.3), we show that $\mathcal{A}$ is continuous in the supremum norm and the image of $\mathcal{A}$ is contained in a compact set. Let $\varphi_n \in \mathcal{M}$ where n is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$\begin{aligned}&|(\mathcal{A}\varphi_n)(t) - (\mathcal{A}\varphi)(t)| \\ &\leq 2\gamma \int_t^{t+T} a(s)|g(\varphi_n(s-\tau(s))) - g(\varphi(s-\tau(s)))| e^{-\int_s^t a(r)dr} ds \\ &+ \gamma \int_t^{t+T} |h(\varphi_n(s), \varphi_n(s-\tau(s))) - h(\varphi(s), \varphi(s-\tau(s)))| e^{-\int_s^t a(r)dr} ds.\end{aligned}$$

Since g and h are continuous, the Dominated Convergence Theorem implies,

$$\lim_{n\to\infty} |(\mathcal{A}\varphi_n)(t) - (\mathcal{A}\varphi)(t)| = 0,$$

then $\mathcal{A}$ is continuous. Now, by (2.3) and (2.4), we obtain

$$\begin{aligned} |g(y)| &\le k_1|y| + |g(0)|, \\ |h(x,y)| &\le k_2|x| + k_3|y| + |h(0,0)|. \end{aligned}$$

Then, let $\varphi_n \in \mathcal{M}$ where n is a positive integer, we have

$$\begin{aligned} &|(\mathcal{A}\varphi_n)(t)| \\ &\le \gamma \int_t^{t+T} [p(s) - a(s)g(\varphi_n(s-\tau(s))) - h(\varphi_n(s), \varphi_n(s-\tau(s)))] e^{-\int_s^t a(r)dr} ds \\ &\le \gamma \int_t^{t+T} [|p(s)| + 2a(s)(k_1\|\varphi_n\| + |g(0)|) \\ &+ k_2\|\varphi_n\| + k_3\|\varphi_n\| + |h(0,0)|] e^{-\int_s^t a(r)dr} ds \\ &\le \gamma\beta T(\mu + 2\lambda(k_1 L + |g(0)|) + k_2 L + k_3 L + |h(0,0)|) \le L, \end{aligned}$$

by (3.6). Next, we calculate $(\mathcal{A}\varphi_n)'(t)$ and show that it is uniformly bounded. By making use of (2.1) we obtain by taking the derivative in (3.3) that

$$\begin{aligned} (\mathcal{A}\varphi_n)'(t) &= -a(t)(\mathcal{A}\varphi_n)(t) \\ &\quad + p(t) - 2a(t)g(\varphi_n(t-\tau(t))) - h(\varphi_n(t), \varphi_n(t-\tau(t))). \end{aligned}$$

Then, by (2.4) and (3.6) we have

$$|(\mathcal{A}\varphi_n)'(t)| \le \lambda L + \mu + 2\lambda(k_1 L + |g(0)|) + k_2 L + k_3 L + |h(0,0)| = Q,$$

Thus the sequence $(\mathcal{A}\varphi_n)$ is uniformly bounded and equicontinuous. Hence by Ascoli–Arzela's theorem $\mathcal{AM}$ is compact.

Second, let $\mathcal{B}$ be defined by (3.4). Then for $\varphi_1, \varphi_2 \in \mathcal{M}$ we have by (2.3)

$$\begin{aligned} |(\mathcal{B}\varphi_1)(t) - (\mathcal{B}\varphi_2)(t)| &= |g(\varphi_1(t-\tau(t))) - g(\varphi_2(t-\tau(t)))| \\ &\le k_1\|\varphi_1 - \varphi_2\|. \end{aligned}$$

Hence $\mathcal{B}$ is contraction because $k_1 < 1$.

Finally, we show that if $\varphi, \phi \in \mathcal{M}$, then $\|A\varphi + B\phi\| \le L$. Let $\varphi, \phi \in \mathcal{M}$ with $\|\varphi\|, \|\phi\| \le L$, then

$$\begin{aligned} \|A\varphi + B\phi\| \le k_1\|\phi\| + |g(0)| + \gamma \int_t^{t+T} [|p(s)| + 2a(s)(k_1\|\varphi\| + |g(0)|) \\ + k_2\|\varphi\| + k_3\|\varphi\| + |h(0,0)|] e^{-\int_s^t a(r)dr} ds \le k_1 L + |g(0)| \\ + \gamma\beta T(\mu + 2\lambda(k_1 L + |g(0)|) + k_2 L + k_3 L + |h(0,0)|) \le L, \end{aligned}$$

by (3.6). Clearly, all the hypotheses of the Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 1 this fixed point is a solution of (1.2). Hence (1.2) has a T-periodic solution. □

Remark 1 Note that, when $\tau(t) = \tau$ is a positive constant, $a(t) = a$ is a strictly positive constant and $g(u(t-\tau(t))) = qu(t-\tau)$ with $|q| < 1$. Theorem 3 reduces to Theorem 2 in [13].

Theorem 4 *Suppose (2.1)–(2.4) hold. If*

$$k_1 + \gamma\beta T(2\lambda k_1 + k_2 + k_3) < 1, \tag{3.7}$$

then equation (1.2) has a unique T-periodic solution.

Proof Let the mapping $\mathcal{P}$ be given by (3.2). For any $\varphi_1, \varphi_2 \in \mathcal{C}_T$, we have

$$\begin{aligned}
&|(\mathcal{P}\varphi_1)(t) - (\mathcal{P}\varphi_2)(t)| \\
&\le |g(\varphi_1(t-\tau(t))) - g(\varphi_2(t-\tau(t)))| \\
&+ 2\gamma \int_t^{t+T} a(s)|g(\varphi_1(s-\tau(s))) - g(\varphi_2(s-\tau(s)))| e^{-\int_s^t a(r)dr} ds \\
&+ \gamma \int_t^{t+T} |h(\varphi_1(s), \varphi_1(s-\tau(s))) - h(\varphi_2(s), \varphi_2(s-\tau(s)))| e^{-\int_s^t a(r)dr} ds \\
&\le k_1\|\varphi_1 - \varphi_2\| + \gamma \int_t^{t+T} (2\lambda k_1 + k_2 + k_3)\|\varphi_1 - \varphi_2\| e^{-\int_s^t a(r)dr} ds \\
&\le [k_1 + \gamma\beta T(2\lambda k_1 + k_2 + k_3)]\|\varphi_1 - \varphi_2\|,
\end{aligned}$$

Since (3.7) hold, the contraction mapping principle completes the proof. □

Corollary 1 *Suppose (2.1)–(2.4) hold and let β, λ, μ be a constant defined in Theorem 3. Let $\mathcal{M}$ defined by (3.1). Suppose there are positive constants k_1^*, k_2^* and k_3^*, such that for any $x, y, z, w \in \mathcal{M}$, we have*

$$|g(x) - g(y)| \le k_1^*\|x - y\| \text{ and } k_1^* < 1, \tag{3.8}$$

$$|h(x, y) - h(z, w)| \le k_2^*\|x - z\| + k_3^*\|y - w\|, \tag{3.9}$$

and

$$k_1^*L + |g(0)| + \gamma\beta T(\mu + 2\lambda(k_1^*L + |g(0)|) + k_2^*L + k_3^*L + |h(0,0)|) \le L. \tag{3.10}$$

If $\|\mathcal{P}\varphi\| \le L$, for any $\varphi \in \mathcal{M}$, then (1.2) has a T-periodic solution in $\mathcal{M}$. Moreover, if

$$k_1^* + \gamma\beta T(2\lambda k_1^* + k_2^* + k_3^*) < 1, \tag{3.11}$$

then (1.2) has a unique solution in $\mathcal{M}$.

Proof Let the mapping $\mathcal{P}$ defined by (3.2). Then the proof follow immediately from Theorem 3 and Theorem 4. □

Notice that the constants k_1^*, k_2^* and k_3^* may depend on L.

Remark 2 Y. Yuan and Z. Guo are not obtained the uniqueness of the solution for the equation (1.1). But here, the equation (1.1) is special case for our results in Theorem 4 and Corollary 1.

4 Existence of positive periodic solution

It is for sure that existence of positive solutions is important for many applied problems. In this Section, by applying the Krasnoselskii's fixed point theorem and some techniques, to establish a set of sufficient conditions which guarantee the existence of positive periodic solutions of (1.2). So, we let $(\mathcal{X}, \|.\|) = (\mathcal{C}_T, \|.\|)$ and $\mathcal{M}(E,K) = \{\varphi \in C_T \colon E \leq \varphi(t) \leq K, \ \forall t \in [0,T]\}$, for any $0 < E < K$. We assume that, there exist constants E, K, a_1, a_2, g_1 and g_2 such that for all $(t,(x,y)) \in [0,T] \times [E,K]^2$ we have

$$0 \leq g_1, \quad 0 \leq g_2 < 1, \quad -g_1 y \leq g(y) \leq g_2 y, \tag{4.1}$$

$$0 < a_1 \leq a(t) \leq a_2, \tag{4.2}$$

$$(E + g_1 K)a_2 \leq p(t) - 2a(t)g(y) - h(x,y) \leq (1 - g_2)Ka_1. \tag{4.3}$$

Theorem 5 *Assume that (2.1)–(2.4) and (4.1)–(4.3) hold. Then (1.2) has at least one positive T-periodic solution in $\mathcal{M}(E,K)$.*

Proof By Lemma 1, it is obvious that (1.2) has a solution φ if and only if the equation $\mathcal{P}\varphi = \varphi$ has a solution φ. Let $\mathcal{A}$, $\mathcal{B}$ defined by (3.3), (3.4) respectively. A change of variable $t \mapsto t+T$ in (3.3) and (3.4) show that for any $\varphi \in \mathcal{M}(E,K)$ and $t \in \mathbb{R}$

$$\mathcal{A}(\mathcal{M}(E,K)) \subseteq \mathcal{C}_T, \qquad \mathcal{B}(\mathcal{M}(E,K)) \subseteq \mathcal{C}_T. \tag{4.4}$$

Arguing as in the Theorem 3, the operator $\mathcal{A}$ is continuous. Next, we claim that $\mathcal{A}$ is compact. It is sufficient to show that $\mathcal{A}(\mathcal{M}(E,K))$ is uniformly bounded and equicontinuous in $[0,T]$. Notice that (4.2) and (4.3) ensure that

$$\begin{aligned}
\|\mathcal{A}\varphi\| \leq& \sup_{t\in[0,T]} |\gamma \int_t^{t+T} [p(s) - 2a(s)g(\varphi(s-\tau(s))) \\
&- h(\varphi(s), \varphi(s-\tau(s)))] e^{-\int_s^t a(r)dr} ds| \\
&\leq (1-g_2)K\gamma a_1 \sup_{t\in[0,T]} \int_t^{t+T} e^{-\int_s^t a(r)dr} ds \\
&\leq (1-g_2)K, \quad \forall \varphi[E,K],
\end{aligned}$$

and

$$\begin{aligned}
|(\mathcal{A}\varphi)'(t)| &\leq a(t)(\mathcal{A}\varphi)(t) + |p(t) - 2a(t)g(\varphi(t-\tau(t))) - h(\varphi(t), \varphi(t-\tau(t)))| \\
&\leq a_2(1-g_1)K + (1-g_1)a_1 K \\
&= (a_1 + a_2)(1-g_1)K, \quad \forall (t,\varphi) \in [0,T] \times [E,K],
\end{aligned}$$

which give that $\mathcal{A}(\mathcal{M}(E,K))$ is uniformly bounded and equicontinuous in $[0,T]$. Hence by Ascoli–Arzela's theorem $\mathcal{A}$ is compact. Next, let $\mathcal{B}$ defined by (3.4), for all $\varphi_1, \varphi_2 \in \mathcal{M}(E,K)$ and $t \in \mathbb{R}$, we obtain by (2.3) $\|\mathcal{B}\varphi_1 - \mathcal{B}\varphi_2\| \leq k_1\|\varphi_1 - \varphi_2\|$.

Thus $\mathcal{B}$ is a contraction. Moreover, by (4.1)–(4.3), we infer that for all $\varphi, \phi \in \mathcal{M}(E, K)$ and $t \in \mathbb{R}$

$$\begin{aligned}&(\mathcal{A}\varphi)(t) + (\mathcal{B}\phi)(t) = g(\phi(t-\tau(t)))\\ &+\gamma\int_t^{t+T}[p(s) - 2a(s)g(\varphi(s-\tau(s))) - h(\varphi(s), \varphi(s-\tau(s)))]e^{-\int_s^t a(r)dr}ds\\ &\le g_2K + (1-g_2)K\gamma\int_t^{t+T} a(s)e^{-\int_s^t a(r)dr} = K,\end{aligned}$$

on the other hand,

$$\begin{aligned}&(\mathcal{A}\varphi)(t) + (\mathcal{B}\phi)(t) \ge g(\phi(t-\tau(t)))\\ &+\gamma\int_t^{t+T}[p(s) - 2a(s)g(\varphi(s-\tau(s))) - h(\varphi(s), \varphi(s-\tau(s)))]e^{-\int_s^t a(r)dr}ds\\ &\ge -g_1K + (E+g_1)K\gamma\int_t^{t+T} a(s)e^{-\int_s^t a(r)dr} = E,\end{aligned}$$

which imply that

$$(\mathcal{A}\varphi)(t) + (\mathcal{B}\phi)(t) \in \mathcal{M}(E, K) \quad \text{for all } \varphi, \phi \in \mathcal{M}(E, K) \text{ and } t \in \mathbb{R}. \tag{4.5}$$

Clearly, all the hypotheses of the Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}(E, K)$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 1 this fixed point is a solution of (1.2). Hence (1.2) has a positive T-periodic solution. This completes the proof. □

Theorem 6 *Assume that (2.1)–(2.4) hold. Suppose that there exist constants E, K, a_1, a_2, g_1, g_2 and $t_0 \in [0, T]$ satisfying (4.1)–(4.3) with*

$$0 \le E < K, \tag{4.6}$$

and either

$$(E + g_1K)a_2 < p(t_0) - 2a(t_0)g(y) - h(x, y), \quad \forall x, y \in [E, K], \tag{4.7}$$

or

$$a(t_0) < a_2. \tag{4.8}$$

Then (1.2) has at least one positive T-periodic solution in $\mathcal{M}(E, K)$ with $E < u(t) \le K$ for each $t \in [0, T]$.

Proof As in the proof of Theorem 5, we conclude similarly that (1.2) has an T-periodic solution $u \in \mathcal{M}(E, K)$. Now we assert that $u(t) > E$ for all $t \in [0, T]$. Otherwise, there exists $t^* \in [0, T]$ satisfying $u(t^*) = E$. In view of (2.5), (3.2) and (4.1), (4.6), we have

$$
\begin{aligned}
E &= g(u(t^* - \tau(t^*))) \\
&+ \gamma \int_{t^*}^{t^*+T} [p(s) - 2a(s)g(u(s-\tau(s))) \\
&- h(u(s), u(s-\tau(s)))]e^{-\int_s^{t^*} a(r)dr} ds \\
&\geq \gamma \int_{t^*}^{t^*+T} [p(s) - 2a(s)g(u(s-\tau(s))) \\
&- h(u(s), u(s-\tau(s)))]e^{-\int_s^{t^*} a(r)dr} ds - g_1 K,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
0 &\geq \gamma \int_{t^*}^{t^*+T} [p(s) - 2a(s)g(u(s-\tau(s))) \\
&- h(u(s), u(s-\tau(s)))]e^{-\int_s^{t^*} a(r)dr} ds - (E + g_1 K) \\
&= \gamma \int_{t^*}^{t^*+T} [p(s) - 2a(s)g(u(s-\tau(s))) \\
&- h(u(s), u(s-\tau(s))) - (E + g_1 K)a(s)]e^{-\int_s^{t^*} a(r)dr} ds \qquad (4.9)
\end{aligned}
$$

Assume that (4.7) holds. By means of (4.2), (4.3), (4.7), and the continuity of h, g, a, p, τ, and u, we get that

$$
\begin{aligned}
&\gamma \int_{t^*}^{t^*+T} [p(s) - 2a(s)g(u(s-\tau(s))) \\
&- h(u(s), u(s-\tau(s))) - (E + g_1 K)a(s)]e^{-\int_s^{t^*} a(r)dr} ds \\
&\geq \int_{t^*}^{t^*+T} e^{-\int_s^{t^*} a(r)dr} [p(s) - 2a(s)g(u(s-\tau(s))) \\
&- h(u(s), u(s-\tau(s))) - (E + g_1 K)a_2] ds > 0,
\end{aligned}
$$

which contradicts (4.9).

Assume that (4.8) holds. In light of (4.2), (4.3), (4.8), and the continuity of h, g, a, p, τ, and u, we get that

$$
\begin{aligned}
&\gamma \int_{t^*}^{t^*+T} [p(s) - 2a(s)g(u(s-\tau(s))) \\
&- h(u(s), u(s-\tau(s))) - (E + g_1 K)a(s)]e^{-\int_s^{t^*} a(r)dr} ds \\
&> \int_{t^*}^{t^*+T} e^{-\int_s^{t^*} a(r)dr} \int_{t^*}^{t^*+T} e^{-\int_s^{t^*} a(r)dr} [p(s) - 2a(s)g(u(s-\tau(s))) \\
&- h(u(s), u(s-\tau(s))) - (E + g_1 K)a_2] ds \geq 0,
\end{aligned}
$$

which contradicts (4.9). This completes the proof. □

Example 1 Consider (1.2), where

$$p(t) = 10 + \frac{\cos t}{10}, \quad a(t) = 1 + \frac{\sin t}{5}, \quad \tau(t) = 3\sin t,$$

$$g(x) = -\frac{x\cos x}{60}, \quad \forall x \in \mathbb{R},$$

$$h(y,x) = -2 - \cos^2 y - \sin^2(y), \quad \forall (y,x) \in \mathbb{R}^2.$$

Let $T = 2\pi$, $K = 40$, $E = 1$, $g_1 = g_2 = \frac{1}{60}$, $a_1 = \frac{4}{5}$, $a_2 = \frac{6}{5}$, $k_1 = \frac{43}{60}$. It is easy to see that (2.3), (2.4) hold. Notice that

$$\begin{aligned}(E + g_1K)a_2 &= \frac{9}{5} < 10 - \frac{1}{10} + \left(2 - \frac{1}{5}\right)\left(\frac{-1}{60}\right) + 2 \\ &\le p(t) - 2a(t)g(y) - h(x,y) \\ &\le 10 + \frac{1}{10} + \left(2 + \frac{1}{5}\right)\frac{1}{60}\cdot 40 + 4 \\ &< \frac{118}{5} = (1 - g_2)Ka_1, \quad \forall (t,x,y) \in \mathbb{R}^3.\end{aligned}$$

That is, (4.3) is satisfied. Thus Theorem 5 yields, that (1.2) has a positive 2π-periodic solution in $\mathcal{M}(1,40)$.

5 Stability of periodic solution

This Section concerned with the stability of a T-periodic solution u^* of (1.2). Let $v = u - u^*$ then (1.2) is transformed as

$$\frac{d}{dt}Dv_t(t) = -a(t)v(t) - a(t)G(v(t-\tau(t))) - H(v(t), v(t-\tau(t))), \tag{5.1}$$

where

$$Dv_t(t) = v(t) - G(v(t-\tau(t))),$$

$$G(v(t-\tau(t))) = g(u^*(t-\tau(t)) + v(t-\tau(t))) - g(u^*(t-\tau(t))),$$

and

$$\begin{aligned}H(v(t), v(t-\tau(t))) = {}& h(u^*(t) + v(t), u^*(t-\tau(t)) + v(t-\tau(t))) \\ & - h(u^*(t) + u^*(t-\tau(t))).\end{aligned}$$

Clearly, (5.1) has trivial solution $v \equiv 0$, and the conditions (2.3), (2.4) hold for G, H respectively. To arrive at the Lemma 1, as in the proof of this Lemma, multiply both sides of the equation (5.1) by $e^{\int_0^t a(r)dr}$ and then integrate from 0 to t, to obtain

$$\begin{aligned}v(t) = {}& (v(0) - G(v(-\tau(0))))e^{-\int_0^t a(r)dr} + G(v(t-\tau(t))) \\ & - \int_0^t [2a(t)G(v(t-\tau(t))) + H(v(s), v(s-\tau(s)))]e^{-\int_s^t a(r)dr}ds.\end{aligned} \tag{5.2}$$

Thus, we see that v is a solution of (5.1) if and only if it satisfies (5.2). Assumed initial function

$$\psi(t) = v(t), \quad t \in [m_0, 0],$$

with $\psi \in C([m_0, 0], \mathbb{R})$, $[m_0, 0] = \{s \le 0 \mid s = t - \tau(t),\ t \ge 0\}$.

For the stability definition we refer the reader to the book [6].

Define the set $\mathcal{S}_\psi$ by

$$\mathcal{S}_\psi = \{\varphi\colon \varphi \in \mathcal{C}_T,\ \|\varphi\| \le R,\ \varphi(t) = \psi(t) \text{ if } t \in [m_0, 0],\ \varphi(t) \to 0 \text{ as } t \to \infty\}, \tag{5.3}$$

for some positive constant R. Then, $(\mathcal{S}_\psi, \|\cdot\|)$ is a complete metric space where $\|\cdot\|$ is the supremum norm.

Theorem 7 *If (2.1), (2.3), (2.4) and*

$$\int_0^t a(r)dr > 0 \text{ and } e^{-\int_0^t a(r)dr} \to 0 \text{ as } t \to \infty, \tag{5.4}$$

$$t - \tau(t) \to \infty \text{ as } t \to \infty, \tag{5.5}$$

$$k_1 + \int_0^t (2\lambda k_1 + k_2 + k_3) e^{-\int_s^t a(r)dr} ds \le \alpha < 1, \tag{5.6}$$

hold. Then every solution $v(t, 0, \psi)$ of (5.1) with small continuous initial function ψ, is bounded and asymptotically stable.

Proof Let the mapping $\mathcal{F}$ defined by $\psi(t)$ if $t \le 0$ and

$$\begin{aligned}(\mathcal{F}\varphi)(t) &= (\psi(0) - G(\psi(-\tau(0))))e^{-\int_0^t a(r)dr} + G(\varphi(t - \tau(t))) \\ &- \int_0^t [2a(s)G(\varphi(s - \tau(s))) + H(\varphi(s), \varphi(s - \tau(s)))] e^{-\int_s^t a(r)dr} ds,\end{aligned} \tag{5.7}$$

if $t \ge 0$. Since H, H, is continuous, it is easy to show that $\mathcal{F}$ is. Let ψ be a small given continuous initial function with $\|\psi\| < \delta$ $(\delta > 0)$. Then using the condition (5.6) and the definition of $\mathcal{F}$ in (5.7), we have for $\varphi \in \mathcal{S}_\psi$

$$\begin{aligned}|(\mathcal{F}\varphi)(t)| &\le k_1 R + |\psi(0) - G(\psi(-\tau(0)))| e^{-\int_0^t a(r)dr} \\ &\quad + R \int_0^t (2\lambda k_1 + k_2 + k_3) e^{-\int_s^t a(r)dr} ds \\ &\le (1 + k_1)\delta + k_1 R + R \int_0^t (2\lambda k_1 + k_2 + k_3) e^{-\int_s^t a(r)dr} ds \\ &\le (1 + k_1)\delta + \alpha R \le R,\end{aligned}$$

which implies $\|\mathcal{F}\varphi\| \le R$, for the right δ. Next we show that $(\mathcal{F}\varphi)(t) \to 0$ as $t \to \infty$. The first term on the right side of (5.7) tends to zero, by condition (5.4). Also, the second term on the right side tends to zero, because of (5.5) and the fact that $\varphi \in \mathcal{S}_\psi$. Let $\epsilon > 0$ be given, then there exists a $t_1 > 0$ such that for $t > t_1$, $\varphi(t - \tau(t)) < \epsilon$. By the condition (5.4), there exists a $t_2 > t_1$ such that for $t > t_2$ implies that

$$e^{-\int_{t_2}^{t} a(r)dr} < \frac{\epsilon}{\alpha R}.$$

Thus for $t > t_2$, we have

$$\begin{aligned}
&\left| \int_0^t [2a(s)G(\varphi(s-\tau(s))) + H(\varphi(s), \varphi(s-\tau(s)))] e^{-\int_s^t a(r)dr} ds \right| \\
&\leq R \int_0^{t_1} (2\lambda k_1 + k_2 + k_3) e^{-\int_s^t a(r)dr} ds \\
&+ \epsilon \int_{t_1}^{t} (2\lambda k_1 + k_2 + k_3) e^{-\int_s^t a(r)dr} ds \\
&\leq R e^{-\int_{t_2}^{t} a(r)dr} \int_0^{t_1} (2\lambda k_1 + k_2 + k_3) e^{-\int_s^{t_2} a(r)dr} ds + \alpha\epsilon \\
&\leq \alpha R e^{-\int_{t_2}^{t} a(r)dr} \alpha + \alpha\epsilon < \alpha\epsilon + \epsilon.
\end{aligned}$$

Hence, $(\mathcal{F}\varphi)(t) \to 0$ as $t \to \infty$. It is natural now to prove that $\mathcal{F}$ is contraction under the supremum norm. Let , $\varphi_1, \varphi_2 \in \mathcal{S}_\psi$. Then

$$\begin{aligned}
&|(\mathcal{F}\varphi_1)(t) - (\mathcal{F}\varphi_2)(t)| \\
&\leq |G(\varphi_1(t-\tau(t))) - G(\varphi_2(t-\tau(t)))| \\
&+ 2\lambda \int_0^t |G(\varphi_1(s-\tau(s))) - G(\varphi_2(s-\tau(s)))| e^{-\int_s^t a(r)dr} ds \\
&+ \int_0^t |H(\varphi_1(s), \varphi_1(s-\tau(s))) - H(\varphi_2(s), \varphi_2(s-\tau(s)))| e^{-\int_s^t a(r)dr} ds \\
&\leq k_1 \|\varphi_1 - \varphi_2\| + \int_0^t (2\lambda k_1 + k_2 + k_3) \|\varphi_1 - \varphi_2\| e^{-\int_s^t a(r)dr} ds \\
&\leq [k_1 + \int_0^t (2\lambda k_1 + k_2 + k_3) e^{-\int_s^t a(r)dr}] \|\varphi_1 - \varphi_2\| \\
&\leq \alpha \|\varphi_1 - \varphi_2\|,
\end{aligned}$$

Hence, the contraction mapping principle implies, $\mathcal{F}$ has a unique fixed point in $\mathcal{S}_\psi$ which solves (5.1), bounded and asymptotically stable. □

Theorem 8 *If (2.1), (2.3), (2.4) and*

$$k_1 + \int_0^t (2\lambda k_1 + k_2 + k_3) e^{-\int_s^t a(r)dr} ds \leq \alpha < 1, \tag{5.8}$$

hold. Then, the zero solution is stable.

Proof The stability of the zero solution of (5.1) follows simply by replacing R by ϵ in the above Theorem. □

Remark 3 Notice that

(i) Our analysis of stability also applies to the more general case, when v is not periodic.

(ii) When $\tau(t) = \tau$ is a positive constant, $a(t) = a$ is a strictly positive constant and

$$g(u(t - \tau(t))) = qu(t - \tau)$$

with $|q| < 1$, Theorems 7 and 8 reduce to Theorems 2 and 4 in [13] respectively.

Remark 4 The authors of this paper have studied the asymptotic stability of the zero solution of (5.1) using fixed point theory. However, the question of uniform and exponential asymptotic stability of the zero solution of (5.1) remains open.

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