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Some Additive $2-(v, 5, \lambda)$ Designs

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Abstract

Given a finite additive abelian group G and an integer k, with $3 \leq k \leq |G|$, denote by $\mathcal{D}_k(G)$ the simple incidence structure whose point-set is G and whose blocks are the k-subsets $C = \{c_1, c_2, \ldots, c_k\}$ of G such that $c_1 + c_2 + \cdots + c_k = 0$. It is known (see [2]) that $\mathcal{D}_k(G)$ is a 2-design, if G is an elementary abelian p-group with p a prime divisor of k. From [3] we know that $\mathcal{D}_3(G)$ is a 2-design if and only if G is an elementary abelian 3-group. It is also known (see [4]) that G is necessarily an elementary abelian 2-group, if $\mathcal{D}_4(G)$ is a 2-design. Here we shall prove that $\mathcal{D}_5(G)$ is a 2-design if and only if G is an elementary abelian 5-group.

Key words: Conformal mapping, geodesic mapping, conformal-geodesic mapping, initial conditions, (pseudo-) Riemannian space.

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1 Introduction and preliminary results

Let v, k, t, λ be positive integers with $v > k > t$. By a $t-$design with parameters v, k, λ (or shortly: a $t-(v, k, \lambda)$ design) one understands a pair $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ where $\mathcal{P}$ is a finite set with v elements (called points) and $\mathcal{B}$ is a set of subsets of $\mathcal{P}$ called blocks such that each block contains k points and any t distinct points are contained in exactly λ common blocks (cf. [1], [5]). We say that a $t-(v, k, \lambda)$ design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is an additive design, if there are a finite abelian group G, written additively, and an injective mapping $\chi\colon \mathcal{P} \to G$ with the property that $\chi(c_1) + \chi(c_2) + \cdots + \chi(c_k) = 0$ whenever $C = \{c_1, c_2, \ldots, c_k\} \in \mathcal{B}$ is a block of $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ (cf. [2]). For every finite additive abelian group G and for any integer $k \in \{3, 4, \ldots, |G| - 1\}$ we denote by $\mathcal{D}_k(G)$ the simple incidence structure the point-set of which is G and the blocks of which are the k-subsets $C = \{c_1, c_2, \ldots, c_k\}$ of G such that $c_1 + c_2 + \cdots + c_k = 0$. Note that each 2-design of the form $\mathcal{D}_k(G)$ is an additive 2-design.

Throughout this paper we shall be concerned only with finite abelian groups, written additively. If G is such a group, the notation that follows will remain fixed: $|G|$ is the order of G; $\langle a \rangle$ is the subgroup of G generated by $a \in G$; if m is a positive integer, mG and G_m are the subgroups of G given by $mG = \{mg \mid g \in G\}$ and $G_m = \{g \in G \mid mg = 0\}$; if $|G| > 4$ and if x, y are distinct elements of G, $N_{x,y}$ denotes the number of pairs $\{c, C\}$ where $c \in G \setminus \{x, y\}$ and C is a block of $\mathcal{D}_5(G)$ through $\{x, y, c\}$.

We state now some preliminary results.

Lemma 1 *If $\mathcal{D}_5(G)$ is a $2-(|G|, 5, \lambda)$ design for some λ, then $N_{x,y}$ is a constant (equal to 3λ).*

Proof Suppose $\mathcal{D}_5(G)$ is a $2-(|G|, 5, \lambda)$ design for some λ. Then there are λ blocks of $\mathcal{D}_5(G)$ through any given two distinct elements $x, y \in G$; on the other hand, each block of $\mathcal{D}_5(G)$ through $\{x, y\}$ contains exactly 3 points distinct from x, y. Therefore $N_{x,y} = 3\lambda$ and the Lemma 1 is proved. □

Proposition 1 *$\mathcal{D}_5(G)$ is not a 2-design if one of the statements below is true:*

1) G is an elementary abelian 2-group;

2) G is direct sum of cyclic groups of order 4;

3) G is direct sum of groups of order 2 and cyclic groups of order 4;

4) G contains just one involution and $2G$ is an elementary abelian 3-group.

Proof We may assume that G has order greater than 4.

1) Suppose G is an elementary abelian 2-group of order $n = 2^\nu \geq 8$. Let $g \in G$, $g \neq 0 \in G$ and let $x \in G \setminus \{0, g\}$. We show that $N_{0,g} \neq N_{x,g}$ and hence, by Lemma 1, $\mathcal{D}_5(G)$ is not a 2-design. There are no blocks of $\mathcal{D}_5(G)$ through $\{0, g, x, g+x\}$, however $\{0, g, x\}$ may be extended to a block $\{0, g, x, y, g+x+y\}$ for every $y \in G \setminus \{0, g, x, g+x\}$. Therefore

$$N_{0,g} = (n-2)\frac{n-4}{2}.$$

There are no blocks of $\mathcal{D}_5(G)$ through $\{g, x, g+x\}$, however there are $\frac{n-4}{2}$ blocks through $\{0, g, x\}$ and $\frac{n-6}{2}$ blocks through $\{x, g, z\}$ for any given $z \in G \setminus \{0, g, x, g+x\}$. Therefore

$$N_{x,g} = \frac{n-4}{2} + (n-4)\frac{n-6}{2}.$$

From $n \neq 4$ it follows $N_{0,g} \neq N_{x,g}$ and hence $\mathcal{D}_5(G)$ is not a 2-design.

2) Suppose G is direct sum of $\nu \geq 2$ cyclic groups of order 4. So G is a finite abelian group of order $n = 4^\nu \geq 16$ and $2G = G_2$ is an elementary abelian 2-group of order $2^\nu \geq 4$.

Let $a \in G_2$, $a \neq 0$ and let $b \in G_4 \setminus G_2$. We show that $N_{0,a} \neq N_{0,b}$ and hence, by Lemma 1, $\mathcal{D}_5(G)$ is not a 2-design.

If $x \in G_2 \setminus \langle a \rangle$, there are no blocks of $\mathcal{D}_5(G)$ through $\{0, a, x, a+x\}$; if $y \in G \setminus G_2$ with $2y \neq a$, any block of $\mathcal{D}_5(G)$ through $\{0, a, y\}$ does not intersect $\{a-y, -y, a+2y\}$. These facts imply:
if $g \in G$ with $2g = a$, then ($g \in G \setminus G_2$ and) there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, a, g\}$;
if $g \in G \setminus G_2$ with $2g \neq a$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, a, g\}$;
if $g \in G_2 \setminus \langle a \rangle$, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, a, g\}$.

Therefore

$$N_{0,a} = |G_2| \cdot \frac{n-4}{2} + (n - 2\,|G_2|) \cdot \frac{n-6}{2} + (|G_2| - 2) \cdot \frac{n-4-|G_2|}{2}$$

can be written as

$$N_{0,a} = 3\,|G_2| - \frac{1}{2}\,|G_2|^2 + \frac{n^2 - 8n + 8}{2}. \tag{1.1}$$

There are no blocks of $\mathcal{D}_5(G)$ containing the group $\langle b \rangle = \{0, b, 2b, -b\}$;
if $g \in b + G_2$ with $b \neq g \neq -b$, there are no blocks of $\mathcal{D}_5(G)$ through $\{0, b, g, -b-g\}$;
if $2b \neq g \in G \setminus b + G_2$, any block of $\mathcal{D}_5(G)$ through $\{0, b, g\}$ does not meet $\{3b-g, 2b-g, 2g-b\}$.

These facts guarantee that:
$\frac{n-2-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{0, b, -b\}$;
there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, b, 2b\}$;
if $g \in b + G_2$ with $b \neq g \neq -b$, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, b, g\}$;
if $g \in G$ with $g \neq 2b \neq 2g$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, b, g\}$.

Therefore

$$N_{0,b} = \frac{n-2-|G_2|}{2} + \frac{n-4}{2} + (|G_2| - 2) \cdot \frac{n-4-|G_2|}{2} + (n - |G_2| - 2) \cdot \frac{n-6}{2}$$

can be written as

$$N_{0,b} = \frac{3}{2} \cdot |G_2| - \frac{1}{2} \cdot |G_2|^2 + \frac{n^2 - 8n + 14}{2} \tag{1.2}$$

Since $|G_2| \neq 2$, (1.1) and (1.2) yield $N_{0,a} \neq N_{0,b}$ and hence $\mathcal{D}_5(G)$ is not a 2-design.

3) Suppose G is direct sum of $h \geq 1$ groups of order 2 and $\nu \geq 1$ cyclic groups of order 4. So G is a finite abelian group of order $n = |G| = 2^h \cdot 4^\nu \geq 8$; $2G$ is an elementary abelian 2-group of order 2^ν; G_2 is an elementary abelian 2-group of order $2^{h+\nu} \geq 4$ which admits $2G$ as a proper subgroup.

Let $a \in G_2 \setminus 2G$ and let $b \in 2G$, $b \neq 0$. We show now that $N_{0,a} \neq N_{0,b}$ and hence, by Lemma 1, $\mathcal{D}_5(G)$ is not a 2-design.
If $a \neq g \in a + 2G$, then $a + g \in 2G$ and there are no blocks of $\mathcal{D}_5(G)$ through $\{0, a, g, a+g\}$;

if $0 \neq g \in G_2 \setminus a + 2G$, then $a + g \notin 2G$ and there are no blocks of $\mathcal{D}_5(G)$ through $\{0, a, g, a + g\}$;
if $g \in G \setminus G_2$, then $a + g \notin 2G$ and any block of $\mathcal{D}_5(G)$ through $\{0, a, g\}$ does not intersect $\{a - g, -g, a + 2g\}$.

From these facts we deduce that:
if $a \neq g \in a + 2G$, then $\frac{n-4-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{0, a, g\}$;
if $g \in G_2$ with $0 \neq g \notin a + 2G$, there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, a, g\}$;
if $g \in G \setminus G_2$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, a, g\}$.

Therefore

$$N_{0,a} = (|2G| - 1) \cdot \frac{n - 4 - |G_2|}{2} + (|G_2| - |2G| - 1) \cdot \frac{n - 4}{2} + (n - |G_2|) \cdot \frac{n - 6}{2}$$

which, since $|2G| \cdot |G_2| = |G| = n$, simplifies to

$$N_{0,a} = \frac{3}{2} \cdot |G_2| + \frac{n^2 - 9n + 8}{2}. \tag{1.3}$$

If $b = 2g$ with $g \in G$, then $b + g \notin 2G$ and there are no blocks of $\mathcal{D}_5(G)$ through $\{0, b, g, -g\}$;
if $g \in 2G \setminus \{0, b\}$, then $b + g \in 2G$ and there are no blocks of $\mathcal{D}_5(G)$ through $\{0, b, g, b + g\}$;
if $g \in G_2 \setminus 2G$, then $b + g \notin 2G$ and there are no blocks of $\mathcal{D}_5(G)$ through $\{0, b, g, b + g\}$;
if $g \in G \setminus G_2$ and $2g \neq b$, then $b + g \notin 2G$ and any block of $\mathcal{D}_5(G)$ through $\{0, b, g\}$ does not meet $\{b - g, -g, b + 2g\}$.

These facts enable us to conclude that:
if $g \in G$ has the property that $2g = b$, there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, b, g\}$;
if $g \in 2G \setminus \{0, b\}$, then $\frac{n-4-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{0, b, g\}$;
if $g \in G_2 \setminus 2G$, there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, b, g\}$;
if $g \in G \setminus G_2$ with $2g \neq b$, then $\frac{n-6}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{0, b, g\}$.

Therefore

$$N_{0,b} =$$

$$= |G_2| \cdot \frac{n - 4}{2} + (|2G| - 2) \cdot \frac{n - 4 - |G_2|}{2} + (|G_2| - |2G|) \cdot \frac{n - 4}{2} + (n - 2|G_2|) \cdot \frac{n - 6}{2}$$

which, since $|2G| \cdot |G_2| = |G| = n$, can be rewritten as

$$N_{0,b} = 3\,|G_2| + \frac{n^2 - 9n + 8}{2}. \tag{1.4}$$

Since $|G_2| \neq 0$, (1.3) and (1.4) give $N_{0,a} \neq N_{0,b}$ and hence $\mathcal{D}_5(G)$ is not a 2-design.

4) In this case $G_2 = \{0, a\}$ is a group of order two and a is the unique involution of G; G can be written as direct sum $G = G_2 \oplus 2G$ and $2G = G_3$ is an elementary abelian 3-group. If $2G = G_3$ has order 3, then G is cyclic of order 6 and clearly $\mathcal{D}_5(G)$ is not a 2-design. Thus we may assume that $|2G| = 3^m$ for some integer $m > 1$. Then G has order $n = |G| = 2|2G| \geq 18$ and we have:
if $a \neq g \in G \setminus 2G$ and $x \in \{2g, -g\}$, there are no blocks of $\mathcal{D}_5(G)$ through $\{0, a, g, x\}$;
if $0 \neq g \in 2G$, any block of $\mathcal{D}_5(G)$ through $\{0, a, g\}$ does not intersect $\{a-g, -g, a-2g\}$.

These facts imply:
if $g \in G \setminus 2G$ with $g \neq a$, there are $\frac{n-4-|G_2|}{2} = \frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, a, g\}$;
if $g \in 2G$ is not equal to $0 \in G$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, a, g\}$.

Therefore

$$N_{0,a} = (|2G|-1) \cdot \frac{n-6}{2} + (|2G|-1) \cdot \frac{n-6}{2}. \tag{1.5}$$

Let $b \in G_3$, $b \neq 0$. Clearly ($b \neq a$ and) we have:
there are no blocks of $\mathcal{D}_5(G)$ containing $\{0, b, -b\}$;
if $g \in G \setminus 2G$, any block of $\mathcal{D}_5(G)$ through $\{0, b, g\}$ does not intersect $\{2b-g, b-g, 2b-2g\}$;
if $g \in 2G \setminus \langle b \rangle$, then $b+g \in 2G$ and any block of $\mathcal{D}_5(G)$ through $\{0, b, g\}$ does not intersect $\{2b-g, b-g, 2b-2g\}$.

These facts imply:
if $g \in G \setminus 2G$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, b, g\}$;
if $g \in 2G \setminus \langle b \rangle$, there are $\frac{n-6-|G_2|}{2} = \frac{n-8}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, b, g\}$.

Therefore

$$N_{0,b} = (n-|2G|) \cdot \frac{n-6}{2} + (|2G|-3) \cdot \frac{n-8}{2}$$

which, since $2|2G| = |G| = n$, simplifies to

$$N_{0,b} = |2G| \cdot (n-6) + 12 - 2n. \tag{1.6}$$

Since $n \neq 6$, (1.5) and (1.6) yield $N_{0,a} \neq N_{0,b}$ and hence $\mathcal{D}_5(G)$ is not a 2-design. This last result completes the proof. □

Lemma 2 *Let G be a finite additive abelian group of even order $n > 4$. If there is $a \in G$ such that $a \notin 2G$ and $2a \neq 0$, then*

$$N_{a,-a} = |G_3| + \frac{n^2 - 9n + 18}{2}.$$

Proof We first note that there are $\frac{n-2-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, 0\}$. We now discuss five cases.

Case (L. 1. 1): $4a = 0$ and $|G_3| = 1$. In this case we have:

$\frac{n-2-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a, -a, 2a\}$;
if $g \in 2G$ with $0 \neq g \neq 2a$, there are $\frac{n-6-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in G - 2G$ with $a \neq g \neq -a$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$.

Therefore

$$N_{a,-a} = 2 \cdot \frac{n-2-|G_2|}{2} + (|2G|-2) \cdot \frac{n-6-|G_2|}{2} + (n-|2G|-2) \cdot \frac{n-6}{2}$$

which, since $|2G| \cdot |G_2| = |G| = n$ and $|G_3| = 1$, can be written as

$$N_{a,-a} = |G_3| + \frac{n^2-9n+18}{2}.$$

Case (L. 1. 2): $4a = 0$ and $|G_3| \neq 1$. In this case we get:
$\frac{n-2-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a, -a, 2a\}$;
if $g \in G_3$ is distinct from 0, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in 2G \setminus G_3$ with $g \neq 2a$, there are $\frac{n-6-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in G \setminus 2G$ with $a \neq g \neq -a$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$.

Therefore

$$N_{a,-a} = 2 \cdot \frac{n-2-|G_2|}{2} + (|G_3|-1) \cdot \frac{n-4-|G_2|}{2}$$
$$+ (|2G|-|G_3|-1) \cdot \tfrac{n-6-|G_2|}{2} + (n-|2G|-2) \cdot \tfrac{n-6}{2}$$

which, since $|2G| \cdot |G_2| = |G| = n$, simplifies to

$$N_{a,-a} = |G_3| + \frac{n^2-9n+18}{2}.$$

Case (L. 1. 3): a has order 6. In this case we have:
if $g \in \{-2a, 2a\}$, then $\frac{n-2-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in G_3 \setminus \{0, -2a, 2a\}$, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in 2G \setminus G_3$, then $\frac{n-6-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ containing $\{a, -a, g\}$;
if $g \in G \setminus 2G$ with $a \neq g \neq -a$, there are $\frac{n-6}{2}$ blocks $\mathcal{D}_5(G)$ including $\{a, -a, g\}$.

Therefore

$$N_{a,-a} = 3 \cdot \frac{n-2-|G_2|}{2} + (|G_3|-3) \cdot \frac{n-4-|G_2|}{2}$$
$$+ (|2G|-|G_3|) \cdot \frac{n-6-|G_2|}{2} + (n-|2G|-2) \cdot \frac{n-6}{2}$$

which, since $|2G|\cdot|G_2|=|G|=n$, gives

$$N_{a,-a}=|G_3|+\frac{n^2-9n+18}{2}.$$

Case (L. 1. 4): $4a\neq 0\neq 6a$ and $|G_3|=1$. In this case we get:
if $g\in\{-2a,2a\}$, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g\in 2G\setminus\{0,-2a,2a\}$, $\frac{n-6-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ including $\{a,-a,g\}$;
if $g\in G\setminus 2G$ with $a\neq g\neq -a$, $\frac{n-6}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$.

Therefore

$$N_{a,-a}=\frac{n-2-|G_2|}{2}+2\cdot\frac{n-4-|G_2|}{2}$$
$$+(|2G|-3)\cdot\frac{n-6-|G_2|}{2}+(n-|2G|-2)\cdot\frac{n-6}{2}$$

which, since $|2G|\cdot|G_2|=|G|=n$ and $|G_3|=1$, can be rewritten as

$$N_{a,-a}=|G_3|+\frac{n^2-9n+18}{2}.$$

Case (L. 1. 5): $4a\neq 0\neq 6a$ and $|G_3|\neq 1$. In this case we obtain:
there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$ if $g\in\{-2a,2a\}$ or $0\neq g\in G_3$;
if $g\in 2G\setminus G_3$ with $2a\neq g\neq -2a$, there are $\frac{n-6-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g\in G\setminus 2G$ with $a\neq g\neq -a$, $\frac{n-6}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$.

Therefore

$$N_{a,-a}=\frac{n-2-|G_2|}{2}+(|G_3|+1)\cdot\frac{n-4-|G_2|}{2}$$
$$+(|2G|-|G_3|-2)\cdot\frac{n-6-|G_2|}{2}+(n-|2G|-2)\cdot\frac{n-6}{2}$$

which, since $|2G|\cdot|G_2|=|G|=n$, simplifies to

$$N_{a,-a}=|G_3|+\frac{n^2-9n+18}{2}.$$

The Lemma 2 is proved. □

Proposition 2 $\mathcal{D}_5(G)$ *is not a* $2-$*design if* G *is a finite abelian group of even order* $n>4$ *with the property that* $2G=4G$.

Proof From $2G = 4G$ it follows $G_2 = G_4$ and this requires that the Sylow 2-subgroup of G is an elementary abelian 2-group. Therefore G can be written as direct sum $G = G_2 \oplus 2G$ and, by Proposition 1, we may assume that $2G$ is a finite abelian group of odd order $|2G| > 1$. Then any $z \in G$ of the form $z = x + y$, with $x \in G_2$ and $y \in 2G$ both distinct from 0, is not equal to $-z$ and does not belong to $2G$. Thus, using Lemma 2 we see that

$$N_{z,-z} = |G_3| + \frac{n^2 - 9n + 18}{2}. \tag{1.7}$$

Choose $a \in 2G$, $a \neq 0$ and let α be the unique element in $2G$ such that $a = 2\alpha$. We shall prove that $N_{a,-a} \neq N_{z,-z}$ and hence, by Lemma 1, $\mathcal{D}_5(G)$ is not a 2-design. We first note that $\frac{n-2-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a, -a, 0\}$. We now discuss five cases.

Case (P. 2. 1): $|G_3| = 1$ and $5a \neq 0$. In this case we have:
if $g \in \{-2a, 2a, -\alpha, \alpha\}$, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $-\alpha \neq g \in G$ with $2g = -a$, then ($g \in -\alpha + G_2$ hence) $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $\alpha \neq g \in G$ with $2g = a$, then ($g \in \alpha + G_2$ hence) $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in 2G \backslash \{a, -a, 0, \alpha, -\alpha, 2a, -2a\}$, there are $\frac{n-6-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in G \setminus 2G$ with $-a \neq 2g \neq a$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$.

Therefore

$$\begin{aligned} N_{a,-a} = {} & \frac{n-2-|G_2|}{2} + 4 \cdot \frac{n-4-|G_2|}{2} + 2(|G_2| - 1) \cdot \frac{n-4}{2} \\ & + (|2G| - 7) \cdot \frac{n-6-|G_2|}{2} + (n - 2 \cdot |G_2| - |2G| + 2) \cdot \frac{n-6}{2} \end{aligned}$$

which, since $|2G| \cdot |G_2| = |G| = n$ and $|G_3| = 1$, can be rewritten as

$$N_{a,-a} = 3|G_2| + |G_3| + \frac{n^2 - 9n + 18}{2}$$

Because $|G_2| \neq 0$, this equality together with (1.7) gives $N_{a,-a} \neq N_{z,-z}$ and hence $\mathcal{D}_5(G)$ is not a 2-design.

Case (P. 2. 2): $|G_3| = 1$ and a has order 5. In this case we have ($\alpha = -2a$ and):
if $g \in \{2a, -2a\}$, there are $\frac{n-2-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $2a \neq g \in G$ with $2g = -a$, then ($g \in 2a + G_2$ hence) $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $-2a \neq g \in G$ with $2g = a$, then ($g \in -2a + G_2$ hence) $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in 2G \setminus \langle a \rangle$, there are $\frac{n-6-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;

if $g \in G \setminus 2G$ with $-a \neq 2g \neq a$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$.

Therefore

$$N_{a,-a} = 3 \cdot \frac{n-2-|G_2|}{2} + 2 \cdot (|G_2|-1)\frac{n-4}{2} + (|2G|-5) \cdot \frac{n-6-|G_2|}{2} + (n-2 \cdot |G_2| - |2G| + 2) \cdot \frac{n-6}{2}$$

which, since $|2G| \cdot |G_2| = |G| = n$ and $|G_3| = 1$, simplifies to

$$N_{a,-a} = 3|G_2| + |G_3| + \frac{n^2-9n+18}{2}$$

Since $|G_2| \neq 0$, this equality together with (1.7) gives $N_{a,-a} \neq N_{z,-z}$ and hence $\mathcal{D}_5(G)$ is not a 2-design.

Case (P. 2. 3): $|G_3| \neq 1$ and a has order 5. In this case we have ($\alpha = -2a$ and):

if $g \in \{2a,-2a\}$, there are $\frac{n-2-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $2a \neq g \in G$ with $2g = -a$, then ($g \in 2a + G_2$ hence) $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $-2a \neq g \in G$ with $2g = a$, then ($g \in -2a + G_2$ hence) $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $0 \neq g \in G_3$, then $\frac{n-4-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g \in 2G \setminus G_3$ with $2a \neq g \neq -2a$, there are $\frac{n-6-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g \in G \setminus 2G$ with $-a \neq 2g \neq a$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$.

Therefore

$$N_{a,-a} = 3 \cdot \frac{n-2-|G_2|}{2} + 2 \cdot (|G_2|-1) \cdot \frac{n-4}{2} + (|G_3|-1) \cdot \frac{n-4-|G_2|}{2} + (|2G|-|G_3|-2) \cdot \frac{n-6-|G_2|}{2} + (n-2 \cdot |G_2| - |2G|) \cdot \frac{n-6}{2}$$

which, since $|2G| \cdot |G_2| = |G| = n$, simplifies to

$$N_{a,-a} = 2|G_2| + |G_3| + \frac{n^2-9n+18}{2}$$

Since $|G_2| \neq 0$, this equality together with (1.7) gives $N_{a,-a} \neq N_{z,-z}$ and hence $\mathcal{D}_5(G)$ is not a 2-design.

Case (P. 2. 4): $|G_3| \neq 1$ and $3a \neq 0 \neq 5a$. In this case we have:

if $g \in \{2a,-2a,\alpha,-\alpha\}$, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $-\alpha \neq g \in G$ with $2g = -a$, then ($g \in -\alpha + G_2$ hence) $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;

if $\alpha \neq g \in G$ with $2g = a$, then ($g \in \alpha + G_2$ hence) $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $0 \neq g \in G_3$, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in 2G \setminus G_3$ and $g \notin \{a, -a, \alpha, -\alpha, 2a, -2a\}$, there are $\frac{n-6-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in G \setminus 2G$ with $-a \neq 2g \neq a$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$.

Therefore

$$N_{a,-a} = \frac{n-2-|G_2|}{2} + 4 \cdot \frac{n-4-|G_2|}{2} + 2 \cdot (|G_2| - 1) \cdot \frac{n-4}{2} + (|G_3| - 1) \cdot \frac{n-4-|G_2|}{2} + (|2G| - |G_3| - 6) \cdot \frac{n-6-|G_2|}{2} + (n - 2 \cdot |G_2| - |2G| + 2) \cdot \frac{n-6}{2}$$

which, since $|2G| \cdot |G_2| = |G| = n$, can be rewritten as

$$N_{a,-a} = 3|G_2| + |G_3| + \frac{n^2 - 9n + 18}{2}$$

Since $|G_2| \neq 0$, this equality together with (1.7) gives $N_{a,-a} \neq N_{z,-z}$ and hence $\mathcal{D}_5(G)$ is not a 2-design.

Case (P. 2. 5): $a \in G_3$. In this case we obtain ($\alpha = -a$ and):
if $a \neq g \in G$ with $2g = -a$, then ($g \in a + G_2$ hence) $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $-a \neq g \in G$ with $2g = a$, then ($g \in -a + G_2$ hence) $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in G_3 \setminus \langle a \rangle$, then $\frac{n-4-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in 2G \setminus G_3$, then $\frac{n-6-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in G \setminus 2G$ with $-a \neq 2g \neq a$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$.

Therefore

$$N_{a,-a} = \frac{n-2-|G_2|}{2} + 2(|G_2| - 1) \cdot \frac{n-4}{2} + (|G_3| - 3) \cdot \frac{n-4-|G_2|}{2} + (|2G| - |G_3|) \cdot \frac{n-6-|G_2|}{2} + (n - 2 \cdot |G_2| - |2G| + 2) \cdot \frac{n-6}{2}$$

which, since $|2G| \cdot |G_2| = |G| = n$, can be rewritten as

$$N_{a,-a} = 3|G_2| - 6 + |G_3| + \frac{n^2 - 9n + 18}{2}$$

This equality together with (1.7) yields $|G_2| = 2$. Such a result and those obtained from the above cases allow as to conclude that: G has just one involution and $2G$ must be an elementary abelian 3-group. Now using Proposition 1 we see that $\mathcal{D}_5(G)$ is not a 2-design, the Proposition is proved. □

Lemma 3 *Suppose G is a finite abelian group of even order $n > 4$ in which $G_4 \neq G \neq G_2 + 2G$ and choose $\alpha \in G$ in such a way that $\alpha \notin G_2 + 2G$, $4\alpha \neq 0$. Then $a = 2\alpha$ and $-a$ are distinct elements of G and*

$$N_{a,-a} = 3|G_2| + |G_3| + \frac{n^2 - 9n + 18}{2}.$$

Proof Clearly, from $a = 2\alpha$ it follows $a \in 2G$, $2a \neq 0$, $a \notin 4G$, $3a \neq 0$, $5a \neq 0$. We first note that: $\frac{n-2-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a, -a, 0\}$; if $g \in G \setminus \{a, -a, 0\}$, any block of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$ does not intersect $\{-a-g, a-g, -2g\}$. We now discuss five cases.

Case (L. 2. 1): $4a = 0$ and $|G_3| = 1$. In this case we have:
$\frac{n-2-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a, -a, 2a\}$;
if $g \in G$ and $2g \in \{a, -a\}$, then $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in 2G \setminus \langle a \rangle$, there are $\frac{n-6-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in G \setminus 2G$ with $-a \neq 2g \neq a$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$.

Therefore

$$\begin{aligned} N_{a,-a} &= 2 \cdot \frac{n-2-|G_2|}{2} + 2 \cdot |G_2| \cdot \frac{n-4}{2} \\ &+ (|2G| - 4) \cdot \frac{n-6-|G_2|}{2} + (n - 2 \cdot |G_2| - |2G|) \cdot \frac{n-6}{2} \end{aligned}$$

which, since $|2G| \cdot |G_2| = |G| = n$ and $|G_3| = 1$, yields

$$N_{a,-a} = 3|G_2| + |G_3| + \frac{n^2 - 9n + 18}{2}.$$

Case (L. 2. 2): $4a = 0$ and $|G_3| \neq 1$. In this case we have:
$\frac{n-2-|G_2|}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a, -a, 2a\}$;
if $g \in G$ and $2g \in \{a, -a\}$, then $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in G_3$ is distinct from 0, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in 2G \setminus G_3$ does not belong to $\langle a \rangle$, there are $\frac{n-6-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in G \setminus 2G$ with $-a \neq 2g \neq a$; there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$.

Therefore

$$N_{a,-a} = 2 \cdot \frac{n-2-|G_2|}{2} + 2 \cdot |G_2| \cdot \frac{n-4}{2} + (|G_3|-1) \cdot \frac{n-4-|G_2|}{2} + (|2G|-|G_3|-3) \cdot \frac{n-6-|G_2|}{2} + (n-2\cdot|G_2|-|2G|) \cdot \frac{n-6}{2}$$

which, since $|2G| \cdot |G_2| = |G| = n$, simplifies to

$$N_{a,-a} = 3|G_2| + |G_3| + \frac{n^2-9n+18}{2}.$$

Case (L. 2. 3): a has order 6. In this case we obtain:
if $g \in \{2a,-2a\}$, there are $\frac{n-2-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g \in G$ and $2g \in \{a,-a\}$, then $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g \in G_3 \setminus \{0,2a,-2a\}$, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g \in 2G \setminus G_3$ with$-a \neq g \neq a$, there are $\frac{n-6-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g \in G \setminus 2G$ with $-a \neq 2g \neq a$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$.

Therefore

$$N_{a,-a} = 3 \cdot \frac{n-2-|G_2|}{2} + 2 \cdot |G_2| \cdot \frac{n-4}{2} + (|G_3|-3) \cdot \frac{n-4-|G_2|}{2} + (|2G|-|G_3|-2) \cdot \frac{n-6-|G_2|}{2} + (n-2\cdot|G_2|-|2G|) \cdot \frac{n-6}{2}$$

which, since $|2G| \cdot |G_2| = |G| = n$, yields

$$N_{a,-a} = 3|G_2| + |G_3| + \frac{n^2-9n+18}{2}.$$

Case (L. 2. 4): $4a \neq 0 \neq 6a$ and $|G_3| = 1$. In this case we get:
if $g \in \{2a,-2a\}$, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g \in G$ and $2g \in \{-a,a\}$, then $g \notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g \in 2G \setminus \{a,-a,0,-2a,2a\}$, there are $\frac{n-6-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g \in G \setminus 2G$ with $-a \neq 2g \neq a$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$.

Therefore

$$N_{a,-a} = \frac{n-2-|G_2|}{2} + 2 \cdot \frac{n-4-|G_2|}{2} + 2 \cdot |G_2| \cdot \frac{n-4}{2} + (|2G|-5) \cdot \frac{n-6-|G_2|}{2} + (n-2\cdot|G_2|-|2G|) \cdot \frac{n-6}{2}$$

which, since $|2G|\cdot|G_2|=|G|=n$ and $|G_3|=1$, yields

$$N_{a,-a}=3|G_2|+|G_3|+\frac{n^2-9n+18}{2}.$$

Case (L. 2. 5): $4a\neq 0\neq 6a$ and $|G_3|\neq 1$. In this case we deduce:
if $g\in\{2a,-2a\}$, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g\in G$ and $2g\in\{a,-a\}$, then $g\notin 2G$ and there are $\frac{n-4}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g\in G_3$ is distinct from 0, there are $\frac{n-4-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g\in 2G\setminus G_3$ and $g\notin\{a,-a,2a,-2a\}$, there are $\frac{n-6-|G_2|}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$;
if $g\in G\setminus 2G$ with $-a\neq 2g\neq a$, there are $\frac{n-6}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a,-a,g\}$.

Therefore

$$\begin{aligned}N_{a,-a}=&\frac{n-2-|G_2|}{2}+(|G_3|+1)\cdot\frac{n-4-|G_2|}{2}+2\cdot|G_2|\cdot\frac{n-4}{2}\\&+(|2G|-|G_3|-4)\cdot\frac{n-6-|G_2|}{2}+(n-2\cdot|G_2|-|2G|)\cdot\frac{n-6}{2}\end{aligned}$$

which, since $|2G|\cdot|G_2|=|G|=n$, yields

$$N_{a,-a}=3|G_2|+|G_3|+\frac{n^2-9n+18}{2}.$$

The Lemma 3 is proved. □

Theorem 1 *If* $\mathcal{D}_5(G)$ *is a* 2*-design, then* $n=|G|$ *must be odd integer.*

Proof We may assume that G is a finite additive abelian group of order $n=|G|>4$. Suppose n is an even integer: so we must show that $\mathcal{D}_5(G)$ is not a 2-design. We discuss five cases.

Case (T. 1): $2g=0$ whenever $g\in G\setminus 2G$. In this case G must be an elementary abelian 2-group and hence, by Proposition 1, $\mathcal{D}_5(G)$ is not a 2-design.

Case (T. 2): G is an abelian group of exponent 4. In this case either G is direct sum of cyclic groups of order 4 or G is direct sum of groups of order 2 and cyclic groups of order 4. Then, by Proposition 1, $\mathcal{D}_5(G)$ is not a 2-design.

Case (T. 3): $2G=4G$. Then Proposition 2 asserts that $\mathcal{D}_5(G)$ is not a 2-design.

Case (T. 4): $2G\neq 4G$ and $4x=0$ for every $x\notin G_2+2G$. Then G must be an abelian group of exponent 4 and hence, by statements *2)* and *3)* of Proposition 1, $\mathcal{D}_5(G)$ is not a 2-design.

Case (T. 5): $2G \neq 4G$ and $G \neq G_4$. Then $G_2 + 2G$ is a proper subgroup of G and there is $\alpha \in G$ such that $\alpha \notin G_2 + 2G$ and $4\alpha \neq 0$. Thus $a = 2\alpha \neq -a$ and, by Lemma 3, we obtain

$$N_{a,-a} = 3|G_2| + |G_3| + \frac{n^2 - 9n + 18}{2}. \tag{1.8}$$

On the other hand, since $2G \neq 4G$ implies that G is not an elementary abelian 2-group, there is $z \in G$ such that $z \notin 2G$ and $2z \neq 0$. Then using Lemma 2 we deduce that

$$N_{z,-z} = |G_3| + \frac{n^2 - 9n + 18}{2}. \tag{1.9}$$

Since $|G_2| \neq 0$, combining (1.8) and (1.9) we deduce that $N_{a,-a} \neq N_{z,-z}$ and hence, by Lemma 1, $\mathcal{D}_5(G)$ is not a 2-design. Now the proof of the theorem is complete. □

2 Main result

Proposition 3 *$\mathcal{D}_5(G)$ is not a 2-design if one of the statements below is true:*

1. *G is a finite abelian group of odd order n divisible by 3;*
2. *G is a finite abelian group of odd order n not divisible by 5.*

Proof

1. Choose $a \in G_3$, $a \neq 0$. Then clearly we have:
$\frac{n-3}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{a, -a, 0\}$;
if $g \in G_3 \setminus \langle a \rangle$, there are $\frac{n-5}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$;
if $g \in G \setminus G_3$, there are $\frac{n-7}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{a, -a, g\}$.

Therefore

$$N_{a,-a} = \frac{n-3}{2} + (|G_3| - 3) \cdot \frac{n-5}{2} + (n - |G_3|) \cdot \frac{n-7}{2}. \tag{2.1}$$

Note that if G is an elementary abelian 3-group, then $G = G_3$ and (2.1) can be rewritten as

$$N_{a,-a} = \frac{n-3}{2} + (n-3) \cdot \frac{n-5}{2}. \tag{2.2}$$

Suppose $|G_5| \neq 1$ and choose $\alpha \in G_5$, $\alpha \neq 0$. Then we obtain:
if $g \in \{0, 2\alpha, -2\alpha\}$, there are $\frac{n-3}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{\alpha, -\alpha, g\}$;
if $0 \neq g \in G_3$, there are $\frac{n-5}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{\alpha, -\alpha, g\}$;
if $g \in G \setminus G_3$ and $g \notin \{\alpha, -\alpha, -2\alpha, 2\alpha\}$, there are $\frac{n-7}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{\alpha, -\alpha, g\}$.

Therefore

$$N_{\alpha,-\alpha} = 3 \cdot \frac{n-3}{2} + (|G_3| - 1) \cdot \frac{n-5}{2} + (n - 4 - |G_3|) \cdot \frac{n-7}{2}. \tag{2.3}$$

Combining (2.1) and (2.3) we deduce that $N_{a,-a} \neq N_{\alpha,-\alpha}$ and hence, by Lemma 1, $\mathcal{D}_5(G)$ is not a 2-design.

Suppose $|G_5|=1$, $G_3 \neq G$ and choose $\beta \in G \setminus G_3$. Then we find:
$\frac{n-3}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{\beta,-\beta,0\}$;
if $0 \neq g \in \{2\beta,-2\beta\}$, there are $\frac{n-5}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{\beta,-\beta,g\}$;
if $\gamma \in G$ with $2\gamma=-\beta$, there are $\frac{n-5}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{\beta,-\beta,\gamma\}$;
there are $\frac{n-5}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{\beta,-\beta,-\gamma\}$;
if $0 \neq g \in G_3$, there are $\frac{n-5}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{\beta,-\beta,g\}$;
if $g \in G \setminus G_3$ and $g \notin \{\beta,-\beta,2\beta,-2\beta,\gamma,-\gamma\}$, there are $\frac{n-7}{2}$ blocks $\mathcal{D}_5(G)$ through $\{\beta,-\beta,g\}$.

Therefore

$$N_{\beta,-\beta}=\frac{n-3}{2}+(|G_3|+3)\cdot\frac{n-5}{2}+(n-6-|G_3|)\cdot\frac{n-7}{2}. \tag{2.4}$$

Combining (2.1) and (2.4) we find $N_{a,-a} \neq N_{\beta,-\beta}$ and hence, by Lemma 1, $\mathcal{D}_5(G)$ is not a 2-design.

We can now assume that $G=G_3$. Then for any $g \in G \setminus \langle a \rangle$ there are $\frac{n-7}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0,a,g\}$. Therefore

$$N_{0,a}=\frac{n-3}{2}+(n-3)\cdot\frac{n-7}{2}. \tag{2.5}$$

Combining (2.2) and (2.5) we find $N_{a,-a} \neq N_{0,a}$ and hence, by Lemma 1, $\mathcal{D}_5(G)$ is not a 2-design.

2. By **1** we may assume that n and 15 are (odd integers) relatively prime. Choose $x \in G$, $x \neq 0$ and let y,z be elements of G such that $2y=x$, $2z=7x$. Then clearly we have:
$\frac{n-3}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through the 3-set $\{x,-x,0\}$;
if $g \in \{2x,-2x,y,-y\}$, there are $\frac{n-5}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{x,-x,g\}$;
if $0 \neq g \in G \setminus \{x,-x,2x,-2x,y,-y\}$, there are $\frac{n-7}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{x,-x,g\}$.

Therefore

$$N_{x,-x}=\frac{n-3}{2}+4\cdot\frac{n-5}{2}+(n-7)\cdot\frac{n-7}{2}. \tag{2.6}$$

On the other hand we have:
if $g \in \{6x,11x,z\}$, there are $\frac{n-5}{2}$ blocks of $\mathcal{D}_5(G)$ through the 3-set $\{x,-4x,g\}$;
if $g \in G \setminus \{x,-4x,6x,11x,z\}$, there are $\frac{n-7}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{x,-4x,g\}$.

Therefore

$$N_{x,-4x}=3\cdot\frac{n-5}{2}+(n-5)\cdot\frac{n-7}{2}. \tag{2.7}$$

Combining (2.6) and (2.7) we obtain $N_{x,-x} \neq N_{x,-4x}$ and hence, by Lemma 1, $\mathcal{D}_5(G)$ is not a 2-design. Now the Proposition 3 is proved. $\square$

We can now state our main result.

Theorem 2 *$\mathcal{D}_5(G)$ is a 2-design if and only if G is an elementary abelian 5-group. When this is so, there are*

$$\lambda=\frac{|G|-3}{2}+\frac{(|G|-5)\cdot(|G|-7)}{6}$$

blocks of $\mathcal{D}_5(G)$ through any given 2-set $\{x,y\} \subset G$.

Proof Suppose $\mathcal{D}_5(G)$ is a 2-design. By Theorem 1 and Proposition 3, $n = |G|$ must be an odd integer multiple of 5 not divisible by 3. Let $a \in G_5$, $a \neq 0$. Then we find: if $g \in \langle a \rangle$ with $0 \neq g \neq a$, then $\frac{n-3}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{0, a, g\}$; if $g \in G \setminus \langle a \rangle$, there are $\frac{n-7}{2}$ blocks of $\mathcal{D}_5(G)$ through $\{0, a, g\}$.

Therefore
$$N_{0,a} = 3 \cdot \frac{n-3}{2} + (n-5) \cdot \frac{n-7}{2} \tag{2.8}$$
Assume that $5b \neq 0$ for some $b \in G$ and let β be the unique element in G such that $2\beta = 7b$. Then ($\beta \in G \setminus \{b, -4b, 6b, 11b\}$ and) we obtain:
if $g \in \{6b, 11b, \beta\}$, then $\frac{n-5}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{b, -4b, g\}$;
if $g \in G \setminus \{b, -4b, 6b, 11b, \beta\}$, then $\frac{n-7}{2}$ is the number of blocks of $\mathcal{D}_5(G)$ through $\{b, -4b, g\}$.

Therefore
$$N_{b,-4b} = 3 \cdot \frac{n-5}{2} + (n-5) \cdot \frac{n-7}{2}$$
and thus, since $\mathcal{D}_5(G)$ is a 2 design, we find (by Lemma 1)
$$3 \cdot \frac{n-3}{2} + (n-5) \cdot \frac{n-7}{2} = N_{0,a} = N_{b,-4b} = 3 \cdot \frac{n-5}{2} + (n-5) \cdot \frac{n-7}{2}$$
and this gives $n - 3 = n - 5$ a contradiction. Such a contradiction shows that $5g = 0$ for all $g \in G$: in other words, G is an elementary abelian 5-group. Furthermore, from Lemma 1 and equation (2.8) we know that
$$3 \cdot \frac{n-3}{2} + (n-5) \cdot \frac{n-7}{2} = N_{0,a} = 3\,\lambda$$
from wich it follows that $\lambda = \frac{|G|-3}{2} + (|G| - 5) \cdot \frac{|G|-7}{6}$ is the number of blocks of $\mathcal{D}_5(G)$ through any given two distinct elements $x, y \in G$.

To finish, assume that G is an elementary abelian 5-group. If we regard G as a vector space over the field with five elements, then we see that the affine group $\mathrm{Aff}(G)$ acts 2-homogeneously on G and the block-set $\mathcal{B}$ of $\mathcal{D}_5(G)$ may be written as $\mathcal{B} = C^{\mathrm{Aff}(G)}$ (i.e. $\mathcal{B} = \{C^\gamma \mid \gamma \in \mathrm{Aff}(G)\}$ is the $\mathrm{Aff}(G)$-orbit of a fixed block $C \in \mathcal{B}$). Hence, by [1, Proposition 4.6], $\mathcal{D}_5(G)$ is a 2-design. The Theorem is proved. □

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