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Orthomodular Posets Can Be Organized as Conditionally Residuated Structures*

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Abstract

It is proved that orthomodular posets are in a natural one-to-one correspondence with certain residuated structures.

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Orthomodular posets are well-known structures used in the foundations of quantum mechanics (cf. e.g. [4], [5], [9], [10] and [11]). They can be considered as effect algebras (see e.g. [6]). Residuated lattices were treated in [7]. In [3] the concept of a conditionally residuated structure was introduced. Since every orthomodular poset is in fact an effect algebra, it follows that also every orthomodular poset can be considered as a conditionally residuated structure. The question is which additional conditions have to be satisfied in order to get a one-to-one correspondence. Contrary to the case of effect algebras, orthomodular posets satisfy also the orthomodular law and a certain condition concerning the orthogonality of their elements.

We start with the definition of an orthomodular poset.

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Definition 1 An *orthomodular poset* (cf. [8], [2] and [12]) is an ordered quintuple $\mathcal{P} = (P, \leq, ^\perp, 0, 1)$ where $(P, \leq, 0, 1)$ is a bounded poset, $^\perp$ is a unary operation on P and the following conditions hold for all $x, y \in P$:

(i) $(x^\perp)^\perp = x$

(ii) If $x \leq y$ then $y^\perp \leq x^\perp$.

(iii) If $x \perp y$ then $x \vee y$ exists.

(iv) If $x \leq y$ then $y = x \vee (y \wedge x^\perp)$.

Here and in the following $x \perp y$ is an abbreviation for $x \leq y^\perp$.

Remark 2 If $(P, \leq)$ is a poset and $^\perp$ a unary operation on P satisfying (i) and (ii) then the so-called de Morgan laws

$$(x \vee y)^\perp = x^\perp \wedge y^\perp \text{ in case } x \perp y \text{ and}$$
$$(x \wedge y)^\perp = x^\perp \vee y^\perp \text{ in case } x^\perp \perp y^\perp$$

hold. Moreover, (iv) is equivalent to the following condition:

(v) If $x \leq y$ then $x = y \wedge (x \vee y^\perp)$.

If $x \leq y$ then $x \perp y^\perp$ and therefore $x \vee y^\perp$ is defined. Hence also $y \wedge x^\perp$ is defined. Moreover, $x \perp y \wedge x^\perp$ which shows that $x \vee (y \wedge x^\perp)$ is defined. Thus the expression in (iv) is well-defined. The same is true for condition (v).

Next we define a partial commutative groupoid with unit.

Definition 3 A *partial commutative groupoid with unit* is a partial algebra $\mathcal{A} = (A, \odot, 1)$ of type $(2, 0)$ satisfying the following conditions for all $x, y \in A$:

(i) If $x \odot y$ is defined so is $y \odot x$ and $x \odot y = y \odot x$.

(ii) $x \odot 1$ and $1 \odot x$ are defined and $x \odot 1 = 1 \odot x = x$.

Now we are ready to define a conditionally residuated structure.

Definition 4 Let $\mathcal{A} = (A, \leq, \odot, \to, 0, 1)$ be an ordered sixtuple such that $(A, \leq, 0, 1)$ is a bounded poset, $(A, \odot, \to, 0, 1)$ is a partial algebra of type $(2, 2, 0, 0)$, $(A, \odot, 1)$ is a partial commutative groupoid with unit and $x \to y$ is defined if and only if $y \leq x$. We write x' instead of $x \to 0$. Moreover, assume that the following conditions are satisfied for all $x, y, z \in A$:

(i) $x \odot y$ is defined if and only if $x' \leq y$.

(ii) If $x \odot y$ and $y \to z$ are defined then $x \odot y \leq z$ if and only if $x \leq y \to z$.

(iii) If $x \to y$ is defined then so is $y' \to x'$ and $x \to y = y' \to x'$.

(iv) If $y \leq x$ and $x', y \leq z$ then $x \to y \leq z$.

Then $\mathcal{A}$ is called a *conditionally residuated structure*.

Remark 5 Condition (ii) is called *left adjointness*, see e.g. [1].

Example 6 Let $M := \{1, \ldots, 6\}$ and $P := \{C \subseteq M \mid |C| \text{ is even}\}$. If one defines for arbitrary $A, B \in P$

$$\begin{aligned}
&A \odot M = M \odot A := A,\\
&A \odot (M \setminus A) := \emptyset,\\
&A \odot B := A \cap B \text{ if } |A| = |B| = 4 \text{ and } A \cup B = M,\\
&A \to \emptyset := M \setminus A,\\
&A \to A := M,\\
&M \to A := A \text{ and}\\
&A \to B := (M \setminus A) \cup B \text{ if } B \subseteq A, |B| = 2 \text{ and } |A| = 4
\end{aligned}$$

then $(P, \subseteq, \odot, \to, \emptyset, M)$ is a conditionally residuated structure.

The following lemma lists some easy properties of conditionally residuated structures used later on.

Lemma 7 *If $\mathcal{A} = (A, \leq, \odot, \to, 0, 1)$ is a conditionally residuated structure then the following conditions hold for all $x, y \in A$:*

(i) $(x')' = x$

(ii) *If $x \leq y$ then $y' \leq x'$.*

(iii) *If $x \odot y$ is defined then $x \odot y = 0$ if and only if $x \leq y'$.*

(iv) *$x \to y = 1$ if and only if $x \leq y$.*

Proof Let $x, y \in A$. We have $x' \leq x'$. Hence $x \odot x'$ exists and therefore also $x' \odot x$ exists which implies $(x')' \leq x$. Moreover, $x' \leq x' = x \to 0$ and hence $x' \odot x \leq 0$ which shows $x' \odot x = 0$ whence $x \odot x' = 0$. Now $x \odot x' \leq 0$ implies $x \leq x' \to 0 = (x')'$. Together we obtain $(x')' = x$. The inequality $x \leq y$ implies that $x' \odot y$ exists. Hence $y \odot x'$ exists wherefrom we conclude that $y' \leq x'$. Moreover, if $x \odot y$ is defined then the following are equivalent: $x \odot y = 0$, $x \odot y \leq 0$, $x \leq y \to 0$, $x \leq y'$. Finally, the following are equivalent: $x \to y = 1$, $1 \leq x \to y$, $1 \odot x \leq y$, $x \leq y$. □

We now introduce two more properties of conditionally residuated structures.

Definition 8 A conditionally residuated structure $\mathcal{A} = (A, \leq, \odot, \to, 0, 1)$ is said to satisfy the *divisibility condition* if $y \leq x$ implies that $x \odot (x \to y)$ exists and $x \odot (x \to y) = y$ and it is said to satisfy the *orthogonality condition* if $x \leq y'$, $y \leq z'$ and $z \leq x'$ together imply $z \leq x' \odot y'$.

In the following theorem we show that an orthomodular poset can be considered as a special conditionally residuated structure.

Theorem 9 *If $\mathcal{P} = (P, \leq, {}^\perp, 0, 1)$ is an orthomodular poset and one defines*

$$x \odot y := x \wedge y \text{ if and only if } x^\perp \leq y \text{ and}$$
$$x \to y := x^\perp \vee y \text{ if and only if } y \leq x$$

for all $x, y \in P$ then $\mathbf{A}(\mathcal{P}) := (P, \leq, \odot, \to, 0, 1)$ is a conditionally residuated structure satisfying both the divisibility and orthogonality condition.

Proof Let $a, b, c \in P$. Of course, $(P, \leq, 0, 1)$ is a bounded poset. The operations $\odot$ and $\to$ are well-defined since $a^\perp \leq b$ implies $a^\perp \perp b^\perp$ and $b \leq a$ implies $a^\perp \perp b$. If $a \odot b$ is defined then $a^\perp \leq b$ and hence $b^\perp \leq a$ which shows that $b \odot a$ is defined and $a \odot b = a \wedge b = b \wedge a = b \odot a$. Since $a^\perp \leq 1$ we have that $a \odot 1$ is defined and $a \odot 1 = a \wedge 1 = a$. Because of $1^\perp = 0 \leq a$ we have that $1 \odot a$ is defined and $1 \odot a = 1 \wedge a = a$ showing that $(P, \odot, 1)$ is a partial commutative groupoid with unit. Now assume that $a \odot b$ and $b \to c$ are defined. Then $a^\perp \leq b$ and $c \leq b$. If $a \odot b \leq c$ then $a \geq b^\perp$ and

$$a = b^\perp \vee (a \wedge b) = b^\perp \vee (a \odot b) \leq b^\perp \vee c = b \to c.$$

If, conversely, $a \leq b \to c$ then $c \leq b$ and

$$a \odot b = a \wedge b \leq (b \to c) \wedge b = (b^\perp \vee c) \wedge b = c.$$

This proves left adjointness. If $b \leq a$ then $a^\perp \leq b^\perp$ and

$$a \to b = a^\perp \vee b = b \vee a^\perp = b^\perp \to a^\perp.$$

If $b \leq a$ and $a^\perp, b \leq c$ then $a \to b = a^\perp \vee b \leq c$. If $b \leq a$ then $a \to b$ exists and $a^\perp \leq a^\perp \vee b = a \to b$ and hence $a \odot (a \to b)$ exists and, by (v) of Remark 2, $a \odot (a \to b) = a \wedge (a^\perp \vee b) = b$ showing that $\mathbf{A}(\mathcal{P})$ satisfies the divisibility condition. Finally, if $a \leq b^\perp$, $b \leq c^\perp$ and $c \leq a^\perp$ then there exists $a^\perp \odot b^\perp = a^\perp \wedge b^\perp$, $c \leq a^\perp$ and $c \leq b^\perp$ and hence $c \leq a^\perp \wedge b^\perp = a^\perp \odot b^\perp$ showing that $\mathbf{A}(\mathcal{P})$ satisfies the orthogonality condition. □

Conversely, we show that certain conditionally residuated structures can be converted in an orthomodular poset.

Theorem 10 *If $\mathcal{A} = (A, \leq, \odot, \to, 0, 1)$ is a conditionally residuated structure satisfying the divisibility and orthogonality condition then $\mathbf{P}(\mathcal{A}) := (A, \leq, {}', 0, 1)$ is an orthomodular poset.*

Proof Let $a, b, c \in A$. Of course, $(A, \leq, 0, 1)$ is a bounded poset. According to Lemma 7, the operation $'$ is an antitone involution of $(A, \leq)$. We show that in case $a \leq b'$ we have $(a' \odot b')' = a \vee b$. If $a \leq b'$ then $a' \odot b'$ and $b' \odot a'$ are defined. Now we have $b' \leq 1 = a' \to a'$ according to Lemma 7, hence $a' \odot b' = b' \odot a' \leq a'$ and therefore $a \leq (a' \odot b')'$. By symmetry $b \leq (a' \odot b')'$ follows. Now, if $a, b \leq c$ then $a \leq b'$, $b \leq c$ and $c' \leq a'$ and hence according to the orthogonality condition $c' \leq a' \odot b'$ whence $c \geq (a' \odot b')'$. This shows $(a' \odot b')' = a \vee b$ in case $a \leq b'$. Since $a \leq (a')'$ we have $a \vee a' = (a' \odot a)' = 0' = 1$

according to Lemma 7. Finally, assume $a \leq b$. Because of $a' \to b' \geq a' \to 0 = a$ and $a' \to b' \geq 1 \to b' = b'$ we have $a' \to b' \geq a \vee b'$. Hence, according to the divisibility condition we obtain

$$a \vee (b \wedge a') = (a' \odot (a \vee b'))' \geq (a' \odot (a' \to b'))' = (b')' = b.$$

Since the converse inequality is obvious, we see that the considered poset is orthomodular. $\square$

Finally, we show that the correspondence described in the last two theorems is one-to-one.

Theorem 11 *If* $\mathcal{P} = (P, \leq, ^\perp, 0, 1)$ *is an orthomodular poset then* $\mathbf{P}(\mathbf{A}(\mathcal{P})) = \mathcal{P}$. *If* $\mathcal{A} = (A, \leq, \odot, \to, 0, 1)$ *is a conditionally residuated structure satisfying the divisibility and orthogonality condition then* $\mathbf{A}(\mathbf{P}(\mathcal{A})) = \mathcal{A}$.

Proof First assume $\mathcal{P} = (P, \leq, ^\perp, 0, 1)$ to be an orthomodular poset and let $\mathbf{A}(\mathcal{P}) = (P, \leq, \odot, \to, 0, 1)$ and $\mathbf{P}(\mathbf{A}(\mathcal{P})) = (P, \leq, ^*, 0, 1)$. Then

$$x^* = x \to 0 = x^\perp \vee 0 = x^\perp$$

for all $x \in P$ and hence $\mathbf{P}(\mathbf{A}(\mathcal{P})) = \mathcal{P}$.

Conversely, assume $\mathcal{A} = (A, \leq, \odot, \to, 0, 1)$ to be a conditionally residuated structure satisfying the divisibility and orthogonality condition and let $\mathbf{P}(\mathcal{A}) = (A, \leq, ', 0, 1)$ and $\mathbf{A}(\mathbf{P}(\mathcal{A})) = (A, \leq, \circ, \Rightarrow, 0, 1)$. Let $a, b, c \in A$. If $a' \leq b$ then $a \circ b = a \wedge b = (a' \vee b')' = a \odot b$ according to the proof of Theorem 10. Finally, if $b \leq a$ then $a \Rightarrow b = a' \vee b = a \to b$ according to the proof of Theorem 10. $\square$

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