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Double Sequence Spaces Defined by a Sequence of Modulus Functions over n-normed Spaces

Sunil K. SHARMA[1], Ayhan ESI[2]

[1] *School of Mathematics, Shri Mata Vaishno Devi University
Katra-182320, J&K, India
e-mail: sunilksharma42@gmail.com*

[2] *Department of Mathematics, Science and Art Faculty
Adiyaman University, 02040, Adiyaman, Turkey
e-mail: aesi23@hotmail.com*

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Abstract

In the present paper we introduce some double sequence spaces defined by a sequence of modulus function $F = (f_{k,l})$ over n-normed spaces. We also make an effort to study some topological properties and inclusion relations between these spaces.

Key words: double sequences, P-convergent, modulus function, paranorm space

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1 Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler [13] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [24]. Since then, many others have studied this concept and obtained various results, see Gunawan ([15], [16]) and Gunawan and Mashadi [17] and references therein. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is the field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $||\cdot, \cdots, \cdot||$ on X^n satisfying the following four conditions:

(1) $\|x_1, x_2, \cdots, x_n\| = 0$ if and only if $x_1, x_2, \cdots, x_n$ are linearly dependent in X;

(2) $\|x_1, x_2, \cdots, x_n\|$ is invariant under permutation;

(3) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \, \|x_1, x_2, \cdots, x_n\|$ for any $\alpha \in \mathbb{K}$;

(4) $\|x + x', x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|x', x_2, \cdots, x_n\|$

is called a n-norm on X, and the pair $(X, \|\cdot, \cdots, \cdot\|)$ is called a n-normed space over the field $\mathbb{K}$. For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean n-norm $\|x_1, x_2, \cdots, x_n\|_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \cdots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \cdots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, \|\cdot, \cdots, \cdot\|)$ be a n-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot, \cdots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \cdots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \cdots, x_{n-1}, a_i\| \colon i = 1, 2, \cdots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k,p\to\infty} \|x_k - x_p, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$

If every cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

The initial works on double sequences are found in Bromwich [8]. Later on, it was studied by Hardy [19], Moricz [25], Moricz and Rhoades [26], Tripathy ([36], [37]), Başarir and Sonalcan [6] and many others. Hardy [20] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [39] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [28] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Subsequently, Mursaleen [27] and Mursaleen and Edely [29] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{k,l})$ into one whose core is a subset of the M-core of x. More recently, Altay and Başar [1] have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{M}_u(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ and $\mathcal{L}_u$, respectively and also examined some properties of these sequence spaces and determined the α-duals of the spaces $\mathcal{BS}$, $\mathcal{BV}$, $\mathcal{CS}_{bp}$ and the $\beta(v)$-duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Now, recently Başar and Sever [7] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. The class of sequences which are strongly Cesàro summable with respect to a modulus function was introduced by Maddox [22] as an extension of the definition of strongly Cesàro summable sequences. Connor [9] further extended this notion to strong A-summability with respect to a modulus where $A = (a_{n,k})$ is a non-negative regular matrix. Using the definition Connor established connections between strong A-summability, strong A-summability with respect to a modulus and A-statistical convergence. In 1900, Pringsheim [30] presented a definition for convergence of double sequences. Following Pringsheim work, Hamilton and Robison in [18] and [33], respectively presented a series of necessary and sufficient conditions on the entries of $A = (a_{m,n,k,l})$ that ensure the preservation of Pringsheim type convergence on the following transformation of double sequences

$$(Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l}.$$

Throughout this paper the four dimensional matrices and double sequences are of real-valued entries unless otherwise specified. Let s'' denote the set of all double sequences of complex numbers. By convergence of a double sequence we shall mean the convergence in the Pringsheim sense, i.e., a double sequence $x = (x_{k,l})$ has Pringsheim limit L denoted by $P - \lim x = L$ if for a given $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ see [30]. We shall also describe such an x more briefly as P-convergent.

The notion of difference sequence spaces was introduced by Kızmaz [21], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Çolak [12] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$. Let w be the space of all complex or real sequences $x = (x_k)$ and let r, s be non-negative integers, then for $Z = l_\infty, c, c_0$ we have sequence spaces

$$Z(\Delta_s^r) = \{x = (x_k) \in w \colon (\Delta_s^r x_k) \in Z\},$$

where $\Delta_s^r x = (\Delta_s^r x_k) = (\Delta_s^{r-1} x_k - \Delta_s^{r-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_s^r x_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} x_{k+sv}.$$

Taking $s = 1$, we get the spaces which were introduced and studied by Et and Çolak [12]. Taking $r = s = 1$, we get the spaces which were introduced and studied by Kızmaz [21].

A modulus function is a function $f \colon [0,\infty) \to [0,\infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0$,

(2) $f(x+y) \leq f(x) + f(y)$, for all $x \geq 0$, $y \geq 0$,

(3) f is increasing and

(4) f is continuous from right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{x}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p$, $0 < p < 1$, then the modulus $f(x)$ is unbounded. Modulus function has been discussed in ([3], [4], [5], [10], [23], [31], [33], [34]) and references therein.

Let X be a linear metric space. A function $p\colon X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$,

(2) $p(-x) = p(x)$, for all $x \in X$,

(3) $p(x+y) \leq p(x) + p(y)$, for all $x, y \in X$,

(4) if (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$, as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$, as $n \to \infty$, then $p(\lambda_n x_n - \lambda x) \to 0$, as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [Theorem 10.4.2, 38]).

Let $A = (a_{m,n,k,l})$ denote a four dimensional summability method that maps the complex double sequences x into the double sequence Ax where the mnth term of Ax is as follows:

$$(Ax)_{m,n} = \sum_{k,l=1,1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}.$$

Let $F = (f_{k,l})$ be a sequence of modulus function and $A = (a_{m,n,k,l})$ be a non-negative four dimensional matrix of real entries with

$$\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty.$$

Let $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers. In the present paper we define the following sequence spaces:

$$w_0''(\Delta_s^r, A, F, u, p, \|\cdot,\cdots,\cdot\|)$$
$$= \left\{x \in s'' \colon P-\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}\, f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)^{p_{k,l}}\right] = 0,\right.$$
$$\left.\rho > 0\right\},$$

$$w''(\Delta_s^r, A, F, u, p, \|\cdot,\cdots,\cdot\|)$$
$$= \left\{x \in s'' \colon P-\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}\, f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l} - L}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)^{p_{k,l}}\right] = 0,\right.$$
$$\left.\text{for some } L,\ \rho > 0\right\}$$

and

$$w_\infty''(\Delta_s^r, A, F, u, p, \|\cdot,\cdots,\cdot\|)$$
$$= \left\{x \in s'' \colon \sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}\, f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)^{p_{k,l}}\right] < \infty,\right.$$
$$\left.\rho > 0\right\}.$$

If $F(x) = x$, we have

$$w_0''(\Delta_s^r, A, u, p, \|\cdot,\cdots,\cdot\|)$$
$$= \left\{x \in s'' \colon P-\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)^{p_{k,l}}\right] = 0,\right.$$
$$\left.\rho > 0\right\},$$

$$w''(\Delta_s^r, A, u, p, \|\cdot,\cdots,\cdot\|)$$
$$= \left\{x \in s'' \colon P-\lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}\left(\left\|\frac{\Delta_s^r x_{k,l} - L}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)^{p_{k,l}}\right] = 0,\right.$$
$$\left.\text{for some } L,\ \rho > 0\right\}$$

and

$$w_\infty''(\Delta_s^r, A, u, p, \|\cdot,\cdots,\cdot\|)$$
$$= \left\{x \in s'' \colon \sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1,\cdots,z_{n-1}\right\|\right)^{p_{k,l}}\right] < \infty,\ \rho > 0\right\}.$$

If we take $p = (p_{k,l}) = 1$, for all $k \in \mathbb{N}$, we have

$$w_0''(\Delta_s^r, A, F, u, \|\cdot, \cdots, \cdot\|)$$
$$= \left\{x \in s'' \colon P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l}\, f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right] = 0,\right.$$
$$\left.\rho > 0\right\},$$
$$w''(\Delta_s^r, A, F, u, \|\cdot, \cdots, \cdot\|)$$
$$= \left\{x \in s'' \colon P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l}\, f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right] = 0,\right.$$
$$\left.\text{for some } L,\ \rho > 0\right\}$$

and

$$w_\infty''(\Delta_s^r, A, F, u, \|\cdot, \cdots, \cdot\|)$$
$$= \left\{x \in s'' \colon \sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l}\, f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)\right] < \infty,\ \rho > 0\right\}.$$

If we take $u = (u_{k,l}) = 1$, for all $k \in \mathbb{N}$, we have

$$w_0''(\Delta_s^r, A, F, p, \|\cdot, \cdots, \cdot\|)$$
$$= \left\{x \in s'' \colon P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} \left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] = 0,\ \rho > 0\right\},$$
$$w''(\Delta_s^r, A, F, p, \|\cdot, \cdots, \cdot\|)$$
$$= \left\{x \in s'' \colon P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} \left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] = 0,\right.$$
$$\left.\text{for some } L,\ \rho > 0\right\}$$

and

$$w_\infty''(\Delta_s^r, A, F, p, \|\cdot, \cdots, \cdot\|)$$
$$= \left\{x \in s'' \colon \sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} \left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] < \infty,\ \rho > 0\right\}.$$

If we take $A = (C, 1, 1)$, we have

$$w_0''(\Delta_s^r, F, u, p, \|\cdot, \cdots, \cdot\|)$$
$$= \left\{ x \in s'' \colon P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} u_{k,l} \left[f_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] = 0, \right.$$
$$\left. \rho > 0 \right\},$$

$$w''(\Delta_s^r, F, u, p, \|\cdot, \cdots, \cdot\|)$$
$$= \left\{ x \in s'' \colon P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} u_{k,l} \left[f_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] = 0, \right.$$
$$\left. \text{for some } L,\ \rho > 0 \right\}$$

and

$$w_\infty''(\Delta_s^r, F, u, p, \|\cdot, \cdots, \cdot\|)$$
$$= \left\{ x \in s'' \colon \sup_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} u_{k,l} \left[f_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] < \infty, \right.$$
$$\left. \rho > 0 \right\}.$$

If we take $A = (C, 1, 1)$ and $F(x) = x$, we have

$$w_0''(\Delta_s^r, u, p, \|\cdot, \cdots, \cdot\|)$$
$$= \left\{ x \in s'' \colon P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} u_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} = 0, \right.$$
$$\left. \rho > 0 \right\},$$

$$w''(\Delta_s^r, u, p, \|\cdot, \cdots, \cdot\|)$$
$$= \left\{ x \in s'' \colon P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} u_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} = 0, \right.$$
$$\left. \text{for some } L,\ \rho > 0 \right\}$$

and

$$w_\infty''(\Delta_s^r, u, p, \|\cdot, \cdots, \cdot\|)$$
$$= \left\{ x \in s'' \colon \sup_{m,n} \frac{1}{mn} \sum_{k,l=0,0}^{m-1,n-1} u_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} < \infty,\ \rho > 0 \right\}.$$

If we take $F(x) = f(x)$, $p = (p_{k,l}) = 1$, $u = (u_{k,l}) = 1$, $r, s = 0$ and $\|\cdot, \cdots, \cdot\| = 1$, then the above spaces reduces to $w''_0(A, f)$, $w''(A, f)$ and $w''_\infty(A, f)$ which were studied by Savaş and Patterson [33].

The following inequality will be used throughout the paper. Let $p = (p_{k,l})$ be a sequence of positive real numbers with $0 \leq p_{k,l} \leq \sup p_{k,l} = H$ and $K = \max(1, 2^{H-1})$ then

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq K\{|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}\} \tag{1.1}$$

for all k, l and $a_{k,l}, b_{k,l} \in \mathbb{C}$. Also $|a|^{p_{k,l}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main purpose of this paper is to study some new type of double sequence spaces defined by a sequence of modulus function and a four dimensional matrix $A = (a_{m,n,k,l})$ of real entries with

$$\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty.$$

We also studied some topological properties and interested inclusion relations between the above defined sequence spaces.

2 Main results

Theorem 2.1 *Let $F = (f_{k,l})$ be a sequence of modulus function, $A = (a_{m,n,k,l})$ be a non negative matrix such that $\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty$, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers, the spaces $w''_0(\Delta^r_s, A, F, u, p, \|\cdot, \cdots, \cdot\|)$, $w''(\Delta^r_s, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ and $w''_\infty(\Delta^r_s, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ are linear over the field of complex numbers $\mathbb{C}$.*

Proof Let $x = (x_{k,l}), y = (y_{k,l}) \in w''_0(\Delta^r_s, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers ρ_1 and ρ_2 such that

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \Big[a_{m,n,k,l}\ f_{k,l}\Big(\Big\|\frac{\Delta^r_s x_{k,l}}{\rho_1}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_{k,l}}\Big] = 0$$

and

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \Big[a_{m,n,k,l}\ f_{k,l}\Big(\Big\|\frac{\Delta^r_s y_{k,l}}{\rho_2}, z_1, \cdots, z_{n-1}\Big\|\Big)^{p_{k,l}}\Big] = 0.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $(f_{k,l})$ is increasing, continuous and so by using inequality (1.1), we have

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r(\alpha x_{k,l}+\beta y_{k,l})}{\rho_3}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$
$$\leq \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\alpha\Delta_s^r x_{k,l}+\beta\Delta_s^r y_{k,l}}{\rho_3}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$
$$\leq K\sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}\ f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho_1}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$
$$+ K\sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}\ f_{k,l}\left(||\frac{\Delta_s^r y_{k,l}}{\rho_2}, z_1, \cdots, z_{n-1}||\right)^{p_{k,l}}\right] \to 0.$$

Thus $\alpha x+\beta y \in w_0''(\Delta^r, A, F, u, p)$. This proves that $w_0''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ is a linear space. Similarly, we can prove that $w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ and $w_\infty''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ are linear spaces. □

Theorem 2.2 *Let $F = (f_{k,l})$ be a sequence of modulus function and $A = (a_{m,n,k,l})$ be a non negative matrix such that* $\sup_{m,n}\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty$, *then*

(i) $w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|) \subset w_\infty''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$;

(ii) $w_0''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|) \subset w_\infty''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$.

Proof (i) Let $x = (x_{k,l}) \in w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$. Then

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$
$$= \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}-L+L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$
$$\leq \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}-L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$
$$+ u_{k,l}\left[f_{k,l}\left(||\frac{L}{\rho}, z_1, \cdots, z_{n-1}||\right)^{p_{k,l}}\right]\sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}.$$

Let there exists an integer M_l such that $\left\| \frac{L}{\rho}, z_1, \cdots, z_{n-1} \right\| \leq M_l$. Thus, we have

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l} f_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right]$$
$$= \sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l} f_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right]$$
$$+ M_l \, u_{k,l} \, f_{k,l}(1) \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}.$$

Since $\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty$ and $x = (x_{k,l}) \in w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$. Thus, we have $x = (x_{k,l}) \in w''_\infty(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ and this completes the proof.

(*ii*) It is easy to prove in view of (*i*) so we omit the details. □

Theorem 2.3 *Let $F = (f_{k,l})$ be a sequence of modulus function, $A = (a_{m,n,k,l})$ be a non-negative matrix such that $\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty$, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers, the spaces $w''_0(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ and $w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ are paranorm with the paranorm defined by*

$$g(x) = \sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l} f_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right].$$

Proof We shall prove the result for $w''_0(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$. Let $x = (x_{k,l}) \in w''_0(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$. It is clear from Theorem 2.2, for each $x = (x_{k,l}) \in w''_0(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$, $g(x)$ exists. Also it is clear that $g(\theta) = 0$, $g(-x) = g(x)$ and $g(x + y) \leq g(x) + g(y)$.

We now show that the scalar multiplication is continuous. First observe the following:

$$g(\lambda x) = \sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l} f_{k,l} \left(\left\| \frac{\lambda \Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right]$$
$$\leq (1 + [|\lambda|]) g(x),$$

where $[|\lambda|]$ denotes the integer part of $|\lambda|$. It is also clear that x and $\lambda \to 0$ implies $g(\lambda x) \to 0$. For fixed λ, if $x \to 0$ then $g(\lambda x) \to 0$. We need to show that for fixed $x, \lambda \to 0$ implies $g(\lambda x) \to 0$. Let $x \in w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ this implies that

$$P - \lim_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l} f_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] = 0.$$

Let $\epsilon > 0$ and choose N such that

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}-L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] < \frac{\epsilon}{4} \tag{2.1}$$

for $m, n > N$. Also, for each m, n with $1 \leq m, n \leq N$, since

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}-L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] < \infty,$$

there exists an integer $M_{m,n}$ such that

$$\sum_{k,l>M_{m,n}} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}-L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] < \frac{\epsilon}{4}.$$

Let

$$M = \max_{1\leq(m,n)\leq N}\{M_{m,n}\}.$$

We have for each m, n with $1 \leq m, n \leq N$

$$\sum_{k,l>M} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}-L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] < \frac{\epsilon}{4}.$$

Also from (2.1), for $m, n > N$ we have

$$\sum_{k,l>M} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}-L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] < \frac{\epsilon}{4}.$$

Thus M is an integer independent of m, n such that

$$\sum_{k,l>M} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}-L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] < \frac{\epsilon}{4}. \tag{2.2}$$

Further for $|\lambda| < 1$ and for all m, n,

$$\begin{aligned}
&\sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda \Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] \\
&= \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda \Delta_s^r x_{k,l} - \lambda L + \lambda L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] \\
&\leq \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda \Delta_s^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] \\
&\quad + \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] \\
&\leq \sum_{k,l>M} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda \Delta_s^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] \\
&\quad + \sum_{k,l\leq M} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda \Delta_s^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] \\
&\quad + \sum_{k\geq M, l<M} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda \Delta^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] \\
&\quad + \sum_{k<M, l\geq M} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda \Delta_s^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] \\
&\quad + \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right].
\end{aligned}$$

For each m, n and by the continuity of f as $\lambda \to 0$ we have the following:

$$\begin{aligned}
&\sum_{k,l\leq M} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda \Delta_s^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] \\
&\quad + \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] \to 0
\end{aligned}$$

in the Pringsheim sense. Now choose $\delta < 1$ such that $|\lambda| < \delta$ implies

$$\begin{aligned}
&\sum_{k,l\leq M} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda \Delta_s^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] \\
&\quad + \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\lambda L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] < \frac{\epsilon}{4}. \qquad (2.3)
\end{aligned}$$

In the same manner we have

$$\sum_{k\geq M,l<M} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\lambda\Delta_s^r x_{k,l}-\lambda L}{\rho},z_1,\cdots,z_{n-1}\right\|\right)^{p_{k,l}}\right]<\frac{\epsilon}{4}, \quad (2.4)$$

and

$$\sum_{k<M,l\geq M} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\lambda\Delta_s^r x_{k,l}-\lambda L}{\rho},z_1,\cdots,z_{n-1}\right\|\right)^{p_{k,l}}\right]<\frac{\epsilon}{4}. \quad (2.5)$$

It follows from equation (2.2), (2.3),(2.4) and (2.5) that

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\lambda\Delta_s^r x_{k,l}-\lambda L}{\rho},z_1,\cdots,z_{n-1}\right\|\right)^{p_{k,l}}\right]<\epsilon \text{ for all } m,n.$$

Thus $g(\lambda x)\to 0$ as $\lambda\to 0$. Therefore $w_0''(\Delta_s^r,A,F,u,p,\|\cdot,\cdots,\cdot\|)$ is a paranormed space. Similarly, we can prove that $w''(\Delta_s^r,A,F,u,p,\|\cdot,\cdots,\cdot\|)$ is a paranormed space. □

Theorem 2.4 *Let $F=(f_{k,l})$ be a sequence of modulus function, $A=(a_{m,n,k,l})$ be a non-negative matrix such that $\sup_{m,n}\sum_{k,l=0,0}^{\infty,\infty}a_{m,n,k,l}<\infty$, $p=(p_{k,l})$ be a bounded sequence of positive real numbers and $u=(u_{k,l})$ be any sequence of strictly positive real numbers, then $w_0''(\Delta_s^r,A,F,u,p,\|\cdot,\cdots,\cdot\|)$ and $w''(\Delta_s^r,A,F,u,p,\|\cdot,\cdots,\cdot\|)$ are complete topological linear spaces.*

Proof Let $(x_{k,l}^s)$ be a cauchy sequence in $w_0''(\Delta_s^r,A,F,u,p,\|\cdot,\cdots,\cdot\|)$. Then, we write $g(x^s-x^t)\to 0$ as $s,t\to\infty$ for all m,n, we have

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}^s-\Delta_s^r x_{k,l}^t}{\rho},z_1,\cdots,z_{n-1}\right\|\right)^{p_{k,l}}\right]\to 0. \quad (2.6)$$

Thus for each fixed k and l as $s,t\to\infty$, since $A=(a_{m,n,k,l})$ is non-negative, we are granted that

$$u_{k,l}\left[f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}^s-\Delta_s^r x_{k,l}^t}{\rho},z_1,\cdots,z_{n-1}\right\|\right)\right]\to 0$$

and by continuity of $F=(f_{k,l})$, $(x_{k,l}^s)$ is a Cauchy sequence in $\mathbb{C}$ for each fixed k and l. Since $\mathbb{C}$ is complete as $t\to\infty$, we have $x_{k,l}^s\to x_{k,l}$ for each (k,l). Now from equation (2.6), we have for $\epsilon>0$, there exists a natural number $\mathbb{N}$ such that

$$\sum_{k,l=0,0}^{\infty,\infty}\sum_{s,t>N} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}^s-\Delta_s^r x_{k,l}^t}{\rho},z_1,\cdots,z_{n-1}\right\|\right)^{p_{k,l}}\right]<\epsilon \quad (2.7)$$

for all m,n. Since for any fixed natural number M we have from equation (2.7)

$$\sum_{k,l\leq M}^{\infty,\infty}\sum_{s,t>N} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(||\frac{\Delta_s^r x_{k,l}^s-\Delta_s^r x_{k,l}^t}{\rho},z_1,\cdots,z_{n-1}||\right)^{p_{k,l}}\right]<\epsilon$$

for all m, n, by letting $t \to \infty$ in the above expression we obtain

$$\sum_{k,l \leq M\, s>N}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l} f_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l}^s - \Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] < \epsilon.$$

Since M is arbitrary, by letting $M \to \infty$ we obtain

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l} f_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l}^s - \Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] < \epsilon$$

for all m, n. Thus $g(x^s - x) \to 0$ as $s \to \infty$. This proves that $w_0''(\Delta^r, A, F, u, p)$ is a complete linear topological space.

Now, we shall show that $w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ is a complete linear topological space. For this, since (x^s) is also a sequence in $w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$, by definition of $w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$, for each s there exists L^s with

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l} f_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l}^s - \Delta_s^r L^s}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] \to 0$$

as $m, n \to \infty$, whence, from the fact that

$$\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty$$

from the definition of modulus function, we have $f_{k,l}(\| \frac{\Delta_s^r L^s - \Delta_s^r L^t}{\rho} \|) \to 0$ as $s, t \to \infty$ and so L^s converges to L. Thus

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l} f_{k,l} \left(\left\| \frac{\Delta_s^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] \to 0$$

as $m, n \to \infty$, thus $x \in w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ and this completes the proof. □

Theorem 2.5 *Let $F = (f_{k,l})$ be a sequence of modulus function and $A = (a_{m,n,k,l})$ be a non negative matrix such that* $\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty$*, then*

(i) $w''(\Delta_s^r, A, u, p, \|\cdot, \cdots, \cdot\|) \subset w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$;

(ii) $w_0''(\Delta_s^r, A, u, p, \|\cdot, \cdots, \cdot\|) \subset w_0''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$;

(iii) $w_\infty''(\Delta_s^r, A, u, p, \|\cdot, \cdots, \cdot\|) \subset w_\infty''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$.

Proof (i) and (ii) are easy to prove so we will prove (iii) only. Let $x = (x_{k,l}) \in w_\infty''(\Delta_s^r, A, u, p, \|\cdot, \cdots, \cdot\|)$ such that

$$\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l} \left[a_{m,n,k,l} \left(\left\| \frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] < \infty.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f_{k,l}(t) < \epsilon$ for $0 \leq t \leq \delta$. Thus, we have

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$

$$= \sum_{\substack{k,l=0,0 \\ \left(\left\|\frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right) \leq \delta}}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$

$$+ \sum_{\substack{k,l=0,0 \\ \left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right) > \delta}}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$

Since $F = (f_{k,l})$ is a sequence of modulus function, we have

$$\sum_{\substack{k,l=0,0 \\ \left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right) \leq \delta}}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right] \leq \epsilon \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l}. \tag{2.8}$$

For $\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right) > \delta$ and the fact that

$$\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right) < \left(\frac{\left|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right|}{\delta}\right) < \left[1 + \left(\frac{\left|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right|}{\delta}\right)\right]$$

where $[t]$ denotes the integer part of t and by the properties of modulus function, we have

$$f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}} < \left(1 + f_{k,l}\left[\frac{\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}}{\delta}\right]\right)$$

$$\leq 2f_k(1)\frac{\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}}{\delta}.$$

Thus

$$\sum_{\substack{k,l=0,0 \\ \left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right) > \delta}}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$

$$\leq \frac{2f_{k,l}(1)}{\delta} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]. \tag{2.9}$$

From equation (2.8) and (2.9) we have

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$

$$\leq \epsilon \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} + \frac{2f_{k,l}(1)}{\delta} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right].$$

Since $\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty$ and $x = (x_{k,l}) \in w''_\infty(\Delta_s^r, A, u, p, \|\cdot, \cdots, \cdot\|)$. Hence, we have $x = (x_{k,l}) \in w''_\infty(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$ and this completes the proof. □

Theorem 2.6 *Let $F = (f_{k,l})$ be a sequence of modulus function and $A = (a_{m,n,k,l})$ be a non negative matrix such that* $\sup_{m,n} \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} < \infty$ *and* $\beta = \lim_{t\to\infty} \frac{f_{k,l}(t)}{t} > 0$*, then*

$$w''(\Delta_s^r, A, u, p, \|\cdot, \cdots, \cdot\|) = w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|).$$

Proof In order to prove that

$$w''(\Delta_s^r, A, u, p, \|\cdot, \cdots, \cdot\|) = w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|).$$

It is sufficient to show that

$$w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|) \subset w''(\Delta_s^r, A, u, p, \|\cdot, \cdots, \cdot\|).$$

Now, let $\beta > 0$. By definition of β we have $f_{k,l}(t) \geq \beta(t)$ for all $t \geq 0$. Since $\beta > 0$, we have $t \leq \frac{1}{\beta} f_{k,l}(t)$ for all $t \geq 0$.

Let $x = (x_{k,l}) \in w''(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|)$. Thus, we have

$$\sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l}\left(\left\|\frac{\Delta_s^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$

$$\leq \frac{1}{\beta} \sum_{k,l=0,0}^{\infty,\infty} u_{k,l}\left[a_{m,n,k,l} f_{k,l}\left(\left\|\frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1}\right\|\right)^{p_{k,l}}\right]$$

which implies that $x = (x_{k,l}) \in w''(\Delta_s^r, A, u, p, \|\cdot, \cdots, \cdot\|)$. This completes the proof. □

Theorem 2.7 *If $A = (a_{m,n,k,l})$ has only positive entries and $B = (b_{m,n,k,l})$ be a non-negative matrix such that $\left\{\frac{b_{m,n,k,l}}{a_{m,n,k,l}}\right\}$ is bounded then*

$$w''_\infty(\Delta_s^r, A, F, u, p, \|\cdot, \cdots, \cdot\|) \subset w''_\infty(\Delta_s^r, B, F, u, p, \|\cdot, \cdots, \cdot\|).$$

Proof It is easy to prove so we omit the details. □

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