Double Sequence Spaces Defined by a Sequence of Modulus Functions over \( n \)-normed Spaces

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Abstract

In the present paper we introduce some double sequence spaces defined by a sequence of modulus function \( F = (f_{k,l}) \) over \( n \)-normed spaces. We also make an effort to study some topological properties and inclusion relations between these spaces.

Key words: double sequences, \( P \)-convergent, modulus function, paranorm space

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1 Introduction and preliminaries

The concept of 2-normed spaces was initially developed by Gähler [13] in the mid of 1960's, while that of \( n \)-normed spaces one can see in Misiak [24]. Since then, many others have studied this concept and obtained various results, see Gunawan ([15], [16]) and Gunawan and Mashadi [17] and references therein. Let \( n \in \mathbb{N} \) and \( X \) be a linear space over the field \( \mathbb{K} \), where \( \mathbb{K} \) is the field of real or complex numbers of dimension \( d \), where \( d \geq n \geq 2 \). A real valued function \( ||\cdot,\cdot,\cdot|| \) on \( X^n \) satisfying the following four conditions:

1. \( ||x_1, x_2, \cdots, x_n|| = 0 \) if and only if \( x_1, x_2, \cdots, x_n \) are linearly dependent in \( X \);
2. \( ||x_1, x_2, \cdots, x_n|| \) is invariant under permutation;
(3) $\|\alpha x_1, x_2, \cdots, x_n\| = |\alpha| \|x_1, x_2, \cdots, x_n\|$ for any $\alpha \in \mathbb{K}$;

(4) $\|x + x', x_2, \cdots, x_n\| \leq \|x, x_2, \cdots, x_n\| + \|x', x_2, \cdots, x_n\|$

is called a $n$-norm on $X$, and the pair $(X, \|\cdot, \cdots, \cdot\|)$ is called a $n$-normed space over the field $\mathbb{K}$. For example, we may take $X = \mathbb{R}^n$ being equipped with the Euclidean $n$-norm $\|x_1, x_2, \cdots, x_n\|_E = \text{the volume of the } n\text{-dimensional parallelopiped spanned by the vectors } x_1, x_2, \cdots, x_n \text{ which may be given explicitly by the formula}$

$$\|x_1, x_2, \cdots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \cdots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots, n$. Let $(X, \|\cdot, \cdots, \cdot\|)$ be a $n$-normed space of dimension $d \geq n \geq 2$ and $\{a_1, a_2, \cdots, a_n\}$ be linearly independent set in $X$. Then the following function $\|\cdot, \cdots, \cdot\|_\infty$ on $X^{n-1}$ defined by

$$\|x_1, x_2, \cdots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \cdots, x_{n-1}, a_i\| : i = 1, 2, \cdots, n\}$$

defines an $(n - 1)$-norm on $X$ with respect to $\{a_1, a_2, \cdots, a_n\}$.

A sequence $(x_k)$ in a $n$-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k \to \infty} \|x_k - L, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$ 

A sequence $(x_k)$ in a $n$-normed space $(X, \|\cdot, \cdots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k,p \to \infty} \|x_k - x_p, z_1, \cdots, z_{n-1}\| = 0 \text{ for every } z_1, \cdots, z_{n-1} \in X.$$ 

If every Cauchy sequence in $X$ converges to some $L \in X$, then $X$ is said to be complete with respect to the $n$-norm. Any complete $n$-normed space is said to be $n$-Banach space.

The initial works on double sequences are found in Bromwich [8]. Later on, it was studied by Hardy [19], Moricz [25], Moricz and Rhoades [26], Tripathy ([36], [37]), Başarir and Sonalcan [6] and many others. Hardy [20] introduced the notion of regular convergence for double sequences. Quite recently, Zeltser [39] in her Ph.D thesis has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [28] have recently introduced the statistical convergence and Cauchy convergence for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Subsequently, Mursaleen [27] and Mursaleen and Edely [29] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the $M$-core for double sequences and determined those four dimensional matrices transforming every bounded double sequences $x = (x_{k, l})$ into one whose core is a subset of the $M$-core of $x$. More recently, Altay and Başar [1] have defined the spaces $BS, BS(t), CS_p, CS_{bp}, CS_r$, and $BV$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $M_u, M_u(t), C_p, C_{bp}, C_r$ and $L_u,$
respectively and also examined some properties of these sequence spaces and
determined the $\alpha$-duals of the spaces $\mathcal{B}_s, \mathcal{B}_v, \mathcal{C}_{s_p}$ and the $\beta(v)$-duals of the
spaces $\mathcal{C}_{s_p}$ and $\mathcal{C}_r$ of double series. Now, recently Başar and Sever [7] have
introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well
known space $\ell_q$ of single sequences and examined some properties of the space
$\mathcal{L}_q$. The class of sequences which are strongly Cesàro summable with respect to
a modulus function was introduced by Maddox [22] as an extension of the deﬁni-
tion of strongly Cesàro summable sequences. Connor [9] further extended this
notion to strong $A$-summability with respect to a modulus where $A = (a_{n,k})$
is a non-negative regular matrix. Using the deﬁnition Connor established con-
nections between strong $A$-summability, strong $A$-summability with respect to
a deﬁnition for convergence of double sequences. Following Pringsheim work,
Hamilton and Robison in [18] and [33], respectively presented a series of neces-
sary and sufﬁcient conditions on the entries of $A = (a_{m,n,k,l})$ that ensure the
preservation of Pringsheim type convergence on the following transformation of
double sequences
\[(Ax)_{m,n} = \sum_{k,l=0,0}^{\infty,\infty} a_{m,n,k,l} x_{k,l}.\]
Throughout this paper the four dimensional matrices and double sequences are
of real-valued entries unless otherwise speciﬁed. Let $s''$ denote the set of all
double sequences of complex numbers. By convergence of a double sequence
we shall mean the convergence in the Pringsheim sense, i.e., a double sequence
$x = (x_{k,l})$ has Pringsheim limit $L$ denoted by $P - \lim x = L$ if for a given $\epsilon > 0$
there exists $n \in \mathbb{N}$ such that $|x_{k,l} - L| < \epsilon$ whenever $k, l > n$ see [30]. We shall
also describe such an $x$ more brieﬂy as $P$-convergent.
The notion of difference sequence spaces was introduced by Kızmaz [21], who
studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was
further generalized by Et and Çolak [12] by introducing the spaces $\ell_\infty(\Delta^n)$,
c($\Delta^n$) and $c_0(\Delta^n)$. Let $w$ be the space of all complex or real sequences $x = (x_k)$
and let $r, s$ be non-negative integers, then for $Z = l_\infty, c, c_0$ we have sequence
spaces
\[Z(\Delta^n_s) = \{x = (x_k) \in w : (\Delta^n_s x_k) \in Z\},\]
where $\Delta^n_s x = (\Delta^n_s x_k) = (\Delta^n_{s-1} x_k - \Delta^n_{s-1} x_{k+1})$ and $\Delta^0 x_k = x_k$ for all $k \in \mathbb{N},$
which is equivalent to the following binomial representation
\[\Delta^n_s x_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} x_{k+sv}.\]
Taking $s = 1$, we get the spaces which were introduced and studied by Et and
Çolak [12]. Taking $r = s = 1$, we get the spaces which were introduced and
studied by Kızmaz [21].
A modulus function is a function $f : [0, \infty) \to [0, \infty)$ such that

(1) $f(x) = 0$ if and only if $x = 0,$
(2) \( f(x + y) \leq f(x) + f(y) \), for all \( x \geq 0, \ y \geq 0 \),

(3) \( f \) is increasing and

(4) \( f \) is continuous from right at 0.

It follows that \( f \) must be continuous everywhere on \([0, \infty)\). The modulus function may be bounded or unbounded. For example, if we take \( f(x) = \frac{x}{x+1} \), then \( f(x) \) is bounded. If \( f(x) = x^p, 0 < p < 1 \), then the modulus \( f(x) \) is unbounded. Modulus function has been discussed in ([3], [4], [5], [10], [23], [31], [33], [34]) and references therein.

Let \( X \) be a linear metric space. A function \( p: X \to \mathbb{R} \) is called paranorm, if

(1) \( p(x) \geq 0 \), for all \( x \in X \),

(2) \( p(-x) = p(x) \), for all \( x \in X \),

(3) \( p(x + y) \leq p(x) + p(y) \), for all \( x, y \in X \),

(4) if \((\lambda_n)\) is a sequence of scalars with \( \lambda_n \to \lambda \), as \( n \to \infty \) and \((x_n)\) is a sequence of vectors with \( p(x_n - x) \to 0 \), as \( n \to \infty \), then \( p(\lambda_n x_n - \lambda x) \to 0 \), as \( n \to \infty \).

A paranorm \( p \) for which \( p(x) = 0 \) implies \( x = 0 \) is called total paranorm and the pair \((X, p)\) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [Theorem 10.4.2, 38]).

Let \( A = (a_{m,n,k,l}) \) denote a four dimensional summability method that maps the complex double sequences \( x \) into the double sequence \( Ax \) where the \( mn \)th term of \( Ax \) is as follows:

\[
(Ax)_{m,n} = \sum_{k,l=1}^{\infty,\infty} a_{m,n,k,l}x_{k,l}.
\]

Let \( F = (f_{k,l}) \) be a sequence of modulus function and \( A = (a_{m,n,k,l}) \) be a non-negative four dimensional matrix of real entries with

\[
\sup_{m,n} \sum_{k,l=0}^{\infty,\infty} a_{m,n,k,l} < \infty.
\]

Let \( p = (p_{k,l}) \) be a bounded sequence of positive real numbers and \( u = (u_{k,l}) \) be any sequence of strictly positive real numbers. In the present paper we define
the following sequence spaces:

\[
\begin{aligned}
& w'' \left( \Delta_s^r, A, F, u, p, \| \cdot \| \right) \\
& = \left\{ x \in s'' : P - \lim_{m,n} \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left( a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right)^{p_{k,l}} = 0, \\
& \quad \rho > 0 \right\}, \\
\end{aligned}
\]

\[
\begin{aligned}
& w'' \left( \Delta_s^r, A, F, u, p, \| \cdot \| \right) \\
& = \left\{ x \in s'' : P - \lim_{m,n} \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left( a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right)^{p_{k,l}} = 0, \\
& \quad \text{for some } L, \rho > 0 \right\}
\end{aligned}
\]

and

\[
\begin{aligned}
& w'' \left( \Delta_s^r, A, F, u, p, \| \cdot \| \right) \\
& = \left\{ x \in s'' : \sup_{m,n} \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left( a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right)^{p_{k,l}} < \infty, \\
& \quad \rho > 0 \right\}
\end{aligned}
\]

If \( F(x) = x \), we have

\[
\begin{aligned}
& w'' \left( \Delta_s^r, A, u, p, \| \cdot \| \right) \\
& = \left\{ x \in s'' : P - \lim_{m,n} \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left( a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right)^{p_{k,l}} = 0, \\
& \quad \rho > 0 \right\}, \\
\end{aligned}
\]

\[
\begin{aligned}
& w'' \left( \Delta_s^r, A, u, p, \| \cdot \| \right) \\
& = \left\{ x \in s'' : P - \lim_{m,n} \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left( a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right)^{p_{k,l}} = 0, \\
& \quad \text{for some } L, \rho > 0 \right\}
\end{aligned}
\]

and

\[
\begin{aligned}
& w'' \left( \Delta_s^r, A, u, p, \| \cdot \| \right) \\
& = \left\{ x \in s'' : \sup_{m,n} \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left( a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right)^{p_{k,l}} < \infty, \rho > 0 \right\}
\end{aligned}
\]
If we take \( p = (p_{k,l}) = 1 \), for all \( k \in \mathbb{N} \), we have

\[
\begin{align*}
& w''_0(\Delta^r_s, A, F, u, \|\cdot\|, \ldots, \|\cdot\|) \\
= & \left\{ x \in s'': P - \lim_{m,n} \infty \sum_{k,l=0} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^r s x_{k,l}}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right] = 0, \\
& \quad \rho > 0 \right\}, \\
& w''(\Delta^r_s, A, F, u, \|\cdot\|, \ldots, \|\cdot\|) \\
= & \left\{ x \in s'': P - \lim_{m,n} \infty \sum_{k,l=0} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^r s x_{k,l} - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right] = 0, \\
& \quad \text{for some } L, \rho > 0 \right\}
\end{align*}
\]

and

\[
\begin{align*}
& w''_\infty(\Delta^r_s, A, F, u, \|\cdot\|, \ldots, \|\cdot\|) \\
= & \left\{ x \in s'': \sup_{m,n} \infty \sum_{k,l=0} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^r s x_{k,l}}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right] < \infty, \rho > 0 \right\}.
\end{align*}
\]

If we take \( u = (u_{k,l}) = 1 \), for all \( k \in \mathbb{N} \), we have

\[
\begin{align*}
& w''_0(\Delta^r_s, A, F, p, \|\cdot\|, \ldots, \|\cdot\|) \\
= & \left\{ x \in s'': P - \lim_{m,n} \infty \sum_{k,l=0} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^r s x_{k,l}}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_{k,l}} \right] = 0, \rho > 0 \right\}, \\
& w''(\Delta^r_s, A, F, p, \|\cdot\|, \ldots, \|\cdot\|) \\
= & \left\{ x \in s'': P - \lim_{m,n} \infty \sum_{k,l=0} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^r s x_{k,l} - L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_{k,l}} \right] = 0, \\
& \quad \text{for some } L, \rho > 0 \right\}
\end{align*}
\]

and

\[
\begin{align*}
& w''_\infty(\Delta^r_s, A, F, p, \|\cdot\|, \ldots, \|\cdot\|) \\
= & \left\{ x \in s'': \sup_{m,n} \infty \sum_{k,l=0} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^r s x_{k,l}}{\rho}, z_1, \ldots, z_{n-1} \right\| \right)^{p_{k,l}} \right] < \infty, \rho > 0 \right\}.
\end{align*}
\]
If we take $A = (C, 1, 1)$, we have

$$w''_0(\Delta^r_s, F, u, p, \|\cdot\|) = \left\{ x \in s'' : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0}^{m-1,n-1} u_{k,l} \left[ f_{k,l} \left( \frac{\Delta^r_{s,k,l}}{\rho}, z_1, \cdots, z_{n-1} \right) \right] = 0, \quad \rho > 0 \right\},$$

$$w''(\Delta^r_s, F, u, p, \|\cdot\|) = \left\{ x \in s'' : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0}^{m-1,n-1} u_{k,l} \left[ f_{k,l} \left( \frac{\Delta^r_{s,k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right) \right] = 0, \quad \text{for some } L, \rho > 0 \right\}.$$

and

$$w''_\infty(\Delta^r_s, F, u, p, \|\cdot\|) = \left\{ x \in s'' : \sup_{m,n} \frac{1}{mn} \sum_{k,l=0}^{m-1,n-1} u_{k,l} \left[ f_{k,l} \left( \frac{\Delta^r_{s,k,l}}{\rho}, z_1, \cdots, z_{n-1} \right) \right] < \infty, \quad \rho > 0 \right\}.$$

If we take $A = (C, 1, 1)$ and $F(x) = x$, we have

$$w''_0(\Delta^r_s, u, p, \|\cdot\|) = \left\{ x \in s'' : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0}^{m-1,n-1} u_{k,l} \left( \frac{\Delta^r_{s,k,l}}{\rho}, z_1, \cdots, z_{n-1} \right) = 0, \quad \rho > 0 \right\},$$

$$w''(\Delta^r_s, u, p, \|\cdot\|) = \left\{ x \in s'' : P - \lim_{m,n} \frac{1}{mn} \sum_{k,l=0}^{m-1,n-1} u_{k,l} \left( \frac{\Delta^r_{s,k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right) = 0, \quad \text{for some } L, \rho > 0 \right\}.$$

and

$$w''_\infty(\Delta^r_s, u, p, \|\cdot\|) = \left\{ x \in s'' : \sup_{m,n} \frac{1}{mn} \sum_{k,l=0}^{m-1,n-1} u_{k,l} \left( \frac{\Delta^r_{s,k,l}}{\rho}, z_1, \cdots, z_{n-1} \right) < \infty, \rho > 0 \right\}.
If we take $F(x) = f(x)$, $p = (p_{k,l}) = 1$, $u = (u_{k,l}) = 1$, $r, s = 0$ and $\|, \cdots, \| = 1$, then the above spaces reduce to $w''_0(A, f)$, $w''(A, f)$ and $w''_\infty(A, f)$ which were studied by Savaş and Patterson [33].

The following inequality will be used throughout the paper. Let $p = (p_{k,l})$ be a sequence of positive real numbers with $0 \leq p_{k,l} \leq \sup p_{k,l} = H$ and $K = \max(1, 2H-1)$ then

$$|a_{k,l} + b_{k,l}|^{p_{k,l}} \leq K \{|a_{k,l}|^{p_{k,l}} + |b_{k,l}|^{p_{k,l}}\} \quad (1.1)$$

for all $k, l$ and $a_{k,l}, b_{k,l} \in \mathbb{C}$. Also $|a|^{p_{k,l}} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

The main purpose of this paper is to study some new type of double sequence spaces defined by a sequence of modulus function and a four dimensional matrix $A = (a_{m,n,k,l})$ of real entries with

$$\sup_{m,n} \sum_{k,l=0}^{\infty, \infty} a_{m,n,k,l} < \infty.$$

We also studied some topological properties and interested inclusion relations between the above defined sequence spaces.

## 2 Main results

**Theorem 2.1** Let $F = (f_{k,l})$ be a sequence of modulus function, $A = (a_{m,n,k,l})$ be a non negative matrix such that $\sup_{m,n} \sum_{k,l=0}^{\infty, \infty} a_{m,n,k,l} < \infty$, $p = (p_{k,l})$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers, the spaces $w''_0(\Delta_s, A, F, u, p, \|, \cdots, \|)$, $w''(\Delta_s, A, F, u, p, \|, \cdots, \|)$ and $w''_\infty(\Delta_s, A, F, u, p, \|, \cdots, \|)$ are linear over the field of complex numbers $\mathbb{C}$.

**Proof** Let $x = (x_{k,l}), y = (y_{k,l}) \in w''_0(\Delta_s, A, F, u, p, \|, \cdots, \|)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive real numbers $\rho_1$ and $\rho_2$ such that

$$\sum_{k,l=0}^{\infty, \infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s x_{k,l}}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] = 0$$

and

$$\sum_{k,l=0}^{\infty, \infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s y_{k,l}}{\rho_2}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] = 0.$$
Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since $(f_{k,l})$ is increasing, continuous and so by using inequality (1.1), we have

$$\sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s^r (\alpha x_{k,l} + \beta y_{k,l})}{\rho_3}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right]$$

$$\leq \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\alpha \Delta_x^r k,l + \beta \Delta_y^r k,l}{\rho_3}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right]$$

$$\leq K \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_x^r k,l}{\rho_1}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right]$$

$$+ K \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\rho_2}{\Delta_y^r k,l}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] \to 0.$$ 

Thus $\alpha x + \beta y \in w''_0(\Delta^r, A, F, u, p)$. This proves that $w''_0(\Delta^r, A, F, u, p, ||\cdot||, \cdots, ||\cdot||)$ is a linear space. Similarly, we can prove that $w''(\Delta^r, A, F, u, p, ||\cdot||, \cdots, ||\cdot||)$ and $w''(\Delta^r, A, F, u, p)$ are linear spaces. □

**Theorem 2.2** Let $F = (f_{k,l})$ be a sequence of modulus function and $A = (a_{m,n,k,l})$ be a non negative matrix such that $\sup_{m,n} \sum_{k,l=0}^{\infty,\infty} a_{m,n,k,l} < \infty$, then

(i) $w''(\Delta^r, A, F, u, p, ||\cdot||, \cdots, ||\cdot||) \subset w''(\Delta^r, A, F, u, p, ||\cdot||, \cdots, ||\cdot||);$ 

(ii) $w''_0(\Delta^r, A, F, u, p, ||\cdot||, \cdots, ||\cdot||) \subset w''(\Delta^r, A, F, u, p, ||\cdot||, \cdots, ||\cdot||).$

**Proof** (i) Let $x = (x_{k,l}) \in w''(\Delta^r, A, F, u, p, ||\cdot||, \cdots, ||\cdot||)$. Then

$$\sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_x^r k,l}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right]$$

$$= \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_x^r k,l - L + L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right]$$

$$\leq \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_x^r k,l - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right]$$

$$+ u_{k,l} \left[ f_{k,l} \left( \left\| \frac{L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] \sum_{k,l=0}^{\infty,\infty} a_{m,n,k,l}.$$
Let there exists an integer $M_l$ such that $\|\frac{L}{\rho}, z_1, \cdots, z_{n-1}\| \leq M_l$. Thus, we have

$$\sum_{k,l=0}^{\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^r_{s} x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}^r}$$

$$= \sum_{k,l=0}^{\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^r_{s} x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}^r} + M_l u_{k,l} f_{k,l} (1) \sum_{k,l=0}^{\infty} a_{m,n,k,l}.$$

Since $\sup_{m,n} \sum_{k,l=0}^{\infty} a_{m,n,k,l} < \infty$ and $x = (x_{k,l}) \in w''(\Delta^r_{s}, A, F, u, p, \|\cdot\|, \|\cdot\|, \|\cdot\|)$. Thus, we have $x = (x_{k,l}) \in w''(\Delta^r_{s}, A, F, u, p, \|\cdot\|, \|\cdot\|) \cap w''(\Delta^r_{s}, A, F, u, p, \|\cdot\|, \|\cdot\|)$ and this completes the proof.

(ii) It is easy to prove in view of (i) so we omit the details. \hfill \Box

**Theorem 2.3** Let $F = (f_{k,l})$ be a sequence of modulus function, $A = (a_{m,n,k,l})$ be a non-negative matrix such that $\sup_{m,n} \sum_{k,l=0}^{\infty} a_{m,n,k,l} < \infty$, $p = (p_{k,l}^r)$ be a bounded sequence of positive real numbers and $u = (u_{k,l})$ be any sequence of strictly positive real numbers, the spaces $w''(\Delta^r_{s}, A, F, u, p, \|\cdot\|, \|\cdot\|, \|\cdot\|)$ and $w''(\Delta^r_{s}, A, F, u, p, \|\cdot\|, \|\cdot\|, \|\cdot\|)$ are paranorm with the paranorm defined by

$$g(x) = \sup_{m,n} \sum_{k,l=0}^{\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^r_{s} x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}^r}.$$

**Proof** We shall prove the result for $w''(\Delta^r_{s}, A, F, u, p, \|\cdot\|, \|\cdot\|, \|\cdot\|)$. Let $x = (x_{k,l}) \in w''(\Delta^r_{s}, A, F, u, p, \|\cdot\|, \|\cdot\|, \|\cdot\|)$. It is clear from Theorem 2.2, for each $x = (x_{k,l}) \in w''(\Delta^r_{s}, A, F, u, p, \|\cdot\|, \|\cdot\|, \|\cdot\|)$, $g(x)$ exists. Also it is clear that $g(\theta) = 0, g(-x) = g(x)$ and $g(x + y) \leq g(x) + g(y)$.

We now show that the scalar multiplication is continuous. First observe the following:

$$g(\lambda x) = \sup_{m,n} \sum_{k,l=0}^{\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\lambda \Delta^r_{s} x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}^r} \leq (1 + ||\lambda||) g(x),$$

where $||\lambda||$ denotes the integer part of $|\lambda|$. It is also clear that $x$ and $\lambda \rightarrow 0$ implies $g(\lambda x) \rightarrow 0$. For fixed $\lambda$, if $x \rightarrow 0$ then $g(\lambda x) \rightarrow 0$. We need to show that for fixed $x, \lambda \rightarrow 0$ implies $g(\lambda x) \rightarrow 0$. Let $x \in w''(\Delta^r_{s}, A, F, u, p, \|\cdot\|, \|\cdot\|)$ this implies that

$$P - \lim_{m,n} \sum_{k,l=0}^{\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^r_{s} x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}^r} = 0.$$
Let $\epsilon > 0$ and choose $N$ such that

$$
\sum_{k,l=0,0}^{\infty,\infty} u_{k,l} a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} < \frac{\epsilon}{4} \tag{2.1}
$$

for $m, n > N$. Also, for each $m, n$ with $1 \leq m, n \leq N$, since

$$
\sum_{k,l=0,0}^{\infty,\infty} u_{k,l} a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} < \infty,
$$

there exists an integer $M_{m,n}$ such that

$$
\sum_{k,l>M_{m,n}} u_{k,l} a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} < \frac{\epsilon}{4}.
$$

Let

$$
M = \max_{1 \leq (m,n) \leq N} \{ M_{m,n} \}.
$$

We have for each $m, n$ with $1 \leq m, n \leq N$

$$
\sum_{k,l>M} u_{k,l} a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} < \frac{\epsilon}{4}.
$$

Also from (2.1), for $m, n > N$ we have

$$
\sum_{k,l>M} u_{k,l} a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} < \frac{\epsilon}{4}.
$$

Thus $M$ is an integer independent of $m, n$ such that

$$
\sum_{k,l>M} u_{k,l} a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_s x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} < \frac{\epsilon}{4}. \tag{2.2}
$$
Further for $|\lambda| < 1$ and for all $m,n$,

$$\sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\lambda \Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right]$$

$$= \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\Delta^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right]$$

$$\leq \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\Delta^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right]$$

$$+ \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\lambda L}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right]$$

$$\leq \sum_{k,l>M} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\Delta^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right]$$

$$+ \sum_{k,l \leq M} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\Delta^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right]$$

$$+ \sum_{k \geq M, l < M} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\Delta^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right]$$

$$+ \sum_{k < M, l \geq M} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\Delta^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right]$$

$$+ \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\lambda L}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right].$$

For each $m,n$ and by the continuity of $f$ as $\lambda \to 0$ we have the following:

$$\sum_{k,l \leq M} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\Delta^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right]$$

$$+ \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\lambda L}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right] \to 0$$

in the Pringsheim sense. Now choose $\delta < 1$ such that $|\lambda| < \delta$ implies

$$\sum_{k,l \leq M} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\Delta^r x_{k,l} - \lambda L}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right]$$

$$+ \sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\{ \frac{\lambda L}{\rho}, z_1, \cdots, z_{n-1} \right\} \right)^{p_{k,l}} \right] < \frac{\epsilon}{4}. \quad (2.3)$$
In the same manner we have
\[
\sum_{k \geq M, l < M} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\lambda \Delta^{r}_{s} x_{k,l} - \lambda L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_{k,l}} < \frac{\epsilon}{4}, \quad (2.4)
\]
and
\[
\sum_{k < M, l \geq M} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\lambda \Delta^{r}_{s} x_{k,l} - \lambda L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_{k,l}} < \frac{\epsilon}{4}. \quad (2.5)
\]
It follows from equation (2.2), (2.3), (2.4) and (2.5) that
\[
\sum_{k,l=0,0}^{\infty, \infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\lambda \Delta^{r}_{s} x_{k,l} - \lambda L}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_{k,l}} < \epsilon \text{ for all } m, n.
\]
Thus \( g(\lambda x) \to 0 \) as \( \lambda \to 0 \). Therefore \( w''(\Delta^{r}_{s}, A, F, u, p, \|\cdot\|, \ldots, \|\cdot\|) \) is a paranormed space. Similarly, we can prove that \( w''(\Delta^{r}_{s}, A, F, u, p, \|\cdot\|, \ldots, \|\cdot\|) \) is a paranormed space.

**Theorem 2.4** Let \( F = (f_{k,l}) \) be a sequence of modulus function, \( A = (a_{m,n,k,l}) \) be a non-negative matrix such that \( \sup_{m,n} \sum_{k,l=0,0}^{\infty, \infty} a_{m,n,k,l} < \infty \), \( p = (p_{k,l}) \) be a bounded sequence of positive real numbers and \( u = (u_{k,l}) \) be any sequence of strictly positive real numbers, then \( w''(\Delta^{r}_{s}, A, F, u, p, \|\cdot\|, \ldots, \|\cdot\|) \) and \( w''(\Delta^{r}_{s}, A, F, u, p, \|\cdot\|, \ldots, \|\cdot\|) \) are complete topological linear spaces.

**Proof** Let \( (x^{s}_{k,l}) \) be a cauchy sequence in \( w''(\Delta^{r}_{s}, A, F, u, p, \|\cdot\|, \ldots, \|\cdot\|) \). Then, we write \( g(x^{s} - x^{t}) \to 0 \) as \( s, t \to \infty \) for all \( m, n \), we have
\[
\sum_{k,l=0,0}^{\infty, \infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^{r}_{s} x^{s}_{k,l} - \Delta^{r}_{s} x^{t}_{k,l}}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \to 0. \quad (2.6)
\]
Thus for each fixed \( k \) and \( l \) as \( s, t \to \infty \), since \( A = (a_{m,n,k,l}) \) is non-negative, we are granted that
\[
u_{k,l} \left[ f_{k,l} \left( \left\| \frac{\Delta^{r}_{s} x^{s}_{k,l} - \Delta^{r}_{s} x^{t}_{k,l}}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right] \to 0
\]
and by continuity of \( F = (f_{k,l}) \), \( (x^{s}_{k,l}) \) is a Cauchy sequence in \( C \) for each fixed \( k \) and \( l \). Since \( C \) is complete as \( t \to \infty \), we have \( x^{s}_{k,l} \to x_{k,l} \) for each \( (k, l) \). Now from equation (2.6), we have for \( \epsilon > 0 \), there exists a natural number \( N \) such that
\[
\sum_{k,l=0,0}^{\infty, \infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^{r}_{s} x^{s}_{k,l} - \Delta^{r}_{s} x^{t}_{k,l}}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_{k,l}} < \epsilon \quad (2.7)
\]
for all \( m, n \). Since for any fixed natural number \( M \) we have from equation (2.7)
\[
\sum_{k,l=0,0}^{\infty, \infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^{r}_{s} x^{s}_{k,l} - \Delta^{r}_{s} x^{t}_{k,l}}{\rho}, z_1, \ldots, z_{n-1} \right\| \right) \right]^{p_{k,l}} < \epsilon
\]
for all \(m, n\), by letting \(t \to \infty\) in the above expression we obtain
\[
\sum_{k,l=0}^{\infty, \infty} a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^s x_{k,l} - \Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} < \epsilon.
\]
Since \(M\) is arbitrary, by letting \(M \to \infty\) we obtain
\[
\sum_{k,l=0}^{\infty, \infty} a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^s x_{k,l} - \Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} < \epsilon
\]
for all \(m, n\). Thus \(g(x^s - x) \to 0\) as \(s \to \infty\). This proves that \(w''(\Delta^r, A, F, u, p)\) is a complete linear topological space.

Now, we shall show that \(w''(\Delta^s, A, F, u, p, \|\cdot\|, \cdots, \|\|)\) is also a sequence in \(w''(\Delta^r, A, F, u, p, \|\cdot\|, \cdots, \|\|)\), for each \(s\) there exists \(L^s\) with
\[
\sum_{k,l=0}^{\infty, \infty} a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^s x_{k,l} - \Delta^r L^s}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \to 0
\]
as \(m, n \to \infty\), whence, from the fact that
\[
\sup_{m,n} \sum_{k,l=0}^{\infty, \infty} a_{m,n,k,l} < \infty
\]
from the definition of modulus function, we have \(f_{k,l} \left( \left\| \frac{\Delta^r L^s - \Delta^s L^r}{\rho} \right\| \right) \to 0\) as \(s, t \to \infty\) and so \(L^s\) converges to \(L\). Thus
\[
\sum_{k,l=0}^{\infty, \infty} a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta^r x_{k,l} - L}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \to 0
\]
as \(m, n \to \infty\), thus \(x \in w''(\Delta^r, A, F, u, p, \|\cdot\|, \cdots, \|\|)\) and this completes the proof.

**Theorem 2.5** Let \(F = (f_{k,l})\) be a sequence of modulus function and \(A = (a_{m,n,k,l})\) be a non negative matrix such that \(\sup_{m,n} \sum_{k,l=0}^{\infty, \infty} a_{m,n,k,l} < \infty\), then

(i) \(w''(\Delta^r, A, u, p, \|\cdot\|, \cdots, \|\|) \subset w''(\Delta^r, A, F, u, p, \|\cdot\|, \cdots, \|\|)\);

(ii) \(w''(\Delta^r, A, u, p, \|\cdot\|, \cdots, \|\|) \subset w''(\Delta^r, A, F, u, p, \|\cdot\|, \cdots, \|\|)\);

(iii) \(w''(\Delta^r, A, u, p, \|\cdot\|, \cdots, \|\|) \subset w''(\Delta^r, A, F, u, p, \|\cdot\|, \cdots, \|\|)\).

**Proof** (i) and (ii) are easy to prove so we will prove (iii) only. Let \(x = (x_{k,l}) \in w''(\Delta^r, A, u, p, \|\cdot\|, \cdots, \|\|)\) such that
\[
\sup_{m,n} \sum_{k,l=0}^{\infty, \infty} a_{m,n,k,l} \left( \left\| \frac{\Delta^r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} < \infty.
\]
Let $\epsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $f_{k,l}(t) < \epsilon$ for $0 \leq t \leq \delta$. Thus, we have

\[
\sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}} = \sum_{k,l=0}^{\infty,\infty} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \leq \delta \\
+ \sum_{k,l=0}^{\infty,\infty} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}} > \delta
\]

Since $F = (f_{k,l})$ is a sequence of modulus function, we have

\[
\sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \leq \epsilon \sum_{k,l=0}^{\infty,\infty} a_{m,n,k,l}. \tag{2.8}
\]

For $\left( \left\| \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) > \delta$ and the fact that

\[
\left( \left\| \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) < \left( \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right) < \left[ 1 + \left( \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right) \right]
\]

where $[t]$ denotes the integer part of $t$ and by the properties of modulus function, we have

\[
f_{k,l} \left( \left\| \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} < \left( 1 + f_{k,l} \left[ \left( \left\| \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}} \right] \right) \]

\[
\leq 2 f_k(1) \frac{\left( \left\| \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right)^{p_{k,l}}}{\delta}.
\]

Thus

\[
\sum_{k,l=0}^{\infty,\infty} u_{k,l} \left[ a_{m,n,k,l} f_{k,l} \left( \left\| \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) \right]^{p_{k,l}} \left( \left\| \frac{\Delta_r x_{k,l}}{\rho}, z_1, \cdots, z_{n-1} \right\| \right) > \delta
\]
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\[ \beta = \lim \]

Theorem 2.7

If \( A = (a_{m,n,k,l}) \) has only positive entries and \( B = (b_{m,n,k,l}) \) is a non-negative matrix such that \( \{ \frac{b_{m,n,k,l}}{a_{m,n,k,l}} \} \) is bounded then

\[ w''(\Delta_s^r, A, F, u, p, ||\cdot, \cdot||) \subset w''(\Delta_s^r, B, F, u, p, ||\cdot, \cdot||). \]

Proof

It is easy to prove so we omit the details.

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