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Stability of Tangential Locally Conformal Symplectic Forms*

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Abstract

In this paper we firstly define a tangential Lichnerowicz cohomology on foliated manifolds. Next, we define tangential locally conformal symplectic forms on a foliated manifold and we formulate and prove some results concerning their stability.

Key words: foliated manifold, tangential Lichnerowicz cohomology, tangential locally conformal symplectic structure, stability

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1 Introduction and preliminaries

1.1 Introduction

The locally conformal symplectic (l.c.s.) structures were introduced by Lee [11] and Vaisman [24]. The fundamental properties of these structures have been studied extensively by Vaisman, Banyaga, de Leon, Bande, Kotschick and many others, see for instance [1, 2, 3, 12, 24] and the references given there for a more thorough discussion.

An important tool in the study of locally conformal symplectic structures is the Lichnerowicz cohomology, also known in literature as Morse-Novikov cohomology, which is a cohomology defined for a smooth manifold M and a closed 1-form θ. It is defined by twisting the usual differential of the de Rham complex $\Omega^\bullet(M)$ of M; namely, the Lichnerowicz cohomology is the cohomology of a complex $(\Omega^\bullet(M), d_\theta)$, where d_θ is defined by $d_\theta\varphi = d\varphi - \theta\wedge\varphi$. This cohomology

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was originally defined by Lichnerowicz and Novikov in the context of Poisson geometry and Hamiltonian mechanics, respectively. Lichnerowicz cohomology is naturally defined for a l.c.s. manifold with its canonical closed 1-form called the Lee form, [1, 2].

The aim of the present paper is to study the stability of tangential locally conformal symplectic forms giving a tangential analogue of some Moser's type stability results for locally conformal symplectic structures [1, 3]. We also notice that in the tangential (leafwise) symplectic setting Moser stability may be found for instance in [5, 9, 20]. In this sense, in the preliminary section, following [14, 15], we make a short review on the tangential de Rham theory of foliated manifolds. In the second section we define a tangential Lichnerowicz cohomology of foliated manifolds and in a similar manner to [23], we present a tangentially version of a de Rham theorem for tangential Lichnerowicz cohomology. In the last section, we define tangential locally conformal symplectic forms on foliated manifolds and following some arguments from [3] and [5] we formulate and prove some results concerning the stability of tangential locally conformal symplectic forms. The methods used here are similarly to those used by [1, 3, 5] and are closely related to those used in tangential geometry and cohomology of foliations [14, 15, 17].

1.2 Preliminaries

Let M be a smooth manifold of dimension $n = p + q$ endowed with a regular foliation $\mathcal{F}$ of codimension q. Denote the space of smooth differential r-forms on M by $\Omega^r(M)$. We define as in [14], for each $r \geq 0$,

$$I^r(\mathcal{F}) = \{\varphi \in \Omega^r(M) \colon \varphi|_{\mathcal{F}} = 0\},$$

where $\varphi|_{\mathcal{F}} = 0$ means that for any $x \in M$, $\varphi(x)(v_1, \ldots, v_r) = 0$ for all $v_1, \ldots, v_r$ in $T_x\mathcal{F}$. Clearly, $I^r(\mathcal{F})$ is a linear subspace of $\Omega^r(M)$ and $I^0(\mathcal{F}) = \Omega^0(M) = C^\infty(M)$. Moreover, $I(\mathcal{F}) = \oplus_{r\geq 0} I^r(\mathcal{F})$ is a graded ideal of the de Rham complex $\Omega^\bullet(M)$. Let $\Omega^r(\mathcal{F}) = \Omega^r(M)/I^r(\mathcal{F})$ and $q\colon \Omega^r(M) \to \Omega^r(\mathcal{F})$ denote the quotient map. The exterior differential operator $d\colon \Omega^r(M) \to \Omega^{r+1}(M)$ induces a morphism $d_{\mathcal{F}} : \Omega^r(\mathcal{F}) \to \Omega^{r+1}(\mathcal{F})$ by $d_{\mathcal{F}}(q\varphi) = q(d\varphi)$, since d maps $I^r(\mathcal{F})$ into $I^{r+1}(\mathcal{F})$.

It can be easily checked that $d_{\mathcal{F}}^2 = 0$ so that $(\Omega^\bullet(\mathcal{F}), d_{\mathcal{F}})$ is a co-chain complex called the *tangential or foliated de Rham complex* of $(M, \mathcal{F})$.

Let $T\mathcal{F}$ denote the subbundle of TM consisting of all vectors which are tangent to the leaves of the foliation and let $T^*\mathcal{F}$ denote the dual of this bundle. Then it may be noted that $\Omega^r(\mathcal{F})$ is isomorphic to the space of sections of the vector bundle $\Lambda^r T^*\mathcal{F}$. With this identification we observe that q is induced by the quotient map $T^*M \to T^*\mathcal{F}$. The commutation relation $d_{\mathcal{F}}(q\varphi) = q(d\varphi)$ says that the quotient map q is a chain map between de Rham complex and tangential de Rham complex. The cohomology of the complex $(\Omega^\bullet(\mathcal{F}), d_{\mathcal{F}})$ is called the *tangential de Rham cohomology* of $(M, \mathcal{F})$ and is denoted by $H^\bullet(\mathcal{F}) = \operatorname{Ker} d_{\mathcal{F}}/\operatorname{Im} d_{\mathcal{F}}$. It follows directly from the above description of the tangential de Rham complex that the tangential cohomology groups vanish in dimensions $r > \dim \mathcal{F}$. For foliated manifolds with single leaf the tangential de Rham complex is the same as the ordinary de Rham complex. Hence the tangential de Rham cohomology is the same as the ordinary de Rham cohomology. If the foliation is by points, the tangential cohomology is 0 in positive degrees.

A form $\varphi \in \Omega^r(\mathcal{F})$ if it can be written locally as

$$\varphi = \sum \varphi_{u_1 \dots u_r}(x^u, x^a) dx^{u_1} \wedge \dots dx^{u_r}, \tag{1.1}$$

where (x^u, x^a), $u = 1, \dots, p$, $a = p{+}1, \dots, p{+}q = \dim M$ are local coordinates in a chart adapted to the foliation and the leaves of the foliation are characterized by $x^a = const$. The coordinates 1-forms dx^u are differentials along the leaves. The tangential differential $d_{\mathcal{F}}$ is defined as one does classically for an r-form with (x^a) playing the role of parameters.

Let us denote $\varphi_{\mathcal{F}} = q\varphi$ for every $\varphi \in \Omega^\bullet(M)$.

Definition 1.1 An r-form φ on M is said to be *tangentially closed* if $d_{\mathcal{F}}\varphi_{\mathcal{F}} = 0$, that is, if φ restricts to a closed form on each leaf of the foliation. Similarly, an r-form φ on M is said to be *tangentially exact* if there exists an $(r-1)$-form ψ on M such that $\varphi_{\mathcal{F}} = d_{\mathcal{F}}\psi_{\mathcal{F}}$ which implies that φ restricts to an exact form on each leaf of the foliation.

Definition 1.2 A vector field X on a foliated manifold $(M, \mathcal{F})$ is said to be a *foliated vector field* if X maps M into $T\mathcal{F}$. The space of all foliated vector fields on $(M, \mathcal{F})$ will be denoted by $\mathcal{X}(\mathcal{F})$. Equivalently, a vector field $Y \in \mathcal{X}(M)$ is said to be foliated if, for every $X \in \Gamma(T\mathcal{F})$ we have $[X, Y] \in \Gamma(T\mathcal{F})$.

A smooth map $f\colon (M, \mathcal{F}) \to (M^{'}, \mathcal{F}^{'})$ between two foliated manifolds is called *foliation preserving* if f send a leaf of $\mathcal{F}$ into a leaf of $\mathcal{F}^{'}$. Such a map f induces a chain map $f^* : \Omega^\bullet(\mathcal{F}^{'}) \to \Omega^\bullet(\mathcal{F})$ and hence a morphism $f^\bullet : H^\bullet(\mathcal{F}^{'}) \to H^\bullet(\mathcal{F})$ in the cohomology level.

It is easy to see that the exterior product $\wedge$ in $\Omega^\bullet(M)$ descends to $\Omega^\bullet(\mathcal{F})$. Also, it is easily checked that $i_X q(\omega) = q\,(i_X\omega)$ and $f^*q(\omega) = q\,(f^*\omega)$ are correct definitions if X is tangent to $\mathcal{F}$ and f is a foliation preserving map. For X tangent to $\mathcal{F}$ we may have also the Lie derivative $\mathcal{L}_X q(\omega) = i_X d_{\mathcal{F}} q(\omega) + d_{\mathcal{F}} i_X q(\omega)$. For basic properties of tangential de Rham cohomology, see for instance Ch. III from [15].

2 Tangential Lichnerowicz cohomology

Let $(M, \mathcal{F})$ be a foliated manifold and $\theta \in \Omega^1(M)$ be a tangentially closed 1-form. Denote by $d^\theta_{\mathcal{F}} : \Omega^r(\mathcal{F}) \to \Omega^{r+1}(\mathcal{F})$ the map $d^\theta_{\mathcal{F}} = d_{\mathcal{F}} - \theta_{\mathcal{F}}\wedge$.

Since $d_{\mathcal{F}}\theta_{\mathcal{F}} = 0$, we easily obtain that $\left(d^\theta_{\mathcal{F}}\right)^2 = 0$. The differential complex $\left(\Omega^\bullet(\mathcal{F}), d^\theta_{\mathcal{F}}\right)$ is called the *tangential Lichnerowicz complex* of $(M, \mathcal{F})$. The cohomology of this complex is called the *tangential Lichnerowicz cohomology* of $(M, \mathcal{F})$ and is denoted by $H^\bullet_\theta(\mathcal{F})$.

This is a tangential (foliated) version of the classical Lichnerowicz cohomology (also referred in the literature as Morse-Novikov cohomology), motivated by Lichnerowicz's work [13] or Lichnerowicz-Jacobi cohomology on Jacobi and locally conformal symplectic manifolds, see [1, 2, 12]. We also notice that Vaisman in [23] studied it under the name of "adapted cohomology" on locally conformal Kähler (l.c.K.) manifolds.

We notice that, locally, the tangential Lichnerowicz cohomology complex becames the tangential de Rham complex after a change $\varphi_{\mathcal{F}} \mapsto e^f \varphi_{\mathcal{F}}$ with f a smooth function on $(M, \mathcal{F})$ which satisfies $d_{\mathcal{F}} f = \theta_{\mathcal{F}}$, namely $d^\theta_{\mathcal{F}}$ is the unique differential in $\Omega^\bullet(\mathcal{F})$ which makes the multiplication by the smooth function e^f an isomorphism of cochain tangential complexes $e^f : (\Omega^\bullet(\mathcal{F}), d^\theta_{\mathcal{F}}) \to (\Omega^\bullet(\mathcal{F}), d_{\mathcal{F}})$.

Proposition 2.1 *The tangential Lichnerowicz cohomology depends only on the tangential cohomology class of θ. In fact, we have the following isomorphism $H^r_{\theta-df}(\mathcal{F}) \approx H^r_\theta(\mathcal{F})$.*

Proof Since $d^\theta_{\mathcal{F}}(e^f \varphi_{\mathcal{F}}) = e^f d^{\theta-df}_{\mathcal{F}} \varphi_{\mathcal{F}}$ it results that the map $[\varphi_{\mathcal{F}}] \mapsto [e^f \varphi_{\mathcal{F}}]$ is an isomorphism between $H^r_{\theta-df}(\mathcal{F})$ and $H^r_\theta(\mathcal{F})$. □ Using the definition of $d^\theta_{\mathcal{F}}$ we easily obtain

$$d^\theta_{\mathcal{F}}(\varphi_{\mathcal{F}} \wedge \psi_{\mathcal{F}}) = d_{\mathcal{F}}\varphi_{\mathcal{F}} \wedge \psi_{\mathcal{F}} + (-1)^{\deg \varphi} \varphi_{\mathcal{F}} \wedge d^\theta_{\mathcal{F}} \psi_{\mathcal{F}}.$$

Also, if θ_1 and θ_2 are two tangential closed 1-forms on $(M, \mathcal{F})$ then

$$d^{\theta_1+\theta_2}_{\mathcal{F}}(\varphi_{\mathcal{F}} \wedge \psi_{\mathcal{F}}) = d^{\theta_1}_{\mathcal{F}}\varphi_{\mathcal{F}} \wedge \psi_{\mathcal{F}} + (-1)^{\deg \varphi} \varphi_{\mathcal{F}} \wedge d^{\theta_2}_{\mathcal{F}} \psi_{\mathcal{F}},$$

which says that the wedge product induces the map

$$\wedge : H^{r_1}_{\theta_1}(\mathcal{F}) \times H^{r_2}_{\theta_2}(\mathcal{F}) \to H^{r_1+r_2}_{\theta_1+\theta_2}(\mathcal{F}).$$

Corollary 2.1 *The wedge product induces the following homomorphism*

$$\wedge : H^r_\theta(\mathcal{F}) \times H^r_{-\theta}(\mathcal{F}) \to H^{2r}(\mathcal{F}).$$

Now, using an argument inspired from [23] we shall prove that the tangential cohomology spaces $H^r_\theta(\mathcal{F})$ can also be obtained as the cohomology spaces of $(M, \mathcal{F})$ with the coefficients in the sheaf of germs of $d^\theta_{\mathcal{F}}$-closed smooth functions. Namely, let us denote by $\Phi^\theta_{\mathcal{F}}(M)$ the sheaf of germs of smooth functions on $(M, \mathcal{F})$ which are such $d^\theta_{\mathcal{F}} f = d_{\mathcal{F}} f - f\theta_{\mathcal{F}} = 0$.

Firstly, we notice that $d^\theta_{\mathcal{F}}$ satisfies a Poincaré type lemma for tangential forms. Indeed, let φ be a local form on $(M, \mathcal{F})$, such that $d^\theta_{\mathcal{F}}\varphi_{\mathcal{F}} = 0$. Since $d_{\mathcal{F}}\theta_{\mathcal{F}} = 0$ by Poincaré lemma for the operator $d_{\mathcal{F}}$ (see for instance Proposition 3.6 from [15] for tangential case or [22] for leafwise case), we may suppose

$$\theta_{\mathcal{F}} = -d_{\mathcal{F}} f / f, \tag{2.1}$$

where f is a nonzero smooth function. Then, $d^\theta_{\mathcal{F}}\varphi_{\mathcal{F}} = 0$ means $d_{\mathcal{F}}(f\varphi_{\mathcal{F}}) = 0$, whence $\varphi_{\mathcal{F}} = d^\theta_{\mathcal{F}}(\psi_{\mathcal{F}}/f)$ for some local form ψ on $(M, \mathcal{F})$. This is exactly the requested result.

Now, if we denote by $\widetilde{\Omega}^r(\mathcal{F})$ the sheaf of germs of tangential r-forms on $(M,\mathcal{F})$, we see that

$$0 \longrightarrow \Phi^\theta_{\mathcal{F}}(M) \xrightarrow{i} \widetilde{\Omega}^0(\mathcal{F}) \xrightarrow{d^\theta_{\mathcal{F}}} \widetilde{\Omega}^1(\mathcal{F}) \xrightarrow{d^\theta_{\mathcal{F}}} \dots \tag{2.2}$$

is a fine resolution of $\Phi^\theta_{\mathcal{F}}(M)$, which leads to the following tangential de Rham type theorem:

Theorem 2.1 *For every foliated manifold $(M,\mathcal{F})$ and every tangentially closed 1-form θ, one has the isomorphisms*

$$H^r(\mathcal{F},\Phi^\theta_{\mathcal{F}}(M)) \approx H^r_\theta(\mathcal{F}).$$

In the end of this section we notice that some other basic properties as: relative cohomology, $\mathcal{F}$-homotopy invariance, Mayer-Vietoris sequence can be formulated in the context of tangential Lichnerowicz cohomology.

3 Tangential locally conformal symplectic forms and their stability

In this section we firstly define tangential locally conformal symplectic forms on a regular foliated manifold $(M,\mathcal{F})$ and next, following some arguments from [3] and [5] we formulate and prove some results concerning the stability of tangential locally conformal symplectic forms.

Definition 3.1 ([5, 9]). A tangentially closed 2-form ω on $(M,\mathcal{F})$ is said to be a *tangential or foliated symplectic form* if ω is non-degenerate on each leaf of $\mathcal{F}$. The foliated manifold $(M,\mathcal{F})$ together with a tangential symplectic form ω is then called a *foliated symplectic manifold.*

Definition 3.2 A *tangential locally conformal symplectic structure* (briefly, tangential l.c.s. structure) on a foliated manifold $(M,\mathcal{F})$ is a 2-form ω, non-degenerate on each leaf of $\mathcal{F}$, which is locally conformal to a tangential symplectic form.

In other words, $(M,\mathcal{F},\omega)$ is a tangential l.c.s. structure if there exists an open covering $\{U_i\}$ of M and a smooth positive function f_i on U_i such that $f_i\omega|_{U_i}$ is a tangential symplectic structure on U_i. Equivalently, there exists on $(M,\mathcal{F})$ a tangentially closed 1-form θ, called *Lee form*, such that ω satisfies the integrability condition

$$d_{\mathcal{F}}\omega_{\mathcal{F}} = \theta_{\mathcal{F}} \wedge \omega_{\mathcal{F}}. \tag{3.1}$$

Indeed, by $d_{\mathcal{F}}\theta_{\mathcal{F}} = 0$ and Poincaré lemma for the operator $d_{\mathcal{F}}$, there is an open cover $\{U_i\}_{i\in I}$ of M and a family $\{\sigma_i\}_{i\in I}$ of C^∞ functions $\sigma_i\colon U_i \to \mathbb{R}$ so that $\theta_{\mathcal{F}} = d_{\mathcal{F}}\sigma_i$ on U_i. Then $\omega_i = e^{-\sigma_i}\omega|_{U_i}$ is a tangential symplectic structure on U_i.

Example 3.1 Let $(M_1, \omega_1, \theta_1)$ and $(M_2, \omega_2, \theta_2)$ two l.c.s. manifolds of dimensions n_1 and n_2, respectively. Then the product manifold $M = M_1 \times M_2$ carries two complementary foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ by copies of M_1 and M_2, respectively. If we consider $\omega = pr_1^*\omega_1 + pr_2^*\omega_2$ and $\theta = pr_1^*\theta_1 + pr_2^*\theta_2$, where $pr_1 \colon M \to M_1$ and $pr_2 : M \to M_2$, then (ω, θ) is a tangential l.c.s. structure on M with respect to both foliations $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively.

We assume throughout that the dimension of $\mathcal{F}$ is at least 4. Then the tangential 1-form $\theta_{\mathcal{F}}$, is uniquely determined by $\omega_{\mathcal{F}}$ because the wedge product with a non-degenerate tangential 2-form is injective on tangential 1-forms. When $\theta_{\mathcal{F}}$ vanishes identically, the form ω is tangential symplectic.

Two tangential l.c.s. forms ω and $\omega^{'}$ are said to be *tangential conformally equivalent* if there exists some positive function f such that $\omega = f\omega^{'}$ on $\mathcal{F}$, that is $\omega_{\mathcal{F}} = f\omega^{'}_{\mathcal{F}}$. A tangential l.c.s. structure is an equivalence class of tangential l.c.s. forms for this relation. Note that the tangential de Rham cohomology class of the tangential Lee form is an invariant of the tangential l.c.s. structure because a conformal rescaling of $\omega_{\mathcal{F}}$ changes $\theta_{\mathcal{F}}$ by the addition of an exact tangential form.

If an tangential l.c.s. structure contains a tangential symplectic representative, then it is *tangential globally conformal symplectic structure*. This is the case if and only if the tangential Lee form is tangentially exact.

In the case when θ is the Lee form of a tangential l.c.s. form ω, equation (3.1) shows that $\omega_{\mathcal{F}}$ is $d^{\theta}_{\mathcal{F}}$-closed and so defines a class in $H^2_{\theta}(\mathcal{F})$, which is a tangential analogue of Morse–Novikov class of l.c.K. manifolds, see [19]. If we consider the tangential l.c.s. structure defined by ω, and $\omega^{'}_{\mathcal{F}} = f\omega_{\mathcal{F}}$, then the tangential Lee form of $\omega^{'}$ is just

$$\theta^{'}_{\mathcal{F}} = \theta_{\mathcal{F}} + d_{\mathcal{F}} \ln f$$

and the class $[\omega_{\mathcal{F}}] \in H^2_{\theta}(\mathcal{F})$ is mapped to $[\omega^{'}_{\mathcal{F}}] \in H^2_{\theta'}(\mathcal{F})$ by Proposition 2.1.

In the following we consider families ω_t of tangential l.c.s. forms depending smoothly on a parameter $t \in [0, 1]$. The uniqueness of the tangential Lee form $q(\theta_t) = \theta_t|_{\mathcal{F}}$ implies that this depends smoothly on t as well. Using the tangential Lichnerowicz operator, by analogy with [3], we obtain the following result concerning the stability of ω_t:

Proposition 3.1 *Let ω_t be a smooth family of tangential l.c.s. forms on a foliated manifold $(M, \mathcal{F})$ such that the corresponding Lee forms θ_t have the same tangential de Rham cohomology class. Suppose there exists a smooth family of 1-forms ψ_t such that $\omega_t = d\psi_t - \theta_t \wedge \psi_t$ on $\mathcal{F}$. Then there exists a foliation preserving isotopy ϕ_t such that $\phi_t^*\omega_t$ is tangential conformally equivalent to ω_0 for all t, namely $\phi_t^* q(\omega_t) = f_t q(\omega_0)$, for some positive functions f_t.*

Proof Because the Lee forms θ_t have the same tangential de Rham class, then there is a smooth family of functions h_t such that $q(\dot{\theta}_t) = d_{\mathcal{F}} h_t$. Now, one defines a time-dependent foliated vector field X_t by

$$i_{X_t} q(\omega_t) = -q(\dot{\psi}_t) + h_t q(\psi_t). \tag{3.2}$$

Then its flow ϕ_t satisfies

$$\begin{aligned}\frac{d}{dt}(\phi_t^* q(\omega_t)) &= \phi_t^*\left(q(\dot\omega_t) + i_{X_t} d_{\mathcal{F}} q(\omega_t) + d_{\mathcal{F}}(i_{X_t} q(\omega_t))\right)\\ &= \phi_t^*\left((q(\theta_t)(X_t) + h_t) q(\omega_t)\right),\end{aligned}$$

where we have used $q(\dot\omega_t) = d_{\mathcal{F}}^{\theta_t} q(\dot\psi_t) - q(\dot\theta_t)\wedge q(\psi_t)$ and (3.2). Thus

$$\phi_t^* q(\omega_t) = e^{\int_0^t \phi_s^*(q(\theta_s)(X_s)+h_s)ds} q(\omega_0)$$

which completes the proof. □

In [5] is proved a stability theorem for foliated symplectic forms using an extended Poincaré lemma for the operator $d_{\mathcal{F}}$. In order to obtain an analogue stability theorem for tangential l.c.s. structures, we firstly obtain an analogue extended Poincaré lemma for the operator $d_{\mathcal{F}}^{\theta}$.

Proposition 3.2 *Let $(M, \mathcal{F})$ be a smooth foliated manifold and let $\pi\colon E \to M$ be a vector bundle over M. Let $\widetilde{\mathcal{F}}$ be the foliation on E defined by $\widetilde{\mathcal{F}} = \pi^{-1}(\mathcal{F})$. Let us denote $\widetilde{q}\colon \Omega^\bullet(E) \to \Omega^\bullet(\widetilde{\mathcal{F}})$ and let $\widetilde{\theta}$ be a tangentially closed 1-form on $(E, \widetilde{\mathcal{F}})$. Suppose $\widetilde{\varphi}$ is a tangentially $d_{\widetilde{\mathcal{F}}}^{\widetilde{\theta}}$-closed r-form on $(E, \widetilde{\mathcal{F}})$ such that $i^*\widetilde{\varphi} = 0$ on $\mathcal{F}$, where $i\colon M \to E$ embeds M as the zero section in E. Then there exists a neighbourhood U of M in E with a $(r-1)$-form $\widetilde{\psi}$ on U such that $d_{\widetilde{\mathcal{F}}}^{\widetilde{\theta}} \widetilde{q}(\widetilde{\psi}) = \widetilde{q}(\widetilde{\varphi})$ and $\widetilde{\psi}|_{i(M)} = 0$.*

Proof Let $\widetilde{\varphi}$ as in the hypothesis. As we explained above (in Section 2), from $d_{\widetilde{\mathcal{F}}}^{\widetilde{\theta}} \widetilde{q}(\widetilde{\varphi}) = 0$ it follows that $d_{\widetilde{\mathcal{F}}}(f\widetilde{q}(\widetilde{\varphi})) = 0$, where locally $\widetilde{q}(\widetilde{\theta}) = -d_{\widetilde{\mathcal{F}}} f/f$ and f is a nonzero smooth function on E. Now from $i^*\widetilde{\varphi} = 0$ on $\mathcal{F}$ it easily follows that $i^*(f\widetilde{\varphi}) = 0$ on $\mathcal{F}$ and by Proposition 3.3. from [5] we obtain that there is a neighbourhood U of M in E with a $(r-1)$-form $\widetilde{\psi}'$ on U such that $d_{\widetilde{\mathcal{F}}} \widetilde{q}(\widetilde{\psi}') = f\widetilde{q}(\widetilde{\varphi})$ and $\widetilde{\psi}'|_{i(M)} = 0$ or equivalently, $d_{\widetilde{\mathcal{F}}}^{\widetilde{\theta}}(\widetilde{q}(\widetilde{\psi}'/f)) = \widetilde{q}(\widetilde{\varphi})$ and $(\widetilde{\psi}'/f)|_{i(M)} = 0$. Now, the proposition follows taking $\widetilde{\psi} = \widetilde{\psi}'/f$. □

Proposition 3.3 *Let $(M, \mathcal{F})$ be a smooth foliated manifold and let $\pi\colon E \to M$ be a vector bundle over M. Let $(E, \widetilde{\mathcal{F}})$ be the foliation on E defined by $\widetilde{\mathcal{F}} = \pi^{-1}(\mathcal{F})$. Suppose $\widetilde{\omega}_0$ and $\widetilde{\omega}_1$ are two tangential l.c.s. forms on $(E, \widetilde{\mathcal{F}})$ with the same tangential Lee 1-form $\widetilde{\theta}_{\widetilde{\mathcal{F}}}$, such that $\widetilde{\omega}_1 = \widetilde{\omega}_0$ on $T\widetilde{\mathcal{F}}|_M$. Then there exist a family of l.c.s. structures $\widetilde{\omega}_t$, $t \in [0,1]$ on open neighbourhood U of M in E, some positive smooth functions f_t on U and a foliation preserving isotopy $\phi_t\colon U \to U$, $t \in [0,1]$ such that $d\phi_t = \mathrm{id}$ on $TE|_M$ and $\phi_t^*\widetilde{\omega}_t = f_t\widetilde{\omega}_0$ on $\widetilde{\mathcal{F}}$, that is $\phi_t^*\widetilde{q}(\widetilde{\omega}_t) = f_t\widetilde{q}(\widetilde{\omega}_0)$.*

Proof It follows from the hypothesis that $\widetilde{\omega}_1 - \widetilde{\omega}_0$ is a tangentially $d_{\widetilde{\mathcal{F}}}^{\widetilde{\theta}}$-closed form and $\widetilde{\omega}_1 - \widetilde{\omega}_0 = 0$ on $T\widetilde{\mathcal{F}}|_M$. Therefore, by Proposition 3.2.,

$$\widetilde{q}(\widetilde{\omega}_1 - \widetilde{\omega}_0) = d_{\widetilde{\mathcal{F}}}^{\widetilde{\theta}} \widetilde{q}(\widetilde{\psi}),$$

for some $\widetilde{\psi} \in \Omega^1(E)$ satisfying $\widetilde{\psi}|_{i(M)} = 0$. For $0 \leq t \leq 1$ we define a family of tangentially $d^{\widetilde{\theta}}_{\widetilde{\mathcal{F}}}$-closed forms $\widetilde{\omega}_t$ by

$$\widetilde{\omega}_t = \widetilde{\omega}_0 + td^{\widetilde{\theta}}\widetilde{\psi}.$$

Since $\widetilde{\omega}_1 = \widetilde{\omega}_0$ on $T\widetilde{\mathcal{F}}|_M$, each $\widetilde{\omega}_t$ restricts to a tangential l.c.s. form on some neighbourhood U of M in E. As in the case of tangential symplectic forms [5], a tangential l.c.s. form ω on $(M, \mathcal{F})$ defines a bundle isomorphism $I_\omega : T\mathcal{F} \to T^*\mathcal{F}$ which is given by the correspondence $X \mapsto i_X\omega_{\mathcal{F}}$. Thus I_ω induces a bijection $\mathcal{X}(\mathcal{F}) \to \Omega^1(\mathcal{F})$ which maps a foliated vector field X onto the tangential 1-form $q(i_X\omega)$. Now, for each $t \in [0, 1]$, define a foliated vector field X_t by $X_t = I^{-1}_{\widetilde{\omega}_t}(-\widetilde{q}(\widetilde{\psi}))$, so that $\widetilde{q}(i_{X_t}\widetilde{\omega}_t) = -\widetilde{q}(\widetilde{\psi})$. Let ϕ_t, $t \in [0, 1]$ be the one-parameter family of diffeomorphisms defined on some open neighbourhood of M such that $\phi_0 = \mathrm{id}|_M$ and $d\phi_t/dt = X_t$. Since $\widetilde{\psi} = 0$ on $T\widetilde{\mathcal{F}}|_M$, it follows that $\phi_t|_M = \mathrm{id}|_M$. Moreover, since X_t is a foliated vector field, $\{\phi_t\}$ is a foliation preserving diffeotopy.

Now, by the same technique as above, one gets

$$\begin{aligned}
\frac{d}{dt}(\phi_t^*\widetilde{q}(\widetilde{\omega}_t)) &= \phi_t^*\left(\widetilde{q}(\dot{\widetilde{\omega}}_t) + i_{X_t}d_{\widetilde{\mathcal{F}}}\widetilde{q}(\widetilde{\omega}_t) + d_{\widetilde{\mathcal{F}}}(i_{X_t}\widetilde{q}(\widetilde{\omega}_t))\right)\\
&= \phi_t^*\left(d^{\widetilde{\theta}}_{\widetilde{\mathcal{F}}}\widetilde{q}(\widetilde{\psi}) - d_{\widetilde{\mathcal{F}}}\widetilde{q}(\widetilde{\psi}) + \widetilde{q}(\widetilde{\theta}) \wedge \widetilde{q}(\widetilde{\psi}) + \widetilde{q}(\widetilde{\theta})(X_t)\widetilde{q}(\widetilde{\omega}_t)\right)\\
&= \phi_t^*\left(\widetilde{q}(\widetilde{\theta})(X_t)\widetilde{q}(\widetilde{\omega}_t)\right)\\
&= \phi_t^*(\widetilde{q}(\widetilde{\theta})(X_t))\phi_t^*(\widetilde{q}(\widetilde{\omega}_t)).
\end{aligned}$$

It follows that

$$\phi_t^*\widetilde{q}(\widetilde{\omega}_t) = e^{\int_0^t \phi_s^*(\widetilde{q}(\widetilde{\theta})(X_s))ds}\widetilde{q}(\widetilde{\omega}_0),$$

which completes the proof. □

Finally, as a consequence, we obtain the following tangential stability theorem analogue to the stability result from [1]:

Theorem 3.1 *Let M be a closed manifold and let $\mathcal{F}$ be a regular foliation on M. Let $\{\omega_t\}$, $t \in [0, 1]$, be a smooth family of tangential l.c.s. forms on $(M, \mathcal{F})$ having the same tangential Lee form $\theta_{\mathcal{F}}$ such that $\omega_t = \omega_0 + d^\theta\psi_t$ on $\mathcal{F}$, where ψ_t is a smooth family of 1-forms. Then there exists a smooth foliated diffeotopy $\phi_t : (M, \mathcal{F}) \to (M, \mathcal{F})$, $\phi_0 = \mathrm{id}$, such that $\phi_t^*\omega_t = f_t\omega_0$ on $\mathcal{F}$, for some positive functions f_t.*

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