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Modal Pseudocomplemented De Morgan Algebras

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Abstract

Modal pseudocomplemented De Morgan algebras (or mpM-algebras for short) are investigated in this paper. This new equational class of algebras was introduced by A. V. Figallo and P. Landini ([10]) and they constitute a proper subvariety of the variety of all pseudocomplemented De Morgan algebras satisfying $x \wedge (\sim x)^* = (\sim (x \wedge (\sim x)^*))^*$. Firstly, a topological duality for these algebras is described and a characterization of mpM-congruences in terms of special subsets of the associated space is shown. As a consequence, the subdirectly irreducible algebras are determined. Furthermore, from the above results on the mpM-congruences, the principal ones are described. In addition, it is proved that the variety of mpM-algebras is a discriminator variety and finally, the ternary discriminator polynomial is described.

Key words: pseudocomplemented De Morgan algebras, Priestley spaces, discriminator varieties, congruences

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1 Introduction

There are several generalizations of Boolean algebras in the literature in which negation is replaced by several new unary operations, which satisfy some of the properties of the original operation. One of them are distributive p-algebras whose study was begun by V. Glivenko ([13]) in 1929. Recall that an algebra $\langle L, \wedge, \vee, ^*, 0, 1\rangle$ of type $(2, 2, 1, 0, 0)$ is called a distributive p-algebra if $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded distributive lattice such that for every $a \in L$, the element a^* is the pseudocomplement of a; i.e. $x \leq a^*$ if and only if $a \wedge x = 0$. In 1949, P. Ribenboim ([22]) showed that the class of these algebras constitutes a variety. More precisely, he proved that distributive p-algebras are bounded distributive lattices with an additional unary operation * which satisfies the following identities:

(R1) $x \wedge (x \wedge y)^* = x \wedge y^*$,

(R2) $x \wedge 0^* = x$,

(R3) $0^{**} = 0$.

A particular case of these distributive p-algebras are pseudocomplemented De Morgan algebras which A. Romanowska ([23]) called pM-algebras. An algebra $\langle L, \wedge, \vee, \sim, ^*, 0, 1\rangle$ of type $(2, 2, 1, 1, 0, 0)$ is called a pM-algebra if $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ is a De Morgan algebra ([15], see also [2, 6]) and $\langle L, \wedge, \vee, ^*, 0, 1\rangle$ is a distributive p-algebra. Let us observe that this definition does not establish any relationship between the operations $\sim$ and *.

In 1978, A. Monteiro introduced tetravalent modal algebras (or TM-algebras for short) as algebras $\langle L, \wedge, \vee, \sim, \nabla, 0, 1\rangle$ of type $(2, 2, 1, 1, 0, 0)$ such that $\langle L, \wedge, \vee, \sim, 0, 1\rangle$ are De Morgan algebras which satisfy the following conditions:

(i) $\nabla x \vee \sim x = 1$,

(ii) $\nabla x \wedge \sim x = \sim x \wedge x$.

These algebras arise as a generalization of three-valued Łukasiewicz algebras ([6]) by omitting the identity $\nabla(x \wedge y) = \nabla x \wedge \nabla y$ and they have been studied by different authors (see [9, 10, 11, 17, 18]). In [10], A. Figallo and P. Landini proved that tetravalent modal algebras are polynomially equivalent to De Morgan algebras with an additional unary operation $\urcorner$ which satisfies:

(T1) $x \wedge \urcorner x = 0$,

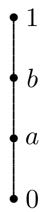
(T2) $x \vee \urcorner x = x \vee \sim x$.

Hence, as a direct consequence of this assertion it follows that pM-algebras which satisfy (T2) are tetravalent modal algebras. More precisely, they are three-valued Łukasiewicz algebras. Thus, in order to find the maximal subclass of pseudocomplemented De Morgan algebras which admit a structure of a TM-algebra, Figallo and Landini considered the subvariety of pM-algebras which satisfies:

(tm) $x \vee \sim x \leq x \vee x^*$,

and they called them modal pseudocomplemented De Morgan algebras (or mpM-algebras). Later, A. Figallo ([9]) showed that every mpM-algebra is a TM-algebra by defining $\nabla x = \sim (\sim x \wedge x^*)$. However, the varieties of mpM-algebras and TM-algebras do not coincide as we will show in Section 3.

On the other hand, it is worth mentioning that these algebras constitute a proper subvariety of the variety $\mathcal{V}_0$ of all pseudocomplemented De Morgan algebras satisfying the identity: $x \wedge (\sim x)^* = (\sim (x \wedge (\sim x)^*))^*$, studied by H. Sankappanavar in [25]. To this end it suffices to consider the algebra L_4 whose Hasse diagram is shown below and where the operations $\sim$ and * are given in the following table:



x	$\sim x$	x^*
0	1	1
a	b	0
b	a	0
1	0	0

where $b = a \vee \sim a \not\leq a \vee a^* = a$.

Here is a summary of our main results. In Section 2, we briefly summarize the main definitions and results needed throughout this article. In Section 3, we describe a topological duality for mpM-algebras and we characterize the congruences on these algebras by means of special subsets of the associated space. In Section 4, we obtain the subdirectly irreducible mpM-algebras taking into account the results established in the above section. Besides, we prove that the variety $\boldsymbol{mpM}$ of mpM-algebras is locally finite, semisimple, residually small and residually finite. In Section 5, we determine the principal congruences and we show that $\boldsymbol{mpM}$ is a discriminator variety. Finally, we obtain the ternary and the dual ternary discriminator polynomials.

2 Preliminaries

We refer the reader to the bibliography listed here as [2, 7, 15, 19, 20, 21] for specific details of the many basic notions and results of universal algebra including distributive lattices, De Morgan algebras and distributive p-algebras considered in this paper. However, in order to simplify the reading, we will summarize the main notions and results we need throughout this work.

If X is a partially ordered set and $Y \subseteq X$, we will denote by $[Y)$ ($(Y]$) the set of all $x \in X$ such that $y \leq x$ ($x \leq y$) for some $y \in Y$, and we will say that Y is increasing (decreasing) if $Y = [Y)$ ($Y = (Y]$). In particular, we will write $[y)$ ($(y]$) instead of $[\{y\})$ ($(\{y\}]$). Furthermore, we will denote by $maxY$ the set of maximum elements of Y.

In [21], H. A. Priestley described a topological duality for distributive p-algebras. For this purpose, the category whose objects are p-spaces and whose morphisms are p-functions was considered. More precisely, a p-space is a Priestley space X ([19, 20]) which satisfies the following condition: $(U]$ is an open subset of X for all $U \in D(X)$, where $D(X)$ denotes the family of increasing, closed and open subsets of X. Furthermore, a p-function f from a p-space X_1 into another one X_2 is an increasing and continuous function (i.e. a Priestley function) such that $f(maxX_1 \cap [x)) = maxX_2 \cap [f(x))$ for each $x \in X_1$. Besides, it is proved

(P1) If A is a distributive p-algebra, then the Priestley space $X(A)$ of all prime filters of A is a p-space. Moreover, $\sigma_A\colon A \to D(X(A))$ defined by $\sigma_A(a) = \{P \in X(A)\colon a \in P\}$ is a p-isomorphism.

(P2) If X is a p-space, then $\langle D(X), \cup, \cap, ^*, \emptyset, X\rangle$ is a distributive p-algebra where $U^* = X \setminus (U]$ for each $U \in D(X)$ and $\varepsilon_X\colon X \to X(D(X))$ defined by $\varepsilon_X(x) = \{U \in D(X)\colon x \in U\}$ is a homeomorphism and an order isomorphism.

Then the category of p-spaces and p-functions is naturally equivalent to the dual of the category of distributive p-algebras and their corresponding homomorphisms, where the isomorphisms σ_L and ε_X are the corresponding natural equivalences.

On the other hand, H. A. Priestley proved that

(P3) the lattice of all closed subsets Y of $X(A)$ with $maxX(A) \cap [Y) \subseteq Y$ is isomorphic to the dual lattice of all congruences on A.

In 1977, W. Cornish and P. Fowler ([8]) restricted Priestley duality for bounded distributive lattices to De Morgan algebras by considering the De Morgan spaces (or m-spaces) as pairs (X, g), where X is a Priestley space and $g\colon X \to X$ is a decreasing and continuous function satisfying $g^2 = id_X$. They also defined the m-functions f from an m-space (X_1, g_1) into another one, (X_2, g_2), as Priestley functions which satisfy the additional condition $f \circ g_1 = g_2 \circ f$.

In order to restrict Priestley duality to the case of De Morgan algebras, these authors defined the unary operation $\sim$ on $D(X)$ by

(P4) $\sim U = X \setminus g(U)$ for each $U \in D(X)$,

and the homeomorphism $g_A\colon X(A) \to X(A)$ by

(P5) $g_A(P) = A \setminus \{\sim x : x \in P\}$.

Then the category of m-spaces and m-functions is naturally equivalent to the dual of the category of De Morgan algebras and their corresponding homomorphisms. In addition, these authors showed that

(P6) the lattice of all involutive closed subsets of $X(A)$ is isomorphic to the dual of the lattice of all congruences on the De Morgan algebra A, where $Y \subseteq X(A)$ is involutive if $g_A(Y) = Y$.

3 A topological duality for *mpM*-algebras

Definition 3.1 A modal pseudocomplemented De Morgan algebra (or mpM-algebra) is a pseudocomplemented De Morgan algebra $\langle A, \wedge, \vee, \sim, ^*, 0, 1\rangle$ which satisfies:

(tm) $x \vee \sim x \leq x \vee x^*$,

where $a \leq b$ if and only if $a = a \wedge b$.

The variety of all mpM-algebras will be denoted by $\boldsymbol{mpM}$. As usual, we are going to denote an algebra of this variety simply by A.

Our next task is to obtain a topological duality for mpM-algebras taking into account the results described in Section 2.

Definition 3.2 A modal pseudocomplemented De Morgan space (or mp_M-space) is a pair (X, g) which is both an m-space and a p-space satisfying the following condition:

(pm1) $x \leq y$ implies $x = y$ or $g(x) = y$.

An mp_M-function from an mp_M-space into another one is both an m-function and a p-function.

Remark 3.1 By virtue of (pm1) we infer that any mp_M-space is the cardinal sum of chains ([3]), each of them with two elements at most. Then, any totally ordered mp_M-space has two elements at most.

Lemma 3.1 plays a relevant role in order to obtain the duality.

Lemma 3.1 *Let (X, g) be is both an m-space and a p-space. Then the following conditions are equivalent:*

(pm1) $x \leq y$ *implies* $x = y$ *or* $g(x) = y$,

(pm2) $(X \setminus U) \cap (U] \subseteq (X \setminus U) \cap g(U)$ *for all* $U \in D(X)$.

Proof (pm1) $\Rightarrow$ (pm2): Let $p \in (X \setminus U) \cap (U]$. Then there is $q \in U$ such that $p \leq q$ and so by (pm1) we have that $p = q$ or $g(p) = q$. If $p = q$, we infer that $p \in U$, which is a contradiction. Therefore, $g(p) = q$ and hence we conclude that $p \in (X \setminus U) \cap g(U)$.

(pm2) $\Rightarrow$ (pm1): Let $x, y \in X$, $x < y$. Then there is $U \in D(X)$ such that $y \in U$ and $x \notin U$. Hence $x \in (X \setminus U) \cap (U]$ and so by (pm2) it follows that $x \in g(U)$. Therefore, there is $z \in U$ and $x = g(z)$. If $y \neq z$, we have that $y \not< z$ or $y < z$.

Suppose that $y \not< z$. Then there is $V \in D(X)$ such that $y \in V$ and $z \notin V$. Let $W = U \cap V \in D(X)$. From the above assertions we conclude that $x \in (X \setminus W) \cap (W]$ and by (pm2) we have that $x \in g(W)$. Hence $z \in W \subseteq V$, which is a contradiction. Therefore $y < z$. This statement and the fact that $x = g(z) < y$ imply that $x < g(y) < z$. Now, $y = g(y)$ or $y \neq g(y)$. If $y = g(y)$, as $y < z$, then $z \not\leq y$, so there is $H \in D(X)$ such that $z \in H$ and $y \notin H$. From these last assertions we conclude that $y \in (X \setminus H) \cap (H]$ and then by (pm2) it follows that $y \in g(H)$. Therefore $y \in H$, a contradiction. Thus $y \neq g(y)$.

If we assume that $y \not\leq g(y)$, there is $S \in D(X)$ such that $y \in S$ and $g(y) \notin S$. Let $R = H \cap S \in D(X)$. Hence, it follows that $z \in R$ and taking into account that $y < z$ and $y \notin H$ we infer that $y \in (X \setminus R) \cap (R]$. Hence, by (pm2), $y \in g(R)$ and thus $g(y) \in S$, a contradiction. On the other hand, in case that $g(y) \not\leq y$, following an analogous reasoning we have a contradiction. Therefore, $y = z$ and so $y = g(x)$. $\square$

Proposition 3.1 *Let* (X, g) *be an* mp_M*-space. Then* $\mathsf{mPM}(X) = \langle D(X), \cup, \cap, \sim, ^*, \emptyset, X\rangle$ *is an mpM-algebra where for all* $U \in D(X)$*,* U^* *and* $\sim U$ *are defined as in* (P2) *and* (P4) *respectively.*

Proof It only remains to prove that $U \cup \sim U \subseteq U \cup U^*$ for all $U \in D(X)$, which is a direct consequence of (pm2). □

Proposition 3.2 *Let* A *be an mpM-algebra. Then* $\mathsf{mp_M}(A) = (X(A), g_A)$ *is an* mp_M*-space where* g_A *is defined as in* (P5)*. Furthermore,* σ_A *defined in* (P1) *is an mpM-isomorphism.*

Proof From the hypothesis, Definition 3.2 and Lemma 3.1 it only remains to prove (pm2). Taking into account (tm) it follows that $\sigma_A(a) \cup \sim \sigma_A(a) = \sigma_A(a \vee \sim a) \subseteq \sigma_A(a \vee a^*) = \sigma_A(a) \cup \sigma_A(a)^*$. Therefore, $U \cup (X(A) \setminus g_A(U)) \subseteq U \cup (X(A) \setminus (U])$ for all $U \in D(X)$, and thus the proof is complete. □

From Proposition 3.1 and Proposition 3.2, using the usual procedures, we conclude Theorem 3.1.

Theorem 3.1 *The category of* mp_M*-spaces and* mp_M*-functions is naturally equivalent to the dual of the category of mpM-algebras and their corresponding homomorphisms.*

Next, taking into account the topological duality described above, we will characterize the lattice $Con(A)$ of all mpM-congruences on A. For this purpose, we will start by showing a property of the involutive subsets of the mp_M-spaces.

Remark 3.2 Let (X, g) be an m-space and let Y be an involutive subset of X. Then Y is increasing if and only if Y is decreasing. Indeed, suppose that $x \in X$, $y \in Y$ and $x \leq y$. Hence, $g(y) \leq g(x)$ and so, taking into account that Y is involutive and increasing, we have that $g(x) \in Y$. Therefore, $x \in Y$. The converse implication is similar.

Lemma 3.2 *Let* (X, g) *be an* mp_M*-space and let* Y *be a non-empty and involutive subset of* X*. Then* Y *is increasing and decreasing.*

Proof Suppose that $x \in X$, $y \in Y$ and $x \leq y$. Then by (pm1) we have that $x = y$ or $x = g(y)$. Since Y is involutive, $g(y) \in Y$. Therefore, Y is decreasing and by Remark 3.2 we conclude that Y is increasing. □

Theorem 3.2 *Let* $A \in \boldsymbol{mpM}$*. Then the lattice* $\mathcal{C}_I(\mathsf{mp_M}(A))$ *of all closed and involutive subsets of* $\mathsf{mp_M}(A)$ *is isomorphic to the dual lattice* $Con(A)$ *and the isomorphism is the function* $\Theta\colon \mathcal{C}_I(\mathsf{mp_M}(A)) \to Con(A)$ *defined by* $\Theta(Y) = \{(a, b) \in A \times A \colon \sigma_A(a) \cap Y = \sigma_A(b) \cap Y\}$.

Proof Notice first that if Y is an involutive subset of $X(A)$, then by Lemma 3.2 we infer that $maxX(A) \cap [Y) = maxX(A) \cap Y \subseteq Y$. Hence, bearing in mind the results established in (P3) and (P6), the proof is complete. □

4 Subdirectly irreducible mpM-algebras

Next, we will apply the results just obtained in order to determine the subdirectly irreducible mpM-algebras. For this purpose, we will characterize the involutive subsets of the mp_M-spaces.

Proposition 4.1 *Let (X, g) be an mp_M-space and Y be a non-empty subset of X. Then the following conditions are equivalent:*

(i) *Y is involutive,*

(ii) *Y is the cardinal sum of a family $\mathcal{C} = \{C_i\}_{i\in I}$ of maximum chains of X such that $g(C_i) \in \mathcal{C}$ for all $i \in I$.*

Proof (i) $\Rightarrow$ (ii): By Remark 3.1 we have that for each $y \in Y$ there is a single maximum chain C_y of X such that $y \in C_y$. Besides, taking into account that Y is involutive, by Lemma 3.2, we obtain that $C_y \subseteq Y$. Hence, $Y = \bigcup_{y\in Y} C_y$. Furthermore, since $g(C_y) = C_{g(y)}$ from the hypothesis, we conclude that $g(C_y) \subseteq Y$.

(ii) $\Rightarrow$ (i): From the hypothesis we have that $Y = \bigcup_{i\in I} C_i$. Then $g(Y) = \bigcup_{i\in I} g(C_i)$ and so $Y = g(Y)$. □

Proposition 4.2 *Let (X, g) be an mp_M-space and let Y be a closed and non-empty subset of X. If $\mathsf{mPM}(X)$ is subdirectly irreducible, then the following conditions are equivalent:*

(i) *Y is involutive,*

(ii) *Y is the cardinal sum of a family of maximum chains and $maxX \subseteq Y$.*

Proof (i) $\Rightarrow$ (ii): Since Y is a non-empty involutive subset of X, by Proposition 4.1, we conclude that Y is a cardinal sum of maximum chains of X. Suppose that $maxX \nsubseteq Y$. Hence, for each $x \in maxX \setminus Y$, there are maximum chains C_x and $C_{g(x)}$ in X such that $x \in C_x$ and $g(x) \in C_{g(x)}$. Besides, from Remark 3.1, we infer that $C_x = \{x\}$ and $C_{g(x)} = \{g(x)\}$, or $C_x = \{x, g(x)\} = C_{g(x)}$. Therefore, $W_x = C_x \cup C_{g(x)}$ is a non-empty, closed and involutive subset of X and taking into account that Y is involutive we have that $W_x \cap Y = \emptyset$. Then there are at least two non-trivial, closed and involutive subsets of X. This last assertion and the fact that $X = \bigcup_{x\in maxX} C_x$ imply that $X = Y \cup \bigcup_{x\in maxX\setminus Y} W_x$, and so a maximum closed, involutive and proper subset of X does not exist. From this statement and Theorem 3.2 we conclude that $\mathsf{mPM}(X)$ is not a subdirectly irreducible mpM-algebra, which is a contradiction.

(ii) $\Rightarrow$ (i): From the hypothesis, it follows that $Y = X$ and so Y is involutive. □

Theorem 4.1 *Let (X, g) be an mp_M-space. Then the following conditions are equivalent:*

(i) *$\mathsf{mPM}(X)$ is subdirectly irreducible,*

(ii) *$\mathsf{mPM}(X)$ is simple.*

Proof (i) $\Rightarrow$ (ii): Let Y be a non-empty, closed and involutive subset of X. Hence, by Proposition 4.2 and Lemma 3.2 we have that $Y = X$. Therefore, the only closed and involutive subsets of X are the trivial ones and thus, by Theorem 3.2, we conclude that $\mathsf{mPM}(X)$ is simple. □

Proposition 4.3 *Let A be an mpM-algebra and let $\mathsf{mp_M}(A)$ be the mp_M-space associated with A. If $X(A)$ is an antichain with more than two elements, then A is not simple.*

Proof If g_A is the identity, for all $P \in X(A)$ we have that $\{P\}$ is a non-trivial, closed and involutive subset of $X(A)$. On the other hand, if g_A is not the identity, there is $P \in X(A)$ such that $g_A(P) \neq P$ and so $\{P, g_A(P)\}$ is a proper, closed and involutive subset of $X(A)$. Hence, in both cases, by Theorem 3.2, we conclude that A is not simple. □

Proposition 4.4 *Let A be an mpM-algebra and let $\mathsf{mp_M}(A)$ be the mp_M-space associated with A. If $X(A)$ is not an antichain and $|X(A)| > 2$, then A is not simple.*

Proof From the hypothesis and Remark 3.1, there are $P, Q \in X(A)$ such that $P \neq Q$, $P \subset g_A(P)$ and $Q \neq g_A(P)$. Hence, $\{P, g_A(P)\}$ is a non-trivial, closed and involutive subset of $X(A)$ and so, by Theorem 3.2, A is not simple. □

Theorem 4.2 is the main result of this section.

Theorem 4.2 *Let A be an mpM-algebra and let $\mathsf{mp_M}(A)$ be the mp_M-space associated with A. Then the following conditions are equivalent:*

(i) *A is simple,*

(ii) *$|X(A)| \leq 2$ and $X(A)$ is a chain or $X(A)$ is an antichain where g_A is not the identity.*

Proof (i) $\Rightarrow$ (ii): If we suppose that $|X(A)| > 2$, then by Remark 3.1, we infer that $X(A)$ is not a chain. Hence, by Proposition 4.3 and Proposition 4.4, we conclude that A is not simple, which is a contradiction. Therefore, $|X(A)| \leq 2$ and by Remark 3.1 we infer that $X(A)$ is a chain or an antichain with two elements. In the latter case, g_A is not the identity. Indeed, if $X(A) = \{P, Q\}$ where $P \not\subseteq Q$ and $Q \not\subseteq P$ and g_A is the identity, we have that $\{P\}$ is a proper, closed and involutive subset of $X(A)$. Thus, by Theorem 3.2 we conclude that A is not simple, which is a contradiction.

(ii) $\Rightarrow$ (i) If $X(A) = \{P, Q\}$ where $P \subset Q$, by (pm1) it follows that $g_A(P) = Q$. On the other hand, if $X(A) = \{P\}$, then $g_A(P) = P$. Furthermore, if $X(A) = \{P, Q\}$ where $P \not\subseteq Q$ and $Q \not\subseteq P$, then g_A is not the identity. Hence, in all cases, the closed and involutive subsets of $\mathsf{mp_M}(A)$ are the trivial ones and so, by Theorem 3.2, we have that A is simple. □

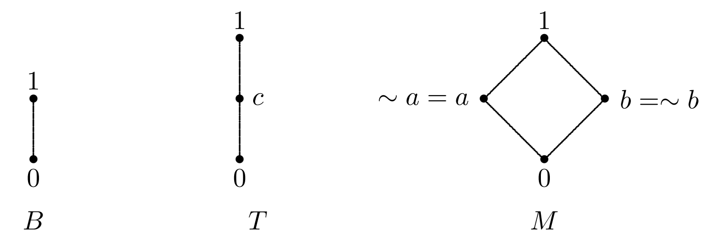
As a direct consequence of Theorem 4.2 we obtain the following description of the subdirectly irreducible *mpM*-algebras.

Corollary 4.1 *The subdirectly irreducible mpM-algebras are, up to isomorphism, the algebras B, T and M described below:*

(a) $B = \{0, 1\}$ *where* $0 < 1$, $\sim 0 = 0^* = 1$, $\sim 1 = 1^* = 0$,

(b) $T = \{0, c, 1\}$, *where* $0 < c < 1$, $\sim c = c$, $c^* = 0$, $\sim 0 = 0^* = 1$, $\sim 1 = 1^* = 0$,

(c) $M = \{0, a, b, 1\}$ *where* $a \not\leq b$, $b \not\leq a$ *and* $0 < a, b < 1$, $\sim b = a^* = b$, $\sim a = b^* = a$, $\sim 0 = 0^* = 1$, $\sim 1 = 1^* = 0$.



Remark 4.1 From Corollary 4.1 it follows that T is not a subalgebra of M because $c^* = 0$ and $a^* = b$. This fact enables us to assert that ***mpM*** is different from the variety of tetravalent modal algebras.

The above results allow us to obtain certain properties of the variety of mpM-algebras.

Theorem 4.3 ***mpM*** *is locally finite, semisimple, residually small and residually finite.*

Proof It is a direct consequence of Theorem 4.2, Corollary 4.1 and well-known results of universal algebra ([7, Theorem 10.16, Lemma 12.2] and [27, Section 2.4]). □

5 Principal congruences

The following version of Theorem 3.2 facilitates the determination of principal congruences of mpM-algebras. It is based on two easily checked facts: (i) $Y \subseteq X(A)$ is closed (open) involutive if and only if $X(A) \setminus Y$ is open (closed) involutive; (ii) $\sigma_A(a) \cap Y = \sigma_A(b) \cap Y$ if and only if $\sigma_A(a) \triangle \sigma_A(b) \subseteq X(A) \setminus Y$.

Theorem 5.1 *Let* $A \in$ ***mpM***. *Then the lattice* $\mathcal{O}_I(\mathsf{mp_M}(A))$ *of all open and involutive subsets of* $\mathsf{mp_M}(A)$ *is isomorphic to the lattice* $Con(A)$; *and the isomorphism is the mapping* $\Theta_{OI} \colon \mathcal{O}_I(\mathsf{mp_M}(A)) \to Con(A)$ *defined by* $\Theta_{OI}(G) = \{(a, b) \in A \times A \colon \sigma_A(b) \triangle \sigma_A(a) \subseteq G\}$.

Remark 5.1 Let us observe that if $a, b \in A$ and $a \leq b$, then $\sigma_A(b) \triangle \sigma_A(a) \subseteq G$ if and only if $\sigma_A(b) \setminus \sigma_A(a) \subseteq G$.

Our next task is to determine the elements of $\mathcal{O}_I(\mathsf{mp_M}(A))$ corresponding to the principal congruences on A. Let $a, b \in A$ and $\theta(a,b)$ be the principal congruence on A generated by (a,b). Since $\theta(a,b) = \theta(a \wedge b, a \vee b)$ there is no loss of generality in assuming that $a \leq b$.

Proposition 5.1 *Let $A \in \boldsymbol{mpM}$ and let $a, b \in A$ be such that $a \leq b$. Then the following conditions are equivalent:*

(i) $\Theta_{OI}(G) = \theta(a,b)$,

(ii) *G is the smallest subset of $\mathcal{O}_I(\mathsf{mp_M}(A))$, in the sense of set inclusion, which contains $\sigma_A(b) \setminus \sigma_A(a)$.*

Proof (i) $\Rightarrow$ (ii): From the hypothesis and Remark 5.1 we have that $\sigma_A(b) \setminus \sigma_A(a) \subseteq G$. Moreover, if $H \in \mathcal{O}_I(\mathsf{mp_M}(A))$ is such that $\sigma_A(b) \setminus \sigma_A(a) \subseteq H$, then by Remark 5.1 we infer that $(a,b) \in \Theta_{OI}(H)$. Hence, $\Theta_{OI}(G) \subseteq \Theta_{OI}(H)$ and so by Theorem 5.1 we conclude that $G \subseteq H$.

(ii) $\Rightarrow$ (i): By Theorem 5.1 and Remark 5.1 we have that $(a,b) \in \Theta_{OI}(G)$. Besides, if $\varphi \in Con(A)$ and $(a,b) \in \varphi$, by Theorem 5.1 there is $H \in \mathcal{O}_I(\mathsf{mp_M}(A))$ such that $\Theta_{OI}(H) = \varphi$ from which it results that $\sigma_A(b) \setminus \sigma_A(a) \subseteq H$. Thus, by (ii) we infer that $G \subseteq H$. Hence, Theorem 5.1 allows us to assert that $\Theta_{OI}(G) \subseteq \varphi$ and so we conclude that $\Theta_{OI}(G) = \theta(a,b)$. □

In what follows, we will describe explicitly the subsets of Proposition 5.1 (ii).

Proposition 5.2 *Let $A \in \boldsymbol{mpM}$ and let $a, b \in A$ be such that $a \leq b$. Then these conditions are equivalent:*

(i) $\Theta_{OI}(G) = \theta(a,b)$,

(ii) $G = (\sigma_A(b) \setminus \sigma_A(a)) \cup g_A(\sigma_A(b) \setminus \sigma_A(a))$,

(iii) *there is a closed and open subset R of $X(A)$ such that $G = R \cup g_A(R)$.*

Proof (i) $\Rightarrow$ (ii): From the hypothesis and Proposition 5.1 we have that G is the smallest open and involutive subset of $\mathcal{O}_I(\mathsf{mp_M}(A))$ which contains $\sigma_A(b) \setminus \sigma_A(a)$. Furthermore, since G is involutive, $g_A(\sigma_A(b) \setminus \sigma_A(a)) \subseteq G$ from which it follows that $(\sigma_A(b) \setminus \sigma_A(a)) \cup g_A(\sigma_A(b) \setminus \sigma_A(a)) \subseteq G$. On the other hand, as $(\sigma_A(b) \setminus \sigma_A(a)) \cup g_A(\sigma_A(b) \setminus \sigma_A(a))$ is open, involutive and it contains $\sigma_A(b) \setminus \sigma_A(a)$, we conclude that $G = (\sigma_A(b) \setminus \sigma_A(a)) \cup g_A(\sigma_A(b) \setminus \sigma_A(a))$.

(ii) $\Rightarrow$ (i): From the hypothesis, it follows that G satisfies item (ii) in Proposition 5.1 and so the proof is complete.

(i) $\Leftrightarrow$ (iii): It is a direct consequence of [1, Lemmas 2, 3], taking into account that Remark 3.1 implies that all the subsets of an mp_M-space are convex. □

Finally, the above results of this section enable us to characterize the principal mpM-congruences as shown in Theorem 5.2.

Theorem 5.2 *Let $A \in \boldsymbol{mpM}$. Then the lattice $\mathcal{CO}_I(\mathsf{mp_M}(A))$ of all closed, open and involutive subsets of $\mathsf{mp_M}(A)$ is isomorphic to the lattice $Con_P(A)$ of all principal mpM-congruences on A; and the isomorphism, which we denote by Θ_{COI}, is the restriction of Θ_{OI} to $\mathcal{CO}_I(\mathsf{mp_M}(A))$.*

Proof If we suppose that $G \in \mathcal{CO}_I(\mathsf{mp_M}(A))$, then $G = G \cup g_A(G)$. This last assertion and Proposition 5.2 imply that $\Theta_{OI}(G) \in Con_P(A)$. Conversely, if $\rho \in Con_P(A)$, by Proposition 5.2 there is $G \in \mathcal{O}_I(\mathsf{mp_M}(A))$ such that $\rho = \Theta_{OI}(G)$ and $G = R \cup g_A(R)$ for some closed and open subset R of $\mathsf{mp_M}(A)$. Besides, considering that g_A is an involutive homeomorphism, we have that $G \in \mathcal{CO}_I(\mathsf{mp_M}(A))$ and so the proof is completed. □

Corollary 5.1 *Let $A \in$ **mpM**. Then*

(i) *$Con_P(A)$ is a Boolean algebra,*

(ii) *the intersection of a finite number of principal congruences is a principal one,*

(iii) *$Con_P(A) = Con_C(A)$, where $Con_C(A)$ denotes the set of all compact congruences on A.*

Proof (i) Let $\rho \in Con_P(A)$. Then, by Theorem 5.2, there is $G \in \mathcal{CO}_I(\mathsf{mp_M}(A))$ such that $\rho = \Theta_{COI}(G)$. Taking into account that $X \setminus G \in \mathcal{CO}_I(\mathsf{mp_M}(A))$, we have that $\phi = \Theta_{COI}(X \setminus G) \in Con_P(A)$ is the Boolean complement of ρ.

(ii) It follows from (i).

(iii) It is well-known that the compact congruences are the finitely generated members of $Con(A)$ and by [7, pp. 38] the latter are suprema of finite sets of principal congruences. Hence, by (i) we conclude that $Con_C(A) \subseteq Con_P(A)$. The converse follows immediately. □

Corollary 5.2 ***mpM*** *has permutable principal congruences.*

Proof Let $\varphi_1,\ \varphi_2 \in Con_P(A)$. Then, by Theorem 5.2, there are $Y_1, Y_2 \in \mathcal{CO}_I(\mathsf{mp_M}(A))$ such that $\varphi_1 = \Theta_{COI}(Y_1)$ and $\varphi_2 = \Theta_{COI}(Y_2)$. Suppose now that $(x, y) \in \varphi_2 \circ \varphi_1$, so there is $z \in A$ such that $(x, z) \in \varphi_1$ and $(z, y) \in \varphi_2$. These last assertions imply that $\sigma_A(x) \cap Y_1 = \sigma_A(z) \cap Y_1$ and $\sigma_A(z) \cap Y_2 = \sigma_A(y) \cap Y_2$. Let $U = (\sigma_A(x) \cap Y_1 \cap Y_2) \cup (\sigma_A(x) \cap (Y_2 \setminus Y_1)) \cup (\sigma_A(y) \cap (Y_1 \setminus Y_2))$. Hence, from Lemma 3.2, we conclude that $U \in D(X(A))$. Therefore, $w = \sigma_A^{-1}(U) \in A$. Moreover, it is easy to check that $\sigma_A(w) \cap Y_2 = \sigma_A(x) \cap Y_2$ and $\sigma_A(w) \cap Y_1 = \sigma_A(y) \cap Y_1$. Thus, we have that $(x, w) \in \varphi_2$ and $(w, y) \in \varphi_1$, and so $(x, y) \in \varphi_1 \circ \varphi_2$. Therefore, $\varphi_2 \circ \varphi_1 \subseteq \varphi_1 \circ \varphi_2$. The other inclusion is similar. □

Corollary 5.3 ***mpM*** *has equationally definable principal congruences.*

Proof It is a direct consequence of Corollary 5.1 (i) and [4, Theorem 0.3]. □

Corollary 5.4 ***mpM*** *is filtral.*

Proof It follows from [5, Corollary 3.7], bearing in mind Corollary 5.3 and the fact that ***mpM*** is semisimple. □

Corollary 5.5 *Let $A \in$ **mpM**. Then the following conditions are equivalent:*

(i) *A is simple,*

(ii) $B(A) = \{0, 1\}$ *where $B(A)$ is the set of Boolean elements of A.*

Proof (i) $\Rightarrow$ (ii): It is a direct consequence of Corollary 4.1.

(ii) $\Rightarrow$ (i): Suppose that A is not simple. Then there is a principal congruence $\theta(a, b)$ such that $\theta(a, b) \neq Id_A$ and $\theta(a, b) \neq A \times A$. Hence, by Theorem 5.2, we have that $\theta(a, b) = \Theta_{COI}(G)$ for some closed, open and involutive subset G of $X(A)$. This statement, Corollary 5.1 (i) and Lemma 3.2 allow us to assert that $G \in B(\mathsf{mPM}(X(A)))$ and so, by the hypothesis and the fact that σ_A is an *mpM*-isomorphism, we conclude that $G = \emptyset$ or $G = X(A)$. Therefore, $\theta(a, b) = Id_A$ or $\theta(a, b) = A \times A$, which is a contradiction. □

Proposition 5.3 *Each directly indecomposable mpM-algebra A is simple.*

Proof Let $\rho \in Con(A)$, $\rho \neq Id_A$. Then there are $a, b \in A$, $a \neq b$ such that $(a, b) \in \rho$ which implies that $\theta(a, b) \subseteq \rho$. Furthermore, from Corollary 5.1 (i) and Corollary 5.2, we infer that $\theta(a, b)$ is a factor congruence and so, by [7, pp. 53], we conclude that $\theta(a, b) = A \times A$. Therefore, $\rho = A \times A$ which completes the proof. □

Theorem 5.3 ***mpM*** *is directly representable.*

Proof From Proposition 5.3 and Corollary 4.1 we conclude that ***mpM*** has only finitely many finite directly indecomposable members. Then ***mpM*** is directly representable. □

Now, by virtue of the results established in [7, pp. 188–189] and Theorem 5.3, we can assert that

Corollary 5.6 *Finite members of* ***mpM*** *have uniform congruences.*

Theorem 5.4 ***mpM*** *is a discriminator variety.*

Proof It is a direct consequence of Theorem 4.3, Corollary 5.2, Corollary 5.3 and the results established in [4, Corollary 3.4]. □

Recall that the ternary discriminator function t on a set X is defined by the conditions:

$$t(x, y, z) = \begin{cases} z & \text{if } x = y, \\ x & \text{otherwise.} \end{cases}$$

In the sequel, we determine the ternary discriminator polynomial for ***mpM*** (i.e. a polynomial p that coincides with the ternary discriminator function on each subdirectly irreducible *mpM*-algebra) which enables us to obtain an equational description of the principal congruences. For this purpose, we define two unary operations on A as follows:

$$\triangle x = (\sim x)^* \wedge x,$$

$$\nabla x = \sim \triangle \sim x,$$

from which we introduce a new binary operation $\oplus$ on A by means of the formula:

$$x \oplus y = (\triangle(x \wedge y) \vee \sim \triangle(x \vee y)) \wedge (\nabla(x \wedge y) \vee \sim \nabla(x \vee y)).$$

Proposition 5.4 *Let $A \in \boldsymbol{mpM}$. Then it holds*

(S1) $x = y$ *if and only if* $x \oplus y = 1$,

(S2) $x \oplus y = y \oplus x$,

(S3) $x \oplus 1 = \triangle x$,

(S4) $(x \oplus y) \wedge x = (x \oplus y) \wedge y$,

(S5) $\triangle(x \oplus y) = x \oplus y$,

(S6) $\nabla(x \oplus y) = x \oplus y$,

(S7) $\sim (x \oplus y)$ *and* $x \oplus y$ *are Boolean complements.*

Proof It is routine. □

Theorem 5.5 *The ternary discriminator polynomial for* $\boldsymbol{mpM}$ *is*

$$p(x, y, z) = ((x \oplus y) \wedge z) \vee (\sim (x \oplus y) \wedge x).$$

Proof From (S1) we have that $p(x, x, z) = z$. If $x \neq y$, then by (S1) we infer that $x \oplus y \neq 1$ and so, by (S7) and Corollary 5.5, we conclude that $x \oplus y = 0$. Hence $p(x, y, z) = x$. □

Lemma 5.1 *Let $A \in \boldsymbol{mpM}$ and let $a, b \in A$ be such that $a \leq b$. Then the following conditions are equivalent:*

(i) $((a \oplus b) \wedge x) \vee (\sim (a \oplus b) \wedge a) = ((a \oplus b) \wedge y) \vee (\sim (a \oplus b) \wedge a)$,

(ii) $(a \oplus b) \wedge x = (a \oplus b) \wedge y$.

Proof We will only prove (i) $\Rightarrow$ (ii). Let $x, y \in A$ be such that (i) is satisfied. Then, by virtue of Theorem 5.5 and [27, Theorem 2.2 (5)], we infer that $(x, y) \in \theta(a, b)$, which implies that $(\sim x, \sim y) \in \theta(a, b)$ and so $((a \oplus b) \wedge \sim x) \vee (\sim (a \oplus b) \wedge a) = ((a \oplus b) \wedge \sim y) \vee (\sim (a \oplus b) \wedge a)$. Hence, $((a \oplus b) \wedge \sim x) \vee \sim (a \oplus b) = ((a \oplus b) \wedge \sim y) \vee \sim (a \oplus b)$ and therefore, by (S7), we have that $\sim (a \oplus b) \vee \sim x = \sim (a \oplus b) \vee \sim y$, from which we conclude the proof. □

Next, we obtain the equational characterization of the principal congruences we were looking for.

Theorem 5.6 *Let $A \in \boldsymbol{mpM}$ and let $a, b \in A$ be such that $a \leq b$. Then*

$$\theta(a, b) = \{(x, y) \in A \times A : x \wedge (a \oplus b) = y \wedge (a \oplus b)\}.$$

Proof It is a direct consequence of Theorems 5.5, [27, Theorem 2.2 (5)] and Lemma 5.1. □

On the other hand, bearing in mind the results established in [12], Theorem 5.5 and Theorem 5.6, we conclude that

Theorem 5.7 ***mpM*** *is a dual discriminator variety and the dual ternary discriminator polynomial is*

$$q(x, y, z) = (\sim (x \oplus y) \wedge z) \vee ((x \oplus y) \wedge x).$$

Furthemore, if $A \in \boldsymbol{mpM}$ *and* $a, b \in A$ *are such that* $a \leq b$*, then each coprincipal congruence on* A *generated by* (a, b) *is*

$$\gamma(a, b) = \{(x, y) \in A \times A : \sim (a \oplus b) \wedge x = \sim (a \oplus b) \wedge y\}.$$

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