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The Role of Halaš Identity in Orthomodular Lattices*

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Abstract

We prove that a certain identity introduced by R. Halaš for classifying basic algebras can be used for characterizing orthomodular lattices in the class of ortholattices with antitone involutions on every principal filter.

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In order to characterize certain basic algebras which are not horizontal sums of chains (see e.g. [3]), R. Halaš considered the identity

$$(x \oplus x) \oplus y = x \oplus (x \oplus y) \tag{I1}$$

which is satisfied e.g. in every MV-algebra or in every commutative basic algebra which is a chain with respect to the induced order. Since every orthomodular lattice can be organized into a basic algebra (see [2], [5]), we can ask if this identity holds true also for orthomodular lattices and, moreover, if it can characterize orthomodular lattices among ortholattices with antitone involutions on principal filters, see [2], [4], [5] and [6].

At first, we recall the mentioned concepts.

By a *basic algebra* (see [2], [4], [6]) is meant an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $(2, 1, 0)$ satisfying the following axioms:

(B1) $x \oplus 0 = x$,
(B2) $\neg\neg x = x$,
(B3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,
(B4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$, where $1 = \neg 0$.

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It is elementary to prove that every basic algebra satisfies also $0 \oplus x = x$ and $x \oplus \neg x = 1 = \neg x \oplus x$, see e.g. [6]. Let us mention that a basic algebra is an MV-algebra if and only if $\oplus$ is associative, i.e. if it satisfies the identity $x \oplus (y \oplus z) = (x \oplus y) \oplus z$.

As shown in [4], [5], [6], every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ can be converted into a bounded lattice $\mathcal{L}(\mathcal{A}) = (A; \vee, \wedge, 0, 1)$ where

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y = \neg(\neg x \vee \neg y)$$

whose induced order $\leq$ is given by

$$x \leq y \text{ if and only if } \neg x \oplus y = 1$$

and such that every principal filter $[p, 1]$ is equipped with an antitone involution $a \mapsto a^p$, i.e. for any $a, b \in [p, 1]$ we have

$$a^{pp} = a \quad \text{and} \quad a \leq b \Rightarrow b^p \leq a^p,$$

where $a^p = \neg a \oplus p$ (expressed in the operations of $\mathcal{A}$).

Also conversely, every bounded lattice $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ in which every principal filter $[p, 1]$ is equipped with an antitone involution $a \mapsto a^p$ for $a \in [p, 1]$, can be organized into a basic algebra $\mathcal{A}(L) = (L; \oplus, \neg, 0)$ as follows:

$$\neg x = x^0 \quad \text{and} \quad x \oplus y = (x^0 \vee y)^y.$$

Since $x^0 \vee y \in [y, 1]$ for any $x, y \in L$, the operation $\oplus$ is correctly defined on L.

By an *ortholattice* (see e.g. [1]) is meant a bounded lattice $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ with a unary operation $x \mapsto x'$ which is an *orthocomplementation*, i.e.

$$x \vee x' = 1 \quad \text{and} \quad x \wedge x' = 0,$$
$$x'' = x \quad \text{and} \quad x \leq y \Rightarrow y' \leq x'.$$

Hence, orthocomplementation is an antitone involution on the whole L. An ortholattice $\mathcal{L}$ is called *orthomodular* (see [1], [7]) if it satisfies the orthomodular law

$$x \leq y \implies x \vee (x' \wedge y) = y \qquad \text{(OML)}$$

which is equivalent to the identity

$$x \vee (x' \wedge (x \vee y)) = x \vee y$$

or to the dual of (OML)

$$x \leq y \implies y \wedge (y' \vee x) = x.$$

Observation 1 Let $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ be an orthomodular lattice and $p \in L$. Then the mapping $a \mapsto a^p = a' \vee p$ is an antitone involution (which is in fact an orthocomplementation) on the principal filter $[p, 1]$.

Of course, the mapping $a \mapsto a^p = a' \vee p$ is antitone because the orthocomplementation $'$ has this property and it maps $[p, 1]$ into itself. Due to (OML) and DeMorgan laws we have

$$a^{pp} = (a' \vee p)' \vee p = (a \wedge p') \vee p = a$$

since $a \in [p, 1]$, i.e. $p \leq a$. Thus it is an involution on $[p, 1]$. Evidently, $a \vee a^p = a \vee a' \vee p = 1$ and $a \wedge a^p = a \wedge (a' \vee p) = p$ due to the dual of (OML).

Hence, every orthomodular lattice $\mathcal{L}$ can be organized into a basic algebra $\mathcal{A}(L) = (L; \oplus, \neg, 0)$ where $\neg x = x'$ and

$$x \oplus y = (x' \vee y)' \vee y = (x \wedge y') \vee y. \tag{1}$$

It was shown in [2], [4] that a basic algebra is represented as an orthomodular lattice if and only if it satisfies the identity

$$y \oplus (x \wedge y) = y.$$

In what follows, we are going to study ortholattices having antitone involutions in every its principal filter. Such a lattice need not be orthomodular, see the following example.

Example 1 Consider the lattice visualized in Fig. 1.

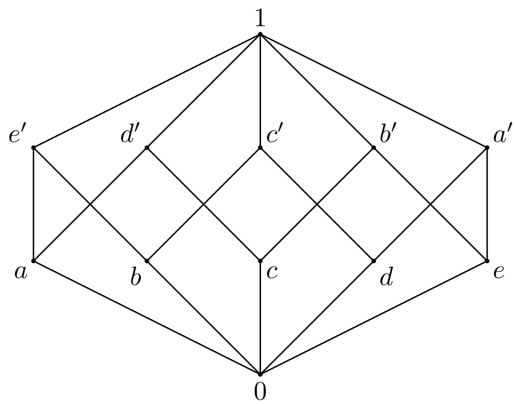

Fig. 1

This is evidently an ortholattice. However, it is not orthomodular because e.g. $a \leq e'$ but $a \vee (a' \wedge e') = a \vee 0 = a \neq e'$.

On the other hand, every its principal filter can be equipped with an antitone involution, e.g. for $a \in L$ we can take $a^a = 1$, $1^a = a$, $(e')^a = d'$ and $(d')^a = e'$. Alternatively, we can take $a^a = 1$, $1^a = a$, $(e')^a = e'$ and $(d')^a = d'$. Similarly it can be done for the remaining four-element principal filters and uniquely for principal filters with at most two elements.

For an ortholattice $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$, by a *subalgebra* of $\mathcal{L}$ we mean a sublattice of $\mathcal{L}$ containing 0 and 1 and being closed with respect to orthocomplementation.

Now, let $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ be an ortholattice every principal filter $[p, 1]$ of which is equipped with an antitone involution $a \mapsto a^p$ (for $a \in [p, 1]$). Then $\mathcal{L}$ can be organized into a basic algebra $\mathcal{A}(L) = (L; \oplus, \neg, 0)$, where

$$\neg x = x' \quad \text{and} \quad x \oplus y = (x' \vee y)^y. \tag{2}$$

Observation 2 If $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ is an ortholattice which is not orthomodular, then the mapping $x \mapsto x' \vee p$ need not be an antitone involution on the principal filter $[p, 1]$.

For example, in the lattice from Example 1 for $a \in L$ and $e' \in [a, 1]$ we have

$$(e'' \vee a)' \vee a = (e \vee a)' \vee a = 0 \vee a = a \neq e'$$

thus $x \mapsto x' \vee a$ is not an involution on $[a, 1]$. Hence, for the construction of binary operation $\oplus$ we cannot use formula (1), but we have to apply (2).

Lemma 1 *Let $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ be an ortholattice whose every principal filter is equipped with an antitone involution. Define $\oplus$ by* (2)*. Then $\mathcal{A}(L) = (L; \oplus, \neg, 0)$ satisfies the identity $x \oplus x = x$.*

Proof It is elementary to see that for every $x \in L$ we compute $x \oplus x = (x' \vee x)^x = 1^x = x$ (without regard what involution $a \mapsto a^x$ is choosen in the principal filter $[x, 1]$). □

Using Lemma 1 for orthomodular lattices, our identity (I1) can be reduced in the equivalent form

$$x \oplus y = x \oplus (x \oplus y). \tag{I2}$$

Theorem 1 *Let $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ be an orthomodular lattice and let $\mathcal{A}(L) = (L; \oplus, \neg, 0)$ be the corresponding basic algebra whose operations are defined by* (1)*. Then $\mathcal{A}(L)$ satisfies the identity* (I2)*.*

Proof Using (1), we compute $x \oplus y = (x' \vee y)' \vee y = (x \wedge y') \vee y$. Hence

$$\begin{aligned} x \oplus (x \oplus y) &= (x' \vee ((x \wedge y') \vee y))' \vee ((x \wedge y') \vee y) \\ &= (x \wedge ((x' \vee y) \wedge y')) \vee ((x \wedge y') \vee y) \\ &= ((x \wedge y') \wedge (x' \vee y)) \vee ((x \wedge y') \vee y). \end{aligned}$$

However,

$$(x \wedge y') \wedge (x' \vee y) \leq x \wedge y' \leq (x \wedge y') \vee y$$

thus $x \oplus (x \oplus y) = (x \wedge y') \vee y = x \oplus y$. □

Remark 1 The assertion of Theorem 1 need not be true if another involution than $x^p = x' \vee p$ is considered in the principal filter $[p, 1]$ of an orthomodular lattice, see e.g. the following example.

Example 2 Consider an orthomodular lattice $\mathcal{L} = \mathrm{OM}2\times\underline{2}$ visualized in Fig. 2.

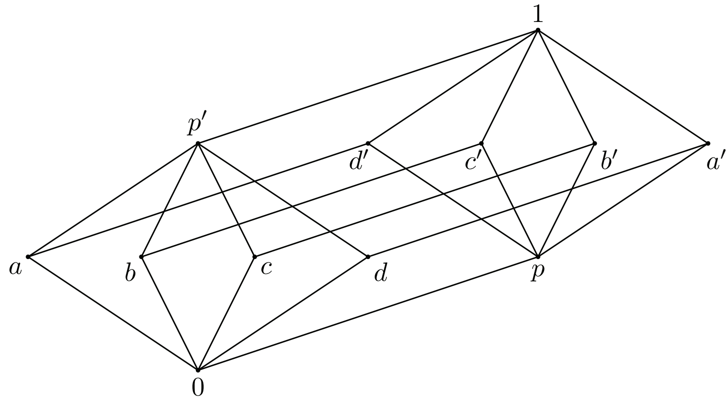

Fig. 2

Let us consider the following antitone involution on the principal filter $[b, 1]$: $(p')^b = p'$, $(c')^b = c'$, $b^b = 1$, $1^b = b$. Then we compute

$$c \oplus b = (c' \vee b)^b = (c')^b = c'$$

but $c \oplus (c \oplus b) = c \oplus c' = 1 \neq c'$, thus (I2) is not satisfied despite the fact that the lattice $\mathcal{L}$ is orthomodular.

On the other hand, we can prove the following.

Theorem 2 *Let $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ be an ortholattice whose every principal filter can be equipped with an antitone involution. If $\mathcal{L}$ is not orthomodular, then there exists a subalgebra $\mathcal{S}$ of $\mathcal{L}$ whose every principal filter is equipped with an antitone involution such that the identity* (I2) *with $\oplus$ being defined by* (2) *does not hold in $\mathcal{S}$ for any possible choice of these involutions.*

Proof Let $\mathcal{L}$ be an ortholattice from the assumption. Since $\mathcal{L}$ is not orthomodular then, by [1] or [7], it contains a subalgebra $\mathcal{S}$ depicted in Fig. 3.

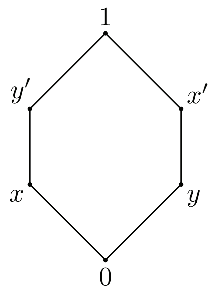

Fig. 3

One can note immediately that there is just one possible antitone involution in the principal filter $[y, 1]$, namely $y^y = 1$, $(x')^y = x'$ and $1^y = y$. However, by (2) we get $x \oplus y = (x' \vee y)^y = (x')^y = x'$, but $x \oplus (x \oplus y) = x \oplus x' = 1 \neq x'$. □

Since every subalgebra of an orthomodular lattice is again an orthomodular lattice, and hence it can be endowed by antitone involutions on every its principal filter as given by Obsevation 1, we can conclude:

Corollary 1 *Let* $\mathcal{L} = (L; \vee, \wedge, ', 0, 1)$ *be an ortholattice whose every principal filter can be equipped with an antitone involution. Then* $\mathcal{L}$ *is orthomodular if and only if every its subalgebra can be organized into a basic algebra satisfying the identity*

$$x \oplus (x \oplus y) = x \oplus y,$$

with $\oplus$ *being defined by* (2).

Remark 2 In the ortholattice $\mathcal{L}$ of Example 1 (see Fig. 1), such a subalgebra in which fails (I2) for any possible antitone involution on principal filters is e.g. $\{0, a, a', e, e', 1\}$.

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