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Statistical Inference about the Drift Parameter in Stochastic Processes*

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Abstract

In statistical inference on the drift parameter a in the Wiener process with a constant drift $Y_t = at + W_t$ there is a large number of options how to do it. We may, for example, base this inference on the properties of the standard normal distribution applied to the differences between the observed values of the process at discrete times. Although such methods are very simple, it turns out that more appropriate is to use the sequential methods. For the hypotheses testing about the drift parameter it is more proper to standardize the observed process, and to use the sequential methods based on the first time when the process reaches either B or $-B$, where $B > 0$, until some given time. These methods can be generalized to other processes, for instance, to the Brownian bridges.

Key words: Wiener process, Brownian bridge, symmetric process, sequential methods

2010 Mathematics Subject Classification: 60G15, 62F03, 62L10

1 Introduction

Random processes provide a useful tool for describing a large number of practical situations. Since we are unable to observe such processes continuously, we can observe the processes only in a set of discrete time points or only observe the time points when the processes reaches some boundary.

We will work with the Wiener process W_t (or with some other Gaussian continuous centered processes) with a constant drift a and constant variance $b^2 > 0$, i.e.

$$Y_t = at + bW_t. \tag{1.1}$$

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This process has been used as a first approximation to other processes with approximately constant drift rate and approximately constant instantaneous variance (see [8]) or can be used for hospital modeling events (see [3]) or for many other applications.

Therefore our objective is to make some inference on the parameters a and b (or b^2). First we formulate the problem in the second section, and recall the definition of the general Brownian bridge. In the third section we will show how the classical statistical approach can be used for the estimating of the parameters, and the hypotheses testing about them. In the fourth section we will consider the sequential methods for the hypotheses testing about the drift parameter a, and discuss its advantages and disadvantages against the classical approach. Generalizations to the Brownian bridge are discussed in the fifth section. In the last section all the results are summarized.

2 Problem formulation and notation

Consider a random process

$$Y_t = at + bB_t, \tag{2.1}$$

where $\{B_t; t \geq 0\}$ is either the Wiener process on the interval $[0, \infty)$ or the Brownian bridge on some fixed interval [0,T], where a is a drift rate, and $b^2 > 0$ is an instantaneous variance. Our goal is to make an inference on these parameters, especially we are interested in estimating them, and in doing hypotheses testing about them. In the hypotheses testing we are also interested in the time-efficiency of a decision, i.e. in shortening the average time needed for decision with some precision. The inference is often based on observing the process at discrete time points, i.e. on observations $Y_{t_1}, \dots, Y_{t_n}$, where $0 \leq t_1 < \cdots < t_n$ is a set of time points. Alternative approach is to observe the first hitting time of the process (2.1), i.e. $\tau_B = \inf\{t \geq 0; Y_t \geq B\}$, where $B > 0$ is a pre-specified boundary point. We may also consider more pre-specified boundary points.

General Brownian bridge

We say that the random process B_t is a general Brownian bridge on the interval $[T_1, T_2]$ with an initial value d_1 and final value d_2 if

- $B_{T_1} = d_1$, $B_{T_2} = d_2$;
- $\{B_t, t \in [T_1, T_2]\}$ has continuous sample paths;
- the finite dimensional distributions of the process B_t are Gaussian;
- the expected value and the covariance function are given by

  $$\mathsf{E}B_t = d_1 + \frac{t - T_1}{T_2 - T_1}(d_2 - d_1), \quad \mathsf{Cov}(B_s, B_t) = \frac{(T_2 - t)(s - T_1)}{T_2 - T_1}$$

  for $s \leq t$, $s, t \in [T_1, T_2]$.

The general Brownian bridge can be denoted by $\left\{B_t^{T_1,T_2}(d_1,d_2), t\in[T_1,T_2]\right\}$. Note that if $T_1=0$ and $T_2=T$ (for some $T>0$), then the Brownian bridge can be represented as follows

$$B_t^{0,T}(d_1,d_2) \stackrel{\mathrm{D}}{=} d_1 + \frac{t}{T}(d_2-d_1) + W_t - \frac{t}{T}W_T,$$

where W_t is the Wiener process and $\stackrel{\mathrm{D}}{=}$ stands for the equality in distribution.

In this paper we will consider the case $d_1=d_2=0$, $T_1=0$, and $T_2=T$ (for some $T>0$). The Brownian bridge on the interval $[0,T]$ (for some $T>0$) with $d_1=d_2=0$ will be denoted by $\left\{B_t^{0,T}, t\in[0,T]\right\}$ or B_t^T, for simplicity. In this case we have

$$B_0^T = B_T^T = 0, \quad \mathsf{E}B_t^T = 0,$$
$$\mathsf{Cov}\left(B_s^T, B_t^T\right) = \mathsf{E}B_s^T B_t^T = s - \frac{st}{T},$$
$$B_t^T \stackrel{\mathrm{D}}{=} W_t - \frac{t}{T}W_T$$

for $s\le t$, $s,t\in[0,T]$.

3 Classical approach

In this part we assume that the process B_t in (2.1) is the Wiener process W_t, i.e. $B_t=W_t$. For the inference on the parameters a and b we will observe the process Y_t at some times $0<t_1<t_2<\ldots<t_n\equiv T$. Assume, for brevity, that t_i are equidistant, i.e. we choose a constant $c>0$ such that (with the convention $t_0\equiv 0$)

$$t_k - t_{k-1} = c, \qquad k=1,2,\ldots,n,$$

or equivalently

$$t_k = ck, \quad k=0,1,\ldots,n.$$

Let $Y_c, Y_{2c},\ldots,Y_{nc}$ be the realizations of the process (2.1). Define (transform) new n observations as follows

$$Z_k = Y_{t_k} - Y_{t_{k-1}} = Y_{kc} - Y_{(k-1)c} = ac + b\left(W_{kc} - W_{(k-1)c}\right), \; k=1,2,\ldots,n.$$

Since the Wiener process has independent increments (with the mean zero normal distribution and the variance equal to the difference between observed times), then $Z_1,Z_2,\ldots,Z_n$ are independent, identically distributed random variables with $\mathrm{N}(ac,b^2c)$. We can use these observations and the elementary mathematical statistics tools to derive the basic properties of the estimators of the parameters a and b, and to do the hypotheses testing about them. Define

$$\bar{Z}_n = \frac{1}{n}\sum_{k=1}^n Z_k, \quad s^2 = \frac{1}{n-1}\sum_{k=1}^n \left(Z_k - \bar{Z}_n\right)^2, \quad S_{ac}^2 = \sum_{k=1}^n \left(Z_k - ac\right)^2.$$

According to the strong law of large numbers we can use $\bar{Z}_n/c := \frac{1}{nc}\sum_{k=1}^n Z_k = Y_T/T$ (in the variable n or in the variable T) and s^2/c as the strictly consistent estimators of the parameters a and b^2 respectively. The estimator Y_T/T of the parameter a can be also obtained by means of the maximal likelihood procedure applied to the Girsanov theorem (see [6, Theorem 8.6.6] and [4, (17.25)] for the choice $a_t \equiv 1$ or [6, Theorems 6.1.2, 6.2.8 and Example 6.2.11] for the choice $M(t) = N(t) \equiv 1$ for the similar result). It is also very well known that if a and b are the real parameters, then

$$\bar{Z}_n \sim \mathrm{N}\left(ac, \frac{b^2 c}{n}\right), \quad \frac{(n-1)s^2}{b^2 c} \sim \chi^2_{n-1}, \quad \frac{\bar{Z}_n - ac}{s}\sqrt{n} \sim t_{n-1}, \quad \frac{S^2_{ac}}{b^2 c} \sim \chi^2_n, \tag{3.1}$$

and that the statistics $\bar{Z}_n$ and s^2 are independent.

3.1 Hypotheses about the parameters a and b

Let us investigate the hypotheses testing $H_1\colon b^2 \geq b_0^2$, $H_2\colon b^2 \leq b_0^2$, and $H_3\colon b^2 = b_0^2$ against the alternatives $K_1\colon b^2 < b_0^2$, $K_2\colon b^2 > b_0^2$, and $K_3\colon b^2 \neq b_0^2$ respectively. Similarly we may consider the hypotheses testing $H_4\colon a \geq a_0$, $H_5\colon a \leq a_0$, and $H_6\colon a = a_0$ against the alternatives $K_4\colon a < a_0$, $K_5\colon a > a_0$, and $K_6\colon a \neq a_0$ respectively. According to the known distributions (3.1) we can test these hypotheses very simply for all the cases.

3.1.1 Hypotheses about the scale parameter b^2 with known a

This reflects the situation of the hypotheses testing about the variance in the normal distribution when the expected value is known. By the symbol $\mathbb{I}$ we will understand the indicator function, and it will be used for the critical functions of tests. When the critical function attains the value 1, we will reject a considered hypothesis. Otherwise we will not reject such a hypothesis. The critical functions of the tests of $H_1\colon b^2 \geq b_0^2$, $H_2\colon b^2 \leq b_0^2$, and $H_3\colon b^2 = b_0^2$ (on the maximal level α) against the alternatives $K_1\colon b^2 < b_0^2$, $K_2\colon b^2 > b_0^2$, and $K_3\colon b^2 \neq b_0^2$ are $\mathbb{I}\left\{\frac{S^2_{ac}}{b_0^2 c} < \chi^2_n(\alpha)\right\}$, $\mathbb{I}\left\{\frac{S^2_{ac}}{b_0^2 c} > \chi^2_n(1-\alpha)\right\}$, $1 - \mathbb{I}\left\{\frac{S^2_{ac}}{b_0^2 c} \in \left(\chi^2_n(\alpha/2), \chi^2_n(1-\alpha/2)\right)\right\}$ respectively. The powers of these tests are

$$\begin{aligned}
\mathsf{P}_{b_1^2}(\text{reject } H_1) &= F_{\chi^2_n}\left(\frac{b_0^2}{b_1^2}\chi^2_n(\alpha)\right), \quad b_1^2 < b_0^2,\\
\mathsf{P}_{b_1^2}(\text{reject } H_2) &= 1 - F_{\chi^2_n}\left(\frac{b_0^2}{b_1^2}\chi^2_n(1-\alpha)\right), \quad b_1^2 > b_0^2,\\
\mathsf{P}_{b_1^2}(\text{reject } H_3) &= 1 - \left(F_{\chi^2_n}\left(\frac{b_0^2}{b_1^2}\chi^2_n(1-\alpha/2)\right) - F_{\chi^2_n}\left(\frac{b_0^2}{b_1^2}\chi^2_n(\alpha/2)\right)\right), \quad b_1^2 \neq b_0^2,
\end{aligned}$$

where $F_{\chi^2_n}$ is the distribution function of the χ^2_n-distribution, and $\chi^2_n(\alpha)$ is its α-quantile. Note that the powers of the tests do not depend on the parameter a nor on the total observed time $T = cn$. The powers depend mainly on the number of observations n. Similarly we may consider the case when the parameter a is not known. The powers are very similar and so is the conclusion.

3.1.2 Hypotheses about the scale parameter a with known b

This reflects the situation of the hypotheses testing about the mean value in the normal distribution when the variance is known. Let $\Phi^{-1}(\alpha)$ be the α-quantile of the standard normal distribution (and let Φ be its distribution function). The critical functions of the tests of $H_4\colon a \geq a_0$, $H_5\colon a \leq a_0$, and $H_6\colon a = a_0$ (on the level α) against the alternatives $K_4\colon a < a_0$, $K_5\colon a > a_0$, and $K_6\colon a \neq a_0$ are $\mathbb{I}\left\{\frac{\bar{Z}_n - a_0 c}{b\sqrt{c}}\sqrt{n} < \Phi^{-1}(\alpha)\right\}$, $\mathbb{I}\left\{\frac{\bar{Z}_n - a_0 c}{b\sqrt{c}}\sqrt{n} > \Phi^{-1}(1-\alpha)\right\}$, $\mathbb{I}\left\{\frac{|\bar{Z}_n - a_0 c|}{b\sqrt{c}}\sqrt{n} > \Phi^{-1}(1-\alpha/2)\right\}$ respectively. The powers of these tests are

$$
\begin{aligned}
\mathsf{P}_{a_1}\,(\text{reject } H_4) &= \Phi\left(\Phi^{-1}(\alpha) + \frac{(a_0 - a_1)}{b}\sqrt{T}\right), \quad a_1 < a_0,\\
\mathsf{P}_{a_1}\,(\text{reject } H_5) &= 1 - \Phi\left(\Phi^{-1}(1-\alpha) + \frac{(a_0 - a_1)}{b}\sqrt{T}\right), \quad a_1 > a_0,\\
\mathsf{P}_{a_1}\,(\text{reject } H_6) &= 1 - \Phi\left(\Phi^{-1}(1-\alpha/2) + \frac{(a_0 - a_1)}{b}\sqrt{T}\right)\\
&\qquad + \Phi\left(\Phi^{-1}(\alpha/2) + \frac{(a_0 - a_1)}{b}\sqrt{T}\right), \quad a_1 \neq a_0.
\end{aligned}
$$

Note that the powers of these tests do not depend on the value a_0, but only on the difference $a_1 - a_0$. The powers also depend on the total observed time $T = cn$. Similarly we may consider the case when the parameter b is not known. The powers are very similar and so is the conclusion.

4 Sequential methods

Although the inference on the parameters a and b based on the classical approach is very simple, and it is very easy to implement, it is not always the most appropriate method, especially for the hypotheses testing from the time-efficiency point of view. Since the inference on the parameter b is sufficient for the classical approach, as the power does not depend on the total observed time, but mainly on the number of observations n, we will concentrate on the inference on the parameter a. Let us assume, for brevity, that the parameter b is known. Without loss of generality we may assume $b = 1$. Thus the observed process (2.1) will reduce to the process

$$Y_t = at + W_t. \tag{4.1}$$

For a given value of a_0 we are now interested in the hypotheses testing $H_4\colon a \geq a_0$, $H_5\colon a \leq a_0$, and $H_6\colon a = a_0$ against the alternatives $K_4\colon a < a_0$, $K_5\colon a > a_0$, and $K_6\colon a \neq a_0$ respectively. For this purpose we use the sequential methods. For better work we will transform the observed process (4.1) as follows

$$\tilde{Y}_t := Y_t - a_0 t = (a - a_0)t + W_t = \tilde{a}t + W_t,$$

where $\tilde{a} := a - a_0$. Since there is no confusion, we will consider that the transformed process is $Y_t = at + W_t$ rather than the process $\tilde{Y}_t = \tilde{a}t + W_t$. Thus we can only consider the hypotheses testing $H_4'\colon a \geq 0$, $H_5'\colon a \leq 0$, and $H_6'\colon a = 0$ against the alternatives $K_4'\colon a < 0$, $K_5'\colon a > 0$, and $K_6'\colon a \neq 0$ respectively. Since the hypotheses $H_4'\colon a \geq 0$ and $H_5'\colon a \leq 0$ are symmetric (as the Wiener process is), we will consider only the hypotheses H_5' and H_6'.

Let us start with the hypothesis H_5'. For this purpose we can define, for some pre-specified boundary point $B > 0$ and a finite observed time T_B (depending on B), the random time

$$\tau_{-\infty,B}(Y) := \inf\{t \geq 0, Y_t \geq B\} \wedge T_B.$$

The idea of such a test is that if there is a positive trend, then the process Y_t hits some positive boundary B till some suitable time T_B, and otherwise not. So the critical function of the test of H_5' will be $\mathbb{I}\{\tau_{-\infty,B}(Y) < T_B\}$. Let $\alpha, \beta \in (0,1)$ represent the maximal type I error and type II error respectively. From the requirement P_a (reject H_5') $= \mathsf{P}_a(\tau_{-\infty,B}(Y) < T_B) \leq \alpha$ for all $a \leq 0$ we get (by setting $a = 0$) $\mathsf{P}(\tau_{-\infty,B}(W) < T_B) = \alpha$. On the other hand it is very well known (see [2, Theorem 1.5.1] or the formula (8.20) in [1] for more details)

$$\begin{aligned}\mathsf{P}(\tau_{-\infty,B}(W) < T_B) &= \mathsf{P}\left(\sup_{0\leq t\leq T_B} W_t \geq B\right) = 2\mathsf{P}(W_{T_B} \geq B) \\ &= 2\left(1 - \Phi\left(B/\sqrt{T_B}\right)\right).\end{aligned}$$

Simple calculation shows $T_B = \left(\frac{B}{\Phi^{-1}(1-\alpha/2)}\right)^2$ so whatever B we choose, we can easily compute T_B or compute B from T_B. On the other hand if we want to attain some power $1 - \beta$ at some chosen $a_1 > 0$, we will use the next result. It is known (see the formula (13.9) in [9] or [8] for many types of results) that the first hitting time of a positive boundary B by a shifted Wiener process $Y_t = at + W_t$ with the parameter $a > 0$ is an absolutely continuous a.s. positive random variable with the inverse Gaussian distribution with the parameters $\frac{B}{a}$ and B^2, denoted by $IG\left(\frac{B}{a}, B^2\right)$, i.e. with the density

$$f_{IG}(t) = \frac{B}{\sqrt{2\pi t^3}} \exp\left\{-\frac{(B-at)^2}{2t}\right\}, \ t > 0.$$

Thus to determine such B or T_B we use

$$\mathsf{P}_{a_1}\left(\sup_{0\leq t\leq T_B} Y_t \geq B\right) = \mathsf{P}_{a_1}(\tau_{-\infty,B}(Y) < T_B) = F_{IG(B/a_1,B^2)}(T_B),$$

where $F_{IG(B/a_1,B^2)}$ is the distribution function of the inverse Gaussian distribution with the parameters $\frac{B}{a_1}$ and B^2. According to this formula we can calculate B (for given T_B) or T_B (for given B) large enough to have the power of this test at least $1 - \beta$ at some chosen $a_1 > 0$.

We can also test the hypothesis H_5' against the alternative K_5' differently. We use a testing method which consider two hitting boundaries and a maximal observed time. Let us define

$$\tau_B^{\min}(Y) := \tau_{-B,B}^{\min}(Y) := \inf\{t \geq 0, |Y_t| \geq B\} \wedge T_B, \tag{4.2}$$

where T_B is some suitable chosen time according to the constant B (or B can be chosen from T_B). The idea of such a test is that if there is a positive trend, then the process Y_t will reach first the positive boundary B rather than the negative boundary $-B$ and till a suitable time T_B, and otherwise not. So the critical function of the test of H_5' will be $\mathbb{I}\{Y_{\tau_B^{\min}(Y)} \geq B\}$. Another possibility is to consider two (generally different) boundaries B and $-A$ ($A, B > 0$), to use [5, Theorem 2.49], and to consider the more general formula (9.13) from [1]. For the maximal type I error α we will use the symmetry of the Wiener process to the next result. According to [2, Theorem 1.5.1] or the formula (9.14) in [1] (and the fact $W_t \overset{\mathrm{D}}{=} \sqrt{T} W_{t/T}$ for $T > 0$) we have

$$\begin{aligned} \mathsf{P}\left(W_{\tau_B^{\min}(W)} \geq B\right) &= \frac{1}{2}\mathsf{P}\left(\sup_{0\leq t\leq T_B} |W_t| \geq B\right) = \frac{1}{2} - \frac{1}{2}\mathsf{P}\left(\sup_{0\leq t\leq T_B} |W_t| < B\right) \\ &= \frac{1}{2} - \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \exp\left\{-T_B \pi^2 (2k+1)^2/8B^2\right\}. \end{aligned} \tag{4.3}$$

From the requirement $\mathsf{P}_a\big(Y_{\tau_B^{\min}(Y)} \geq B\big) \leq \alpha$ for all $a \leq 0$ and this equation we can compute T_B numerically for given $B > 0$ and $\alpha \in (0,1)$ (or compute B numerically for given T_B and $\alpha \in (0,1)$). Let us illustrate the procedure on simulated data for the choice $\alpha = 0.05$, $a = 1$, and $B = 7$. According to the equation (4.3) we can approximately compute $T_B = 12.76$.

As we can observe from the Fig. 1 the process hits first the positive boundary B and before time T_B so we would reject the hypothesis $H_5'\colon a \leq 0$.

If we need to choose appropriate $B > 0$ for attaining the power $1 - \beta$ at some chosen $a_1 > 0$, we can use the known result (see [3] the formula (2) or [7] for more details)

$$\begin{aligned} \mathsf{P}_{a_1}\left(Y_{\tau_B^{\min}(Y)} \geq B\right) = 1 - \sum_{k=-\infty}^{\infty} \operatorname{sign}(a_1)\Bigg\{ & e^{-a_1 c_k}\left[\Phi\left(\frac{c_k + B - a_1 T_B}{\sqrt{T_B}}\operatorname{sign}(a_1)\right)\right] \\ & - e^{a_1 d_k}\left[\Phi\left(\frac{-d_k + B - a_1 T_B}{\sqrt{T_B}}\operatorname{sign}(a_1)\right)\right]\Bigg\}, \end{aligned} \tag{4.4}$$

where $c_k = 4kB$ and $d_k = 4kB + 2B$.

Now we concentrate on the hypothesis testing $H_6'\colon a = 0$ against the alternative $K_6'\colon a \neq 0$. For this purpose we will consider the process $\tilde{Y}_t = \tilde{a}t + W_t$ as that for the hypothesis testing H_5' and the random time $\tau_B^{\min}(Y)$ given by (4.2). The idea of such a test is also similar to the tests used for the hypotheses testing at one-sided alternatives—if there is any positive or negative trend, then the absolute value of the process Y_t will grow fast, and otherwise not. The critical function of the test of H_6' will be $\mathbb{I}\left\{\tau_B^{\min}(Y) < T_B\right\}$. For the maximal type I error α we use a similar formula as that for the hypothesis H_5'.

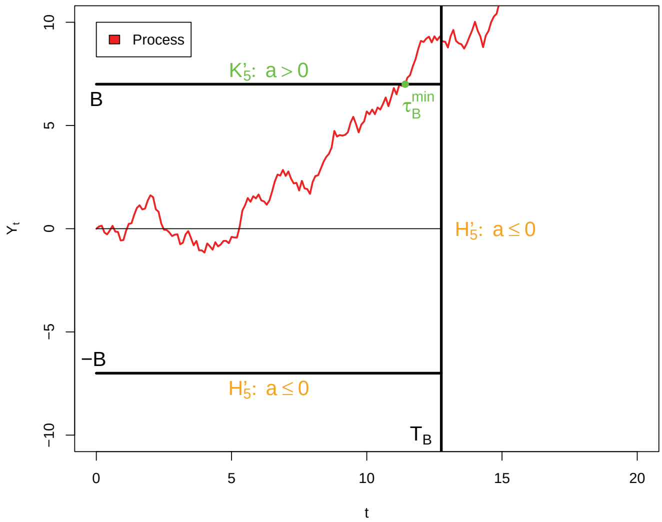

Fig. 1: Sequential method considering 2 boundaries in the process $Y_t = at + W_t$, and for the hypothesis testing H_5': $a \leq 0$ against the alternative K_5': $a > 0$.

Under the hypothesis H_6' the process Y_t is of course the Wiener one, and using the same argumentation as we used in the equation (4.3) we get

$$\begin{aligned}\mathsf{P}(\tau_B^{\min}(W) < T_B) &= \mathsf{P}\left(\sup_{0\leq t\leq T_B} |W_t| \geq B\right) \\ &= 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left(T_B\pi^2(2k+1)^2/8B^2\right).\end{aligned}$$

For given $B > 0$ and $\alpha \in (0,1)$ it is not difficult to compute T_B numerically (or B from $T_B > 0$ and $\alpha \in (0,1)$). But we need to choose appropriate $B > 0$ for the power $1-\beta$ at some chosen $a_1 \neq 0$. For this purpose we can use the formula (1) from [3] (or see [7] for more details) to obtain

$$\begin{aligned}&\mathsf{P}_{a_1}\left(\sup_{0\leq t\leq T_B}|Y_t| \geq B\right) = \\ &= 1 - \sum_{k=-\infty}^{\infty}\left\{e^{-a_1c_k}\left[\Phi\left(\frac{c_k + B - a_1T_B}{\sqrt{T_B}}\right) - \Phi\left(\frac{c_k - B - a_1T_B}{\sqrt{T_B}}\right)\right]\right. \\ &\qquad \left. - e^{a_1d_k}\left[\Phi\left(\frac{-d_k + B - a_1T_B}{\sqrt{T_B}}\right) - \Phi\left(\frac{-d_k - B - a_1T_B}{\sqrt{T_B}}\right)\right]\right\}, \qquad (4.5)\end{aligned}$$

where c_k and d_k are the same as in the equation (4.4), i.e. $c_k = 4kB$ and $d_k = 4kB + 2B$. Note that such sequential methods considering two boundaries (with the same absolute value) until some given time are usually very appropriate, especially from the time-efficiency point of view. This conclusion is illustrated on the Fig. 2 and Fig. 3 for the power $1-\beta = 0.95$, type I error $\alpha = 0.05$, and $a_1 = a_0 + 0.2$. Instead of using the complicated formulas (4.4) and (4.5), we can also use the simulation methods to attain the needed powers at desirable alternatives. Such a procedure can be applied for more general drift functions.

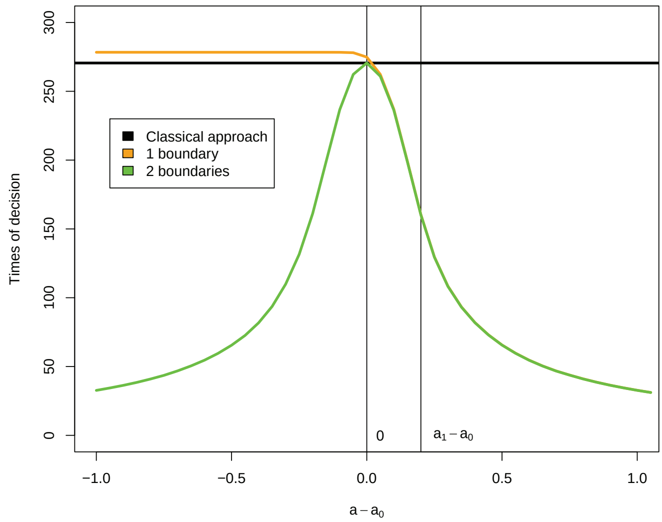

Fig. 2: Average times of decision at $H_5: a \leq a_0$ against $K_5: a > a_0$ in the process $Y_t = at + W_t$, and for the different procedures.

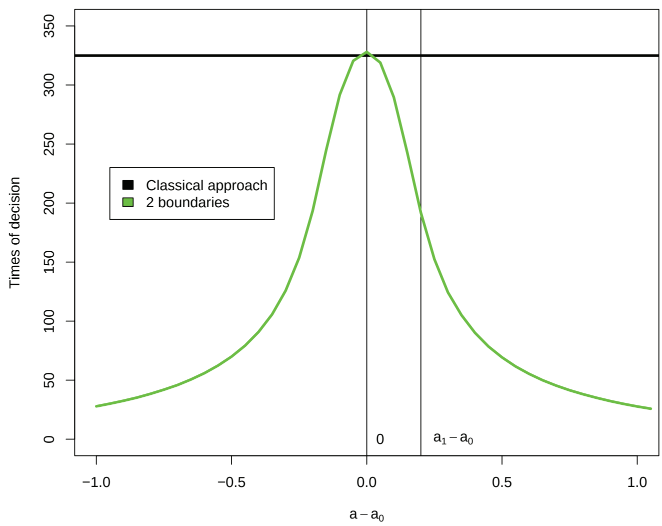

Fig. 3: Average times of decision at $H_6: a = a_0$ against $K_6: a \neq a_0$ in the process $Y_t = at + W_t$, and for the different procedures.

5 Sequential methods for the Brownian bridge

For a given value of $T > 0$ let us assume that we observe a process

$$U_t = at + B_t^T, \quad t \in [0, T],$$

where B_t^T is the Brownian bridge on the interval $[0, T]$. Such a process can be called as the shifted Brownian bridge. Note that the Brownian bridge is usually considered on the interval $[0, 1]$, and it is not difficult to show

$$B_t^T \overset{\mathrm{D}}{=} \sqrt{T} B_{t/T}^1 \quad \text{or} \quad B_t^1 \overset{\mathrm{D}}{=} \frac{1}{\sqrt{T}} B_{tT}^T.$$

Using similar techniques it can be easily shown (for chosen $T, S > 0$) the more general formula

$$B_t^S \overset{\mathrm{D}}{=} \sqrt{\frac{S}{T}} B_{tT/S}^T. \tag{5.1}$$

Our goal is to do the hypotheses testing about the unknown parameter a, i.e. for some chosen a_0 we want to test $H_4\colon a \geq a_0$, $H_5\colon a \leq a_0$, and $H_6\colon a = a_0$ against the alternatives $K_4\colon a < a_0$, $K_5\colon a > a_0$, and $K_6\colon a \neq a_0$ respectively. Note that we need to consider the hypotheses testing only if we are unable to wait until time T or if we are more interested in shortening the time of decision. Otherwise we can use the obvious equality $a = \frac{U_T}{T}$ (as $B_T^T = 0$), which may lead us to the strictly consistent estimator $\frac{U_t}{t}$ (as $t \to T$) of the parameter a, and decide with certainty. For the hypotheses testing we first transform the observed process into more convenient process—like before we transform

$$\tilde{U}_t := U_t - a_0 t = (a - a_0)t + B_t^T = \tilde{a}t + B_t^T, \quad t \in [0, T],$$

where $\tilde{a} := a - a_0$. We will consider that the transformed process is $U_t = at + B_t^T$, as in the process (4.1), rather than the process $\tilde{U}_t = \tilde{a}t + B_t^T$. We can similarly consider the equivalent hypotheses $H_4'\colon a \geq 0$, $H_5'\colon a \leq 0$, and $H_6'\colon a = 0$ against the alternatives $K_4'\colon a < 0$, $K_5'\colon a > 0$, and $K_6'\colon a \neq 0$ respectively. Since the hypotheses H_4' and H_5' are symmetric (as the Brownian bridge is), we will concentrate only on the hypotheses H_5' and H_6' against the alternatives K_5' and K_6' respectively.

Let us start with the hypothesis testing H_5', and use the similar idea as we used in the case of the Wiener process in the previous section. Let us consider (for some chosen $B > 0$) the random time

$$\tau_{-\infty,B}(U) := \inf\{t \geq 0, U_t \geq B\} \wedge T.$$

The idea of testing is also the same as that for the process (4.1), i.e. if there is some positive trend, the process U_t hits some suitably chosen boundary $B > 0$ before time T, and otherwise not. The critical function of the test of H_5' will be $\mathbb{I}\left(\{\tau_{-\infty,B}(U) < T\} \cup \{U_T > 0\}\right)$. To choose the suitable boundary $B > 0$ we need the condition for the maximal type I error. Let $\alpha \in (0,1)$ be such a given maximal error, and we calculate such a probability under the null hypothesis $a = 0$. According to [2, Theorem 1.5.1] or the formula (9.41) in [1] (and the equation (5.1)) we get

$$\alpha = \mathsf{P}\left(\tau_{-\infty,B}\left(B^T\right) < T\right) = \mathsf{P}\left(\sup_{0\le t\le T} B_t^T \ge B\right) = \exp\left(-2B^2/T\right).$$

For α and T we can compute $B = \sqrt{-\frac{\log\alpha}{2}T}$. We see that the choice of such B is limited, and gives us only one possibility how to choose B. We also see that for the value of time T there holds $T = \frac{-2B^2}{\log\alpha}$ and so $\sqrt{T} = \sqrt{\frac{-2}{\log\alpha}}B$. Let us evaluate the process U_t at the last time, and see the chance of hitting or exceeding the boundary B before time T

$$U_T = aT = a\sqrt{T}\sqrt{T} = a\sqrt{T}\sqrt{\frac{-2}{\log\alpha}}B.$$

From this equation we see that we can be sure of detecting the alternative before time T if

$$U_T > B \Leftrightarrow a\sqrt{T}\sqrt{\frac{-2}{\log\alpha}} > 1 \Leftrightarrow a > \frac{1}{\sqrt{T}}\sqrt{\frac{-\log\alpha}{2}} = \frac{B}{T}. \tag{5.2}$$

We see the advantages and disadvantages of the procedure if we are unable to wait the whole time T. For small $a > 0$ the time-efficiency of this test can not be improved, and we need to wait the whole time T. On the other hand we are able to detect the positive trend before time T with certainty for $a > 0$ which is greater than the bound in (5.2). We see that this bound can be exactly computed, and depends on the level α, and on the maximal observed time T. The fact that we need to wait until time T for small a should not be surprising, because on the bounded intervals we are not usually able to detect all positive trends. However, the Brownian bridge is not wild, i.e. it is continuous, tied down to the origin and at $t = T$ (as $B_0^T = B_T^T = 0$) so we are able to detect larger a before time T.

The hypothesis H_5' can be also tested by means of the two boundaries as that for the Wiener process in the previous section. Let us recall the random time

$$\tau_B^{\min}(U) := \tau_{-B,B}^{\min}(U) := \inf\left\{t \ge 0, |U_t| \ge B\right\} \wedge T.$$

The idea of testing is the same as that for the shifted Wiener process (4.1), i.e. if there is a positive trend, then the process U_t will reach first the positive boundary B rather than the negative boundary $-B$, and before time T, and otherwise not. Thus the critical function of the test of H_5' will be

$$\mathbb{I}\left(\left\{U_{\tau_B^{\min}(U)} \ge B\right\} \cup \left(\left\{\tau_B^{\min}(U) = T\right\} \cap \left\{U_T > 0\right\}\right)\right).$$

Let us consider the maximal type I error α. Since the Brownian bridge has similar important properties as the Wiener process—symmetry, continuity, the knowledge of the distribution of the supremum of the absolute value, we can use a similar calculation as we used in (4.3) and [2, Theorem 1.5.1] (or the formula (9.40) in [1]) to get

$$\alpha = \mathsf{P}\left(B^T_{\tau_B^{\min}(B^T)} \geq B\right) = \frac{1}{2}\mathsf{P}\left(\sup_{0\leq t\leq T} |B^T_t| \geq B\right) = \frac{1}{2} - \frac{1}{2}\mathsf{P}\left(\sup_{0\leq t\leq T} |B^T_t| < B\right)$$
$$= \frac{1}{2}\sum_{k\neq 0}(-1)^{k+1}\exp\left\{-2k^2B^2/T\right\}. \tag{5.3}$$

From this equation we can calculate the boundary $B > 0$, and test the hypothesis H'_5. Note that the test can not reveal all (small) trends before time T because of the same reason as before (finite observed time). We see that T is of the form $\sqrt{T} = c(\alpha)B$, where $c(\alpha)$ is a positive constant given by (5.3), depending only on α, and satisfying $\lim_{\alpha\to 0} c(\alpha) = 0$, $\lim_{\alpha\to 1} c(\alpha) = \infty$. If we evaluate the process U_t at the last time, we will get

$$U_T = aT = a\sqrt{T}\sqrt{T} = a\sqrt{T}c(\alpha)B.$$

To hit or exceed the boundary B before time T we need

$$a > \frac{1}{\sqrt{T}c(\alpha)} = \frac{B}{T}. \tag{5.4}$$

Although the test is usually (or more conveniently) more efficient for one-sided alternatives, there is always a positive probability that the process U_t hits first the negative boundary $-B$, and we can not be certainly sure of detecting any trend. However, behind the bound (5.4) there is a high probability that we detect the positive trend before time T. Let us illustrate the procedure on simulated data for the choice $\alpha = 0.05$, $a = 1$, and $T = 1$. According to the equation (5.3) we can approximately compute $B = 1.22$.

As we can observe from the Fig. 4 the process hits first the positive boundary B and before time $T = 1$ so we would decide correctly before time T, i.e. we would reject the hypothesis H'_5.

Now we will investigate the hypothesis testing $H'_6\colon a = 0$ against the both-sided alternative $K'_6\colon a \neq 0$. Similarly as in the case of the Wiener process we use the random time

$$\tau_B^{\min}(U) := \tau_{-B,B}^{\min}(U) := \inf\left\{t \geq 0, |U_t| \geq B\right\} \wedge T.$$

The idea of the test is also similar to the tests considered for the hypotheses testing at two-sided alternatives with the Wiener process—if there is any positive or negative trend, then the absolute value of the process U_t will grow fast, and otherwise not. The critical function of the test of H'_6 will be

$$\mathbb{I}\left(\left\{\tau_B^{\min}(U) < T\right\} \cup \left\{U_T \neq 0\right\}\right).$$

From the requirement for the maximal type I error α and the equation (5.3) we get

$$\alpha = \mathsf{P}\left(\tau_B^{\min}\left(B^T\right) < T\right) = \mathsf{P}\left(\sup_{0\le t\le T}\left|B_t^T\right| \ge B\right) = \sum_{k\neq 0}(-1)^{k+1}\exp\left\{-2k^2B^2/T\right\}. \qquad (5.5)$$

From this equation we can calculate the boundary $B > 0$, and test the hypothesis H_6'.

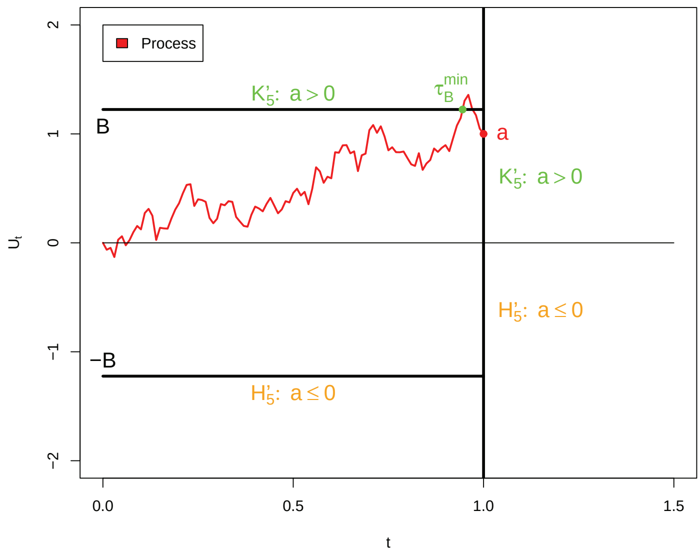

Fig. 4: Sequential method considering 2 boundaries in the process $U_t = at + B_t^1$, and for the hypothesis testing H_5': $a \le 0$ against the alternative K_5': $a > 0$.

According to the equation (5.5) we have T of the form $\sqrt{T} = d(\alpha)B$, where $d(\alpha)$ is a positive constant given by (5.5), depending only on α, and satisfying $\lim_{\alpha\to 0} d(\alpha) = 0$, $\lim_{\alpha\to 1} d(\alpha) = \infty$. If we evaluate the process U_t at the last time, we will get

$$U_T = aT = a\sqrt{T}\sqrt{T} = a\sqrt{T}d(\alpha)B.$$

To hit or exceed the absolute value of the boundary B before time T we need

$$|a| > \frac{1}{\sqrt{T}d(\alpha)} = \frac{B}{T}. \qquad (5.6)$$

We see that the test has similar properties as that for one-sided alternatives if we are unable to wait the whole time T. We are not able to detect all (small) trends before time T for the small absolute value of a. However, we are able to detect the positive and negative trends before time T with certainty for a which has the absolute value greater than the bound (5.6). This bound can be numerically computed, and depends on the level α and on the maximal observed time T. The sequential methods considering two boundaries for the Brownian bridge are more appropriate, as that for the Wiener process, from the time-efficiency point of view, than the sequential methods considering one boundary.

6 Conclusions and extensions

The hypotheses about the parameter a in the process $Y_t = at + B_t$, where B_t is either the Wiener process or the Brownian bridge, can be successfully tested by means of the sequential methods with two boundaries until some finite time. Although the method is a little complicated, its time-efficiency is more appropriate then in the classical statistical approach. The consistent estimators of the parameter a can be obtained by different methods with similar results. For the hypotheses testing the sequential methods can be generalized to different processes B_t. We usually require for such processes to be continuous and symmetric. Moreover, if the process is a martingale, we can use the maximal martingale inequality to extend the sequential procedure. This holds, for example, for the process $B_t = \int_0^t b(s)\,\mathrm{d}W_s$, where $b(s)$ is a given deterministic function, and W_t is the Wiener process.

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