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On the Kluvánek Construction of the Lebesgue Integral*

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Abstract

I. Kluvánek suggested to built the Lebesgue integral on a compact interval in the real line by the help of the length of intervals only. In the paper a modification of the Kluvánek construction is presented applicable to abstract spaces, too.

Key words: Lebesgue integral, rings of sets, measures on compact interval

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1 Introduction

Let $\mathcal{A}$ be the family of all subintervals of a given interval $[a, b]$. If $A \in \mathcal{A}$, then $\mu(A)$ is the lenght of A, i.e. $\mu([c, d]) = \mu([c, d)) = \mu((c, d]) = \mu((c, d)) = d - c$. In the following definition the Kluvánek construction is presented.

Definition 1 $f \in \mathcal{K} \Longleftrightarrow$

$$\exists \alpha_i \in R,\ \exists A_i \in \mathcal{A},\ \sum_{i=1}^{\infty} |\alpha_i| \mu(A_i) < \infty$$

$$\sum_{i=1}^{\infty} |\alpha_i| \chi_{A_i}(x) < \infty \Rightarrow f(x) = \sum \alpha_i \chi_{A_i}(x).$$

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The only problem is in the proof of the independence of the integral of a function f

$$\int_a^b f(x)\,dx = \sum_{i=1}^{\infty} \alpha_i \mu(A_i)$$

on the presentation of f in the form

$$f(x) = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}(x).$$

Mainly, Kluvánek's proof depends on some properties of the real line.

Therefore we suggest a modification of the construction considering first non-negative functions only. In the first part of our paper the equality of two definitions is shown. In the second part it is shown the independence of the sum

$$\sum_{i=1}^{\infty} \alpha_i \mu(A_i)$$

$(\alpha_i \geq 0)$ on the representation of f in the form

$$f(x) = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}(x).$$

2 Integrable functions

Definition 2 $f \in \mathcal{P}^+ \iff$

$$\exists \alpha_i \geq 0,\ \exists A_i \in \mathcal{A},\ \sum_{i=1}^{\infty} \alpha_i \mu(A_i) < \infty$$

and

$$f(x) = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}(x).$$

Definition 3

$$f \in \mathcal{P} \iff \exists g, h \in \mathcal{P}^+,\ g(x) < \infty,\ h(x) < \infty \Rightarrow f(x) = g(x) - h(x).$$

Theorem 1 $\mathcal{K} = \mathcal{P}$.

Proof 1. $\mathcal{K} \subset \mathcal{P}$. Let $f \in \mathcal{K}$, i.e.

$$\exists \alpha_i\ \exists A_i \in \mathcal{A},\ \sum_{i=1}^{\infty} |\alpha_i| \mu(A_i) < \infty$$

$$\sum |\alpha_i| \chi_{A_i}(x) < \infty \Rightarrow f(x) = \sum \alpha_i \chi_{A_i}(x)$$

Then

$$\sum_{i=1}^{\infty} \alpha_i^+ \mu(A_i) \le \sum_{i=1}^{\infty} |\alpha_i| \mu(A_i) < \infty,$$

$$\sum_{i=1}^{\infty} \alpha_i^- \mu(A_i) \le \sum_{i=1}^{\infty} |\alpha_i| \mu(A_i) < \infty.$$

$$\sum_{i=1}^{\infty} \alpha_i^+ \chi_{A_i}(x) \le \sum_{i=1}^{\infty} |\alpha_i| \chi_{A_i}(x) < \infty,$$

$$\sum_{i=1}^{\infty} \alpha_i^- \chi_{A_i}(x) \le \sum_{i=1}^{\infty} |\alpha_i| \chi_{A_i}(x) < \infty.$$

Put

$$g(x) = \sum \alpha_i^+ \chi_{A_i}(x), \quad h(x) = \sum \alpha_i^- \chi_{A_i}(x)$$

Then $g, h \in \mathcal{P}^+$, $g(x) < \infty$, $h(x) < \infty$, and

$$f(x) = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}(x) = \sum_{i=1}^{\infty} \alpha_i^+ \chi_{A_i}(x) - \sum_{i=1}^{\infty} \alpha_i^- \chi_{A_i}(x) = g(x) - h(x).$$

Therefore $f \in \mathcal{P}$.

2. $\mathcal{P} \subset \mathcal{K}$. Let $f \in \mathcal{P}$. Then there exist $g, h \in \mathcal{P}^+$ such that

$$g(x) < \infty, h(x) < \infty \Rightarrow f(x) = g(x) - h(x).$$

Evidently $g, h \in \mathcal{K}$. Also $-h \in \mathcal{K}$ by [1, 26.12.2]. Since $|g(x)| + |-h(x)| < \infty$, then by [1, 26.12.4] $f(x) = g(x) + (-h(x))$ is a member from $\mathcal{K}$. □

3 Integral

We want to define the integral of a function $f \in \mathcal{P}^+$

$$f = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}, \quad \alpha_i \ge 0,\ A_i \in \mathcal{A}$$

by the equality

$$\int_a^b f(x)\,dx = \sum_{i=1}^{\infty} \alpha_i \mu(A_i).$$

Of course, it is first necessary to prove the independence of the sum on the representation of f in the form

$$f = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i}, \quad \alpha_i \ge 0,\ A_i \in \mathcal{A}.$$

It is realized in Theorem 4 what is the main result of the paper.

Definition 4 Let $[a, b]$ be an interval, $\mathcal{A}$ be the family af all subintervals of $[a, b]$, $\mu\colon \mathcal{A} \to [0, \infty)$ be a measure. A function $f\colon [a, b] \to \mathbb{R}$ belongs to $\mathcal{P}_0$, if there exist $n \in N$, $\alpha_1, \ldots, \alpha_n \geq 0$, $A_1, \ldots, A_n \in \mathcal{A}$ such that

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}.$$

Theorem 2 *To any* $f \in \mathcal{P}_0$,

$$f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$$

there exist $m \in N$, $\beta_1, \ldots, \beta_m \geq 0$, $B_1, \ldots, B_m \in \mathcal{A}$, $B_i \cap B_j = \emptyset$ $(i \neq j)$ *such that*

$$f = \sum_{j=1}^{m} \beta_j \chi_{B_j}, \quad \textit{and} \quad \sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{j=1}^{m} \beta_j \mu(B_j).$$

Proof By induction. The idea:

$$\alpha_1 \chi_{A_1} + \alpha_2 \chi_{A_2} = \alpha_1 \chi_{A_1 \setminus A_2} + (\alpha_1 + \alpha_2) \chi_{A_1 \cap A_2} + \alpha_2 \chi_{A_2 \setminus A_1},$$
$$\alpha_1 \mu(A_1) + \alpha_2 \mu(A_2) = \alpha_1 \mu(A_1 \setminus A_2) + (\alpha_1 + \alpha_2) \mu(A_1 \cap A_2) + \alpha_2 \mu(A_2 \setminus A_1),$$

Let the assertion hold for some $n \in N$. Then

$$\sum_{i=1}^{n+1} \alpha_i \chi_{A_i} = \sum_{i=1}^{n} \alpha_i \chi_{A_i} + \alpha_{n+1} \chi_{A_{n+1}} = \sum_{j=1}^{m} \beta_j \chi_{B_j} + \alpha_{n+1} \chi_{A_{n+1}}$$

Put $B = \bigcup_{i=1}^{m} B_j$. Then

$$\sum_{i=1}^{n+1} \alpha_i \chi_{A_i} = \sum_{j=1}^{m} \beta_j \chi_{B_j \setminus A_{n+1}} + \sum_{j=1}^{m} (\beta_j + \alpha_{n+1}) \chi_{A_{n+1} \cap B_j} + \alpha_{n+1} \chi_{A_{n+1} \setminus B}.$$

Similarly

$$\sum_{i=1}^{n+1} \alpha_i \mu(A_i) =$$

$$= \sum_{j=1}^{m} \beta_j \mu(B_j \setminus A_{n+1}) + \sum_{j=1}^{m} (\beta_j + \alpha_{n+1}) \mu(A_{n+1} \cap B_j) + \alpha_{n+1} \mu(A_{n+1} \setminus B).$$

□

Theorem 3 *Let* $f \in \mathcal{P}_0$, $f = \sum_{i=1}^{n} \alpha_i \chi_{A_i} = \sum_{j=1}^{m} \beta_j \chi_{B_j}$. *Then*

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{j=1}^{m} \beta_j \mu(B_j).$$

Proof By Theorem 2 there exist nonnegative η_k, δ_l and pairwise disjoint C_k, or D_l resp. such that

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{k=1}^{u} \eta_k \mu(C_k), \qquad \sum_{j=1}^{m} \beta_j \mu(B_j) = \sum_{l=1}^{v} \delta_l \mu(D_l).$$

Since

$$\eta_k \chi_{C_k \cap D_l} = \delta_l \chi_{C_k \cap D_l},$$

we have

$$\eta_k \mu(C_k \cap D_l) = \delta_l \mu(C_k \cap D_l).$$

Therefore

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{k=1}^{u} \eta_k \mu(C_k) = \sum_{k=1}^{u} \eta_k \sum_{l=1}^{v} \mu(C_k \cap D_l)$$

$$= \sum_{l=1}^{v} \delta_l \sum_{k=1}^{u} \mu(C_k \cap D_l) = \sum_{j=1}^{m} \beta_j \mu(B_j).$$

□

Theorem 4 *Let* $f \in \mathcal{P}^+$,

$$f = \sum_{i=1}^{\infty} \alpha_i \chi_{A_i} = \sum_{j=1}^{\infty} \beta_j \chi_{B_j}.$$

Then

$$\sum_{i=1}^{\infty} \alpha_i \mu(A_i) = \sum_{j=1}^{\infty} \beta_j \mu(B_j)).$$

Proof For $f \in \mathcal{P}_0, f = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ put

$$J_0(f) = \sum_{i=1}^{n} \alpha_i \mu(A_i).$$

It is possible by Theorem 3.

Assertion 1 If $f_n \in \mathcal{P}_0$, $f_n \searrow 0$, then $J_0(f_n) \searrow 0$.

Put $\alpha = \max f_1$, $Y = \{x \in [a,b]; f_1(x) > 0\}$. Let $\epsilon > 0$ be arbitrary. Put

$$A_n = \{x \in [a,b]; f_n(x) \geq \epsilon\}.$$

Then $A_n \searrow \emptyset$, hence $\mu(A_n) \searrow 0$.

Let $f_n = \sum_{i=1}^{k} \alpha_i \chi_{C_i}, C_i$ disjoint. Then

$$J_0(f_n) = \sum_{i=1}^{k} \alpha_i \mu(C_i) = \sum_{i=1}^{k} \alpha_i \mu(C_i \cap A_n) + \sum_{i=1}^{k} \alpha_i \mu(C_i \cap A_n{}').$$

If $x \in C_i \cap A_n'$, then $f_n(x) \le \alpha_i < \epsilon$. Therefore

$$J_0(f_n) \le \sum_{i=1}^{k} \alpha\mu(C_i \cap A_n) + \epsilon \sum_{i=1}^{k} \mu(C_i \cap A_n{}')$$

$$\le \alpha\mu(A_n \cap \bigcup_{i=1}^{k} C_i) + \epsilon\mu\left(\bigcup_{i=1}^{k} C_i\right) \le \alpha\mu(A_n) + \epsilon\mu([a,b]).$$

Therefore

$$0 \le \lim_{n\to\infty} J_0(f_n) \le \alpha \lim_{n\to\infty} \mu(A_n) + \epsilon\mu([a,b]) = \epsilon\mu([a,b]).$$

Since the previous inequality holds for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} J_0(f_n) = 0.$$

Assertion 2 If $f_n \nearrow f$, $f_n \in \mathcal{P}_0$, $f \in \mathcal{P}_0$, then $J_0(f_n) \nearrow J_0(f)$.

Put $g_n = f - f_n$. Then $g_n \searrow 0$, hence

$$J_0(f) - J_0(f_n) = J_0(g_n) \searrow 0.$$

Proof of Theorem Let $f \in \mathcal{P}^+$

$$f = \sum_i \alpha_i \chi_{A_i} = \lim_{n\to\infty} \sum_{i=1}^{n} \alpha_i \chi_{A_i} = \sum_i \beta_j \chi_{B_j} = \lim_{m\to\infty} \sum_{j=1}^{m} \beta_j \chi_{B_j}$$

$$= \bigvee_{n=1}^{\infty} f_n = \bigvee_{m=1}^{\infty} g_m,$$

where

$$f_n = \sum_{i=1}^{n} \alpha_i \chi_{A_i}, \qquad g_m = \sum_{j=1}^{m} \beta_j \chi_{B_j}.$$

Then

$$f_n \wedge g_m \nearrow f \wedge g_m = g_m,$$

hence

$$\sum_{i=1}^{\infty} \alpha_i \mu(A_i) = \lim_{n\to\infty} J_0(f_n) \ge \lim_{n\to\infty} J_0(f_n \wedge g_m) = J_0(g_m) = \sum_{j=1}^{m} \beta_j \mu(B_j)$$

for any $m \in N$, and therefore

$$\sum_{i=1}^{\infty} \alpha_i \mu(A_i) \ge \sum_{j=1}^{\infty} \beta_j \mu(B_j).$$

Analogously the oposite inequality can be proved. □

4 Conclusion

We have presented an elementary way how to construct the Lebesgue integral. We suggest to consider any non-empty set X instead of $[a, b]$, any ring $\mathcal{A}$ instead of the family of subintervals of $[a, b]$, and any non-negative σ-additive mapping $\mu\colon \mathcal{A} \to [0, \infty)$ instead of the lenght $\mu(A)$ of the interval A. It could have many aplications mainly in education and consequently in many areas, e.g. in statistics, but also in areas using functional spaces. We have used some ideas concerned in [1–5] and [7]. Recall that another applications of some Kluvánek ideas have been used in [6].

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