

Variance of Plug-in Estimators in Multivariate Regression Models*

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Abstract

Variance components in regression models are usually unknown. They must be estimated and it leads to a construction of plug-in estimators of the parameters of the mean value of the observation matrix. Uncertainty of the estimators of the variance components enlarge the variances of the plug-in estimators. The aim of the paper is to find this enlargement.

Key words: variance components, plug-in estimator, multivariate models

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1 Introduction

A construction of the best linear unbiased estimator (BLUE) of model parameters need a knowledge of the covariance matrix. If variance components are under discussion and they must be estimated, then a plug-in estimator of the model parameters must be used. This enlarges the variance of the BLUE. The aim of the paper is to find this enlargement.

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2 Notation and preliminaries

Four basic structures of multivariate models are under consideration (in more detail see in [2]).

$$\text{vec}(\underline{\mathbf{Y}}) \sim_{nm} \left[(\mathbf{I}_m \otimes \mathbf{X}_{n,k}) \text{vec}(\mathbf{B}_{k,m}), \sum_{i=1}^p \vartheta_i \mathbf{V}_{i,(m,m)} \otimes \mathbf{I}_n \right], \quad (2.1)$$

$$\text{vec}(\underline{\mathbf{Y}}) \sim_{nm} \left[(\mathbf{I}_m \otimes \mathbf{X}_{n,k}) \text{vec}(\mathbf{B}_{k,m}), \mathbf{I}_m \otimes \sum_{i=1}^p \vartheta_i \mathbf{V}_{i,(n,n)} \right], \quad (2.2)$$

$$\text{vec}(\underline{\mathbf{Y}}) \sim_{nr} \left[(\mathbf{Z}'_{r,m} \otimes \mathbf{X}_{n,k}) \text{vec}(\mathbf{B}_{k,m}), \sum_{i=1}^p \vartheta_i \mathbf{V}_{i,(r,r)} \otimes \mathbf{I}_n \right], \quad (2.3)$$

$$\text{vec}(\underline{\mathbf{Y}}) \sim_{nr} \left[(\mathbf{Z}'_{r,m} \otimes \mathbf{X}_{n,k}) \text{vec}(\mathbf{B}_{k,m}), \mathbf{I}_r \otimes \sum_{i=1}^p \vartheta_i \mathbf{V}_{i,(n,n)} \right]. \quad (2.4)$$

Here $\underline{\mathbf{Y}}$ is either $n \times m$ or $n \times r$ random matrix (observation matrix), \mathbf{I}_m is $m \times m$ identity matrix, \mathbf{X} is a given $n \times k$ matrix, \mathbf{Z} is a given $m \times r$ matrix, $\mathbf{V}_1, \dots, \mathbf{V}_p$ are given either $r \times r$ or $n \times n$ symmetric and positive semidefinite matrices and $\vartheta_1, \dots, \vartheta_p$ are unknown variance components.

Because of simplicity all models are considered to satisfy the following conditions.

$r(\mathbf{X}_{n,k}) = k < n$, $\mathbf{V}_1, \dots, \mathbf{V}_p$ are symmetric and positive semidefinite,

$$\vartheta_i > 0, \quad i = 1, \dots, p,$$

$\underline{\vartheta} \in \underline{\vartheta}$ (open set in the p -dimensional Euclidean space E^p),

$\underline{\vartheta} \in \underline{\vartheta} \Rightarrow \sum_{i=1}^p \vartheta_i \mathbf{V}_i$ is positive definite matrix, $r(\mathbf{Z}_{m,r}) = m < r$.

Let ϑ_0 be an approximate value of the vector ϑ and $\Sigma(\vartheta) = \sum_{i=1}^p \vartheta_i \mathbf{V}_i$, $\vartheta_0 = (\vartheta_{1,0}, \dots, \vartheta_{p,0})'$. Then ϑ_0 -LBLUEs (locally best linear unbiased estimator) of the matrix \mathbf{B} are

$$\widehat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}}, \quad \text{Var}_{\vartheta_0} [\text{vec}(\widehat{\mathbf{B}})] = \Sigma(\vartheta_0) \otimes (\mathbf{X}'\mathbf{X})^{-1}$$

in the model (1),

$$\widehat{\mathbf{B}} = [\mathbf{X}'\Sigma^{-1}(\vartheta_0)\mathbf{X}]^{-1}\mathbf{X}'\Sigma^{-1}(\vartheta_0)\underline{\mathbf{Y}}, \quad \text{Var}_{\vartheta_0} [\text{vec}(\widehat{\mathbf{B}})] = \mathbf{I}_m \otimes [(\mathbf{X}'\Sigma^{-1}(\vartheta_0)\mathbf{X})^{-1}]$$

in the model (2),

$$\widehat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}}\Sigma^{-1}(\vartheta_0)\mathbf{Z}'[\mathbf{Z}\Sigma^{-1}(\vartheta_0)\mathbf{Z}']^{-1},$$

$$\text{Var}_{\vartheta_0} [\text{vec}(\widehat{\mathbf{B}})] = [\mathbf{Z}\Sigma^{-1}(\vartheta_0)\mathbf{Z}']^{-1} \otimes (\mathbf{X}'\mathbf{X})^{-1}$$

in the model (3) and

$$\begin{aligned}\widehat{\mathbf{B}} &= [\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{Y}}\mathbf{Z}'(\mathbf{Z}\mathbf{Z}')^{-1}, \\ \text{Var}_{\boldsymbol{\vartheta}_0}[\text{vec}(\widehat{\mathbf{B}})] &= (\mathbf{Z}\mathbf{Z}')^{-1} \otimes [\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}]^{-1}\end{aligned}$$

in the model (4) (in more detail see in [2]).

In fact the model (2) is m -tuple of univariate models. Nevertheless it can be analysed as the multivariate one.

In the following text the symbol

$$\mathbf{M}_X = \mathbf{I} - \mathbf{X}\mathbf{X}^+$$

(here " $^+$ " means the Moore–Penrose generalized inverse of a matrix; in more detail see in [3]) and

$$[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+ = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) - \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)$$

will be used.

If the vector $\boldsymbol{\vartheta}$ of variance components is estimable, then the $\boldsymbol{\vartheta}_0$ -MINQUEs (minimum norm quadratic unbiased estimator; in more detail see in [4], [2], [1]) of $\boldsymbol{\vartheta}$ are

$$\begin{aligned}\widehat{\boldsymbol{\vartheta}} &= \frac{1}{n-k}\mathbf{S}_{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)}^{-1}\widehat{\boldsymbol{\gamma}}, \quad \widehat{\boldsymbol{\gamma}} = (\widehat{\gamma}_1, \dots, \widehat{\gamma}_p)', \\ \widehat{\gamma}_i &= \text{Tr}[\underline{\mathbf{Y}}'\mathbf{M}_X\underline{\mathbf{Y}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)], \quad i = 1, \dots, p, \\ \{\mathbf{S}_{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)}\}_{i,j} &= \text{Tr}[\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)], \quad i, j = 1, \dots, p, \\ \text{Var}_{\boldsymbol{\vartheta}_0}(\widehat{\boldsymbol{\vartheta}}) &= \frac{2}{n-k}\mathbf{S}_{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)}^{-1} \quad (\text{in the case of normality of } \underline{\mathbf{Y}})\end{aligned}$$

in the model (1),

$$\begin{aligned}\widehat{\boldsymbol{\vartheta}} &= \frac{1}{m}\mathbf{S}_{[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+}^{-1}\widehat{\boldsymbol{\gamma}}, \quad \widehat{\boldsymbol{\gamma}} = (\widehat{\gamma}_1, \dots, \widehat{\gamma}_p)' \\ \widehat{\gamma}_i &= \text{Tr}\left\{\underline{\mathbf{Y}}'[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+\mathbf{V}_i[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+\underline{\mathbf{Y}}\right\}, \quad i, j = 1, \dots, p, \\ \{\mathbf{S}_{[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+}\}_{i,j} &= \text{Tr}\left\{\mathbf{V}_i[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+\mathbf{V}_j[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+\right\}, \\ & \quad i, j = 1, \dots, p, \\ \text{Var}_{\boldsymbol{\vartheta}_0}(\widehat{\boldsymbol{\vartheta}}) &= \frac{2}{m}\mathbf{S}_{[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+}^{-1} \quad (\text{in the case of normality of } \underline{\mathbf{Y}})\end{aligned}$$

in the model (2),

$$\begin{aligned}\widehat{\boldsymbol{\vartheta}} &= [(n-k)\mathbf{S}_{\Sigma^{-1}(\boldsymbol{\vartheta}_0)} + k\mathbf{S}_{[M_{Z'}\Sigma(\boldsymbol{\vartheta}_0)M_{Z'}]^+}]^{-1} \widehat{\boldsymbol{\gamma}}, \quad \widehat{\boldsymbol{\gamma}} = (\widehat{\gamma}_1, \dots, \widehat{\gamma}_p)', \\ \widehat{\gamma}_i &= \text{Tr} \left(\underline{\mathbf{Y}}' \mathbf{M}_X \underline{\mathbf{Y}} \Sigma^{-1}(\boldsymbol{\vartheta}_0) \mathbf{V}_i \Sigma^{-1}(\boldsymbol{\vartheta}_0) + \underline{\mathbf{Y}}' \mathbf{P}_X \underline{\mathbf{Y}} \left\{ [M_{Z'}\Sigma(\boldsymbol{\vartheta}_0)M_{Z'}]^+ \right. \right. \\ &\quad \left. \left. \times \mathbf{V}_i [M_{Z'}\Sigma(\boldsymbol{\vartheta}_0)M_{Z'}]^+ \right\} \right), \quad i = 1, \dots, p, \\ \text{Var}_{\boldsymbol{\vartheta}_0}(\widehat{\boldsymbol{\vartheta}}) &= 2 [(n-k)\mathbf{S}_{\Sigma^{-1}(\boldsymbol{\vartheta}_0)} + k\mathbf{S}_{[M_{Z'}\Sigma(\boldsymbol{\vartheta}_0)M_{Z'}]^+}]^{-1} \quad (\text{in the case of} \\ &\quad \text{normality of } \underline{\mathbf{Y}})\end{aligned}$$

in the model (3) and in the model (4) it is valid that

$$\begin{aligned}\widehat{\boldsymbol{\vartheta}} &= [(r-m)\mathbf{S}_{\Sigma^{-1}(\boldsymbol{\vartheta}_0)} + m\mathbf{S}_{[M_X\Sigma(\boldsymbol{\vartheta}_0)M_X]^+}]^{-1} \widehat{\boldsymbol{\gamma}}, \quad \widehat{\boldsymbol{\gamma}} = (\widehat{\gamma}_1, \dots, \widehat{\gamma}_p)', \\ \widehat{\gamma}_i &= \text{Tr} [\underline{\mathbf{Y}} \mathbf{M}_{Z'} \underline{\mathbf{Y}}' \Sigma^{-1}(\boldsymbol{\vartheta}_0) \mathbf{V}_i \Sigma^{-1}(\boldsymbol{\vartheta}_0)] + \text{Tr} \left\{ \underline{\mathbf{Y}} \mathbf{P}_{Z'} \underline{\mathbf{Y}}' [M_X\Sigma(\boldsymbol{\vartheta}_0)M_X]^+ \right. \\ &\quad \left. \times \mathbf{V}_i [M_X\Sigma(\boldsymbol{\vartheta}_0)M_X]^+ \right\}, \quad i = 1, \dots, p, \\ \text{Var}_{\boldsymbol{\vartheta}_0}(\widehat{\boldsymbol{\vartheta}}) &= 2 [(r-m)\mathbf{S}_{\Sigma^{-1}(\boldsymbol{\vartheta}_0)} + m\mathbf{S}_{[M_X\Sigma(\boldsymbol{\vartheta}_0)M_X]^+}]^{-1} \quad (\text{in the case of} \\ &\quad \text{normality of } \underline{\mathbf{Y}}).\end{aligned}$$

Here $\mathbf{P}_X = \mathbf{X}\mathbf{X}^+$.

3 Variance of plug-in estimators

In this section the normality of $\underline{\mathbf{Y}}$ is assumed.

3.1 Model (1)

In the model (1) it is valid that the BLUE of the unbiasedly estimable function $\text{Tr}(\mathbf{H}\mathbf{B})$ for any given $m \times k$ matrix \mathbf{H} is

$$\text{Tr}(\mathbf{H}\widehat{\mathbf{B}}) = \text{Tr} [\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\underline{\mathbf{Y}}], \quad \text{Var}_{\boldsymbol{\vartheta}} [\text{Tr}(\mathbf{H}\widehat{\mathbf{B}})] = \text{Tr} [\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\Sigma(\boldsymbol{\vartheta})],$$

thus the plug-in estimator need not be used and only $\text{Var}_{\boldsymbol{\vartheta}} [\text{Tr}(\mathbf{H}\widehat{\mathbf{B}})]$ must be estimated. The estimator of the dispersion of $\text{Tr}(\mathbf{H}\widehat{\mathbf{B}})$ is

$$\text{Var}_{\boldsymbol{\vartheta}} [\widehat{\text{Tr}(\mathbf{H}\widehat{\mathbf{B}})}] = \text{Tr} \left[\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}' \sum_{i=1}^p \widehat{\boldsymbol{\vartheta}}_i \mathbf{V}_i \right] = \mathbf{g}'\widehat{\boldsymbol{\vartheta}},$$

where $\mathbf{g} = (g_1, \dots, g_p)'$, $g_i = \text{Tr} [\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\mathbf{V}_i]$, $i = 1, \dots, p$.

Thus

$$\text{Var}_{\boldsymbol{\vartheta}} [\widehat{\text{Tr}(\mathbf{H}\widehat{\mathbf{B}})}] = \mathbf{g}'\widehat{\boldsymbol{\vartheta}}, \quad \text{Var}_{\boldsymbol{\vartheta}_0} \left\{ \widehat{\text{Var}_{\boldsymbol{\vartheta}} [\text{Tr}(\mathbf{H}\widehat{\mathbf{B}})]} \right\} = \frac{2}{n-k} \mathbf{g}' \mathbf{S}_{\Sigma^{-1}(\boldsymbol{\vartheta}_0)}^{-1} \mathbf{g}.$$

3.2 Model (2)

Lemma 1 *In the model (2) it is valid that*

$$\frac{\partial \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)]}{\partial \vartheta_i} = -\text{Tr} [\mathbf{H}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{y}}],$$

where $\mathbf{C}(\boldsymbol{\vartheta}_0) = \mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}$, $\underline{\mathbf{y}} = \underline{\mathbf{Y}} - \mathbf{X}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)$.

Thus

$$\begin{aligned} E_{\boldsymbol{\vartheta}_0} \left(\frac{\partial \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)]}{\partial \vartheta_i} \right) &= 0, \quad E_{\boldsymbol{\vartheta}_0} \left(\frac{\partial \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)]}{\partial \vartheta_i} \frac{\partial \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)]}{\partial \vartheta_j} \right) \\ &= \text{Tr} \left\{ \mathbf{H}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i [\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+ \mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{H}' \right\}, \\ &\quad i, j = 1, \dots, p. \end{aligned}$$

Proof

$$\begin{aligned} \frac{\partial \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)]}{\partial \vartheta_i} &= \frac{\partial \text{Tr} \left\{ \mathbf{H}[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{y}} \right\}}{\partial \vartheta_i} \\ &= \text{Tr} \left\{ \mathbf{H}[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}]^{-1} \right. \\ &\quad \left. \times \mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{y}} \right\} - \text{Tr} \left\{ \mathbf{H}[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{y}} \right\} \\ &= -\text{Tr} \left\{ \mathbf{H}[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}]^{-1}\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\underline{\mathbf{y}} \right\}. \end{aligned}$$

Since

$$\text{Var}_{\boldsymbol{\vartheta}_0} [\text{vec}(\underline{\mathbf{y}})] = \mathbf{I} \otimes [\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) - \mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}']$$

and

$$[\mathbf{I} \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)] \text{Var}_{\boldsymbol{\vartheta}_0} [\text{vec}(\underline{\mathbf{y}})] [\mathbf{I} \otimes \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)] = \mathbf{I} \otimes [\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+,$$

it is valid that

$$\begin{aligned} E_{\boldsymbol{\vartheta}_0} \left(\frac{\partial \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)]}{\partial \vartheta_i} \frac{\partial \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)]}{\partial \vartheta_j} \right) \\ = \text{Tr} \left\{ \mathbf{H}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i [\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+ \mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{H}' \right\}. \end{aligned}$$

□

Let $\mathbf{Z}_H(\boldsymbol{\vartheta}_0)$ be $p \times p$ matrix with entries given as

$$\begin{aligned} & \{\mathbf{Z}_H(\boldsymbol{\vartheta}_0)\}_{i,j} = \\ & = \text{Tr} \left\{ \mathbf{H}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+\mathbf{V}_j\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{H}' \right\}, \\ & \quad i, j = 1, \dots, p. \end{aligned}$$

Corollary 1 Let $\widehat{\delta\boldsymbol{\vartheta}} = \widehat{\boldsymbol{\vartheta}} - \boldsymbol{\vartheta}_0$. Since

$$\begin{aligned} & \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] \approx \\ & \approx \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)] - \sum_{i=1}^p \text{Tr} [\mathbf{H}\mathbf{C}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{V}_i\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{y}] \widehat{\delta\vartheta}_i, \end{aligned}$$

and \mathbf{y} and $\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)$ are non-correlated, it is valid that

$$\begin{aligned} & E_{\boldsymbol{\vartheta}_0} \left(\text{Tr} \left\{ [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] \middle| \widehat{\delta\boldsymbol{\vartheta}} \right\} \right) \approx \text{Tr}(\mathbf{H}\mathbf{B}), \\ & \text{Var}_{\boldsymbol{\vartheta}_0} \left(\text{Tr} \left\{ [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] \middle| \widehat{\delta\boldsymbol{\vartheta}} \right\} \right) \approx \text{Tr} [\mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}'\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)] + \widehat{\delta\boldsymbol{\vartheta}}' \mathbf{Z}_H(\boldsymbol{\vartheta}_0) \widehat{\delta\boldsymbol{\vartheta}}. \end{aligned}$$

Lemma 2

$$\begin{aligned} \text{Var}_{\boldsymbol{\vartheta}_0} \left\{ \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] \right\} &= \text{Var}_{\boldsymbol{\vartheta}_0} \left[E_{\boldsymbol{\vartheta}_0} \left(\text{Tr} \left\{ [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] \middle| \widehat{\delta\boldsymbol{\vartheta}} \right\} \right) \right] \\ &+ E_{\boldsymbol{\vartheta}_0} \left[\text{Var}_{\boldsymbol{\vartheta}_0} \left(\text{Tr} \left\{ [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] \middle| \widehat{\delta\boldsymbol{\vartheta}} \right\} \right) \right]. \end{aligned}$$

Proof It is well known. □

With respect to Corollary 1 and Lemma 2 the following statement can be obtained.

Theorem 1

$$\begin{aligned} & \text{Var}_{\boldsymbol{\vartheta}_0} \left\{ \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] \right\} \approx \\ & \approx \text{Tr} \left\{ \mathbf{H}[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{X}]^{-1}\mathbf{H}' \right\} + \delta\boldsymbol{\vartheta}' \mathbf{Z}_H(\boldsymbol{\vartheta}_0) \delta\boldsymbol{\vartheta} + \frac{2}{m} \text{Tr} \left(\mathbf{Z}_H(\boldsymbol{\vartheta}_0) \mathbf{S}_{[\mathbf{M}_X\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{M}_X]^+}^{-1} \right). \end{aligned}$$

Remark 1 If the vector $\boldsymbol{\vartheta}$ is estimated by an iteration, i.e.

$$\begin{aligned} \widehat{\boldsymbol{\vartheta}}^{(i+1)} &= \frac{1}{m} \mathbf{S}_{[\mathbf{M}_X\boldsymbol{\Sigma}(\widehat{\boldsymbol{\vartheta}}^{(i)})\mathbf{M}_X]^+}^{-1} \widehat{\boldsymbol{\gamma}}^{(i)}, \\ \widehat{\boldsymbol{\gamma}}_j^{(i)} &= \text{Tr} \left\{ \mathbf{y}' [\mathbf{M}_X\boldsymbol{\Sigma}(\widehat{\boldsymbol{\vartheta}}^{(i)})\mathbf{M}_X]^+ \mathbf{V}_j [\mathbf{M}_X\boldsymbol{\Sigma}(\widehat{\boldsymbol{\vartheta}}^{(i)})\mathbf{M}_X]^+ \mathbf{y} \right\}, \quad j = 1, \dots, p, \end{aligned}$$

then the estimator

$$\text{Var}_{\boldsymbol{\vartheta}} \left\{ \text{Tr} [\widehat{\mathbf{H}\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})] \right\} \approx \text{Tr} \left\{ \mathbf{H}[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\widehat{\boldsymbol{\vartheta}})\mathbf{X}]^{-1}\mathbf{H}' \right\} + \frac{2}{m} \left(\mathbf{Z}_H(\widehat{\boldsymbol{\vartheta}}) \mathbf{S}_{[\mathbf{M}_X\boldsymbol{\Sigma}(\widehat{\boldsymbol{\vartheta}})\mathbf{M}_X]^+}^{-1} \right)$$

can be used.

3.3 Model (3)

Lemma 3 *In the model (3) it is valid that*

$$\frac{\partial \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)]}{\partial \vartheta_i} = - \text{Tr} \left\{ (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \underline{\mathbf{v}} \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \mathbf{V}_i \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \mathbf{Z}' [\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}']^{-1} \right\},$$

$$i = 1, \dots, p,$$

where $\underline{\mathbf{v}} = \underline{\mathbf{Y}} - \mathbf{X}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)\mathbf{Z}$.

Thus

$$E_{\boldsymbol{\vartheta}_0} \left(\frac{\partial \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)]}{\partial \vartheta_i} \right) = 0, \quad E_{\boldsymbol{\vartheta}_0} \left(\frac{\partial \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)]}{\partial \vartheta_i} \frac{\partial \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)]}{\partial \vartheta_j} \right)$$

$$= \text{Tr} \left\{ \mathbf{V}_i \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \mathbf{Z}' [\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}']^{-1} \mathbf{H}(\mathbf{X}'\mathbf{X})^{-1} \mathbf{H}' [\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}']^{-1} \mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \right.$$

$$\left. \times \mathbf{V}_j [\mathbf{M}_{\mathbf{Z}'} \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) \mathbf{M}_{\mathbf{Z}'}]^{+} \right\}, \quad i, j = 1, \dots, p.$$

Proof It is an analogy of the proof of Lemma 1. The equality

$$\text{Var}_{\boldsymbol{\vartheta}_0} [\text{vec}(\underline{\mathbf{v}})] = \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) \otimes \mathbf{M}_X + \left\{ \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) - \mathbf{Z}' [\mathbf{Z}\boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0)\mathbf{Z}']^{-1} \mathbf{Z} \right\} \otimes \mathbf{P}_X$$

was utilized. Here $\mathbf{P}_X = \mathbf{X}\mathbf{X}^+$. \square

Let $\mathbf{T}_H(\boldsymbol{\vartheta}_0)$ be the $p \times p$ matrix with the entries

$$\{\mathbf{T}_H(\boldsymbol{\vartheta}_0)\}_{i,j} =$$

$$= \text{Tr} \left\{ \mathbf{V}_i \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \mathbf{Z}' [\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}']^{-1} \mathbf{H}(\mathbf{X}'\mathbf{X})^{-1} \mathbf{H}' [\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}']^{-1} \right.$$

$$\left. \times \mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \mathbf{V}_j [\mathbf{M}_{\mathbf{Z}'} \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) \mathbf{M}_{\mathbf{Z}'}]^{+} \right\}, \quad i, j = 1, \dots, p.$$

Corollary 2 *Since*

$$\text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\boldsymbol{\delta}}\boldsymbol{\vartheta})] \approx \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)] - \sum_{i=1}^p \text{Tr} \left\{ (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}' \underline{\mathbf{v}} \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \mathbf{V}_i \right.$$

$$\left. \times \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \mathbf{Z}' [\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}']^{-1} \right\} \widehat{\boldsymbol{\delta}}\boldsymbol{\vartheta}_i$$

and $\underline{\mathbf{v}}$ and $\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)$ are non-correlated, it is valid that

$$E_{\boldsymbol{\vartheta}_0} \left(\text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\boldsymbol{\delta}}\boldsymbol{\vartheta})] \middle| \widehat{\boldsymbol{\delta}}\boldsymbol{\vartheta} \right) \approx \text{Tr}(\mathbf{H}\mathbf{B}),$$

$$\text{Var}_{\boldsymbol{\vartheta}_0} \left(\text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\boldsymbol{\delta}}\boldsymbol{\vartheta})] \middle| \widehat{\boldsymbol{\delta}}\boldsymbol{\vartheta} \right) \approx$$

$$\approx \text{Tr} \left\{ \mathbf{H}(\mathbf{X}'\mathbf{X})^{-1} \mathbf{H}' [\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)\mathbf{Z}']^{-1} \right\} + \widehat{\boldsymbol{\delta}}\boldsymbol{\vartheta}' \mathbf{T}_H(\boldsymbol{\vartheta}_0) \widehat{\boldsymbol{\delta}}\boldsymbol{\vartheta}.$$

Since

$$\text{Var}_{\boldsymbol{\vartheta}_0} \left[E_{\boldsymbol{\vartheta}_0} \left(\text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\boldsymbol{\delta}}\boldsymbol{\vartheta})] \middle| \widehat{\boldsymbol{\delta}}\boldsymbol{\vartheta} \right) \right] \approx 0,$$

and

$$E_{\vartheta_0} \left[\text{Var}_{\vartheta_0} \left(\text{Tr} \left[\mathbf{H}\widehat{\mathbf{B}}(\vartheta_0 + \widehat{\delta\vartheta}) \right] \middle| \widehat{\delta\vartheta} \right) \right] \approx \text{Tr} \left\{ \mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}' \left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\vartheta_0)\mathbf{Z}' \right]^{-1} \right\} \\ + \delta\vartheta' \mathbf{T}_H(\vartheta_0)\delta\vartheta + 2 \text{Tr} \left\{ \mathbf{T}_H(\vartheta_0) \left[(n-k)\mathbf{S}_{\Sigma^{-1}(\vartheta_0)} + k\mathbf{S}_{[M_{Z'}\Sigma(\vartheta_0)M_{Z'}]^+} \right]^{-1} \right\},$$

the following statement is valid.

Theorem 2 In the model (3)

$$\text{Var}_{\vartheta_0} \left\{ \text{Tr} \left[\mathbf{H}\widehat{\mathbf{B}}(\vartheta_0 + \widehat{\delta\vartheta}) \right] \right\} \approx \text{Tr} \left\{ \mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}' \left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\vartheta_0)\mathbf{Z}' \right]^{-1} \right\} \\ + \delta\vartheta' \mathbf{T}_H(\vartheta_0)\delta\vartheta + 2 \text{Tr} \left\{ \mathbf{T}_H(\vartheta_0) \left[(n-k)\mathbf{S}_{\Sigma^{-1}(\vartheta_0)} + k\mathbf{S}_{[M_{Z'}\Sigma(\vartheta_0)M_{Z'}]^+} \right]^{-1} \right\}.$$

Remark 2 If the vector ϑ is estimated by iteration, then

$$\text{Var}_{\vartheta} \left\{ \widehat{\text{Tr}} \left[\widehat{\mathbf{B}}(\widehat{\vartheta}) \right] \right\} \approx \text{Tr} \left\{ \mathbf{H}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{H}' \left[\mathbf{Z}\boldsymbol{\Sigma}^{-1}(\widehat{\vartheta})\mathbf{Z}' \right]^{-1} \right\} \\ + 2 \text{Tr} \left\{ \mathbf{T}_H(\widehat{\vartheta}) \left[(n-k)\mathbf{S}_{\Sigma^{-1}(\widehat{\vartheta})} + k\mathbf{S}_{[M_{Z'}\Sigma(\widehat{\vartheta})M_{Z'}]^+} \right]^{-1} \right\}.$$

3.4 Model (4)

Lemma 4 In the model (4) it is valid that

$$\frac{\partial \text{Tr} \left[\mathbf{H}\widehat{\mathbf{B}}(\vartheta_0) \right]}{\partial \vartheta_i} = - \text{Tr} \left\{ \mathbf{V}_i \boldsymbol{\Sigma}^{-1}(\vartheta_0) \underline{\mathbf{v}} \mathbf{Z}' (\mathbf{Z}\mathbf{Z}')^{-1} \mathbf{H} \left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\vartheta_0)\mathbf{X} \right]^{-1} \mathbf{X}'\boldsymbol{\Sigma}^{-1}(\vartheta_0) \right\}, \\ i = 1, \dots, p,$$

where $\underline{\mathbf{v}} = \underline{\mathbf{Y}} - \mathbf{X}\widehat{\mathbf{B}}(\vartheta_0)\mathbf{Z}$. Thus

$$E_{\vartheta_0} \left(\frac{\partial \text{Tr} \left[\mathbf{H}\widehat{\mathbf{B}}(\vartheta_0) \right]}{\partial \vartheta_i} \right) = 0, \quad i = 1, \dots, p, \\ \text{cov}_{\vartheta_0} \left(\frac{\partial \text{Tr} \left[\mathbf{H}\widehat{\mathbf{B}}(\vartheta_0) \right]}{\partial \vartheta_i}, \frac{\partial \text{Tr} \left[\mathbf{H}\widehat{\mathbf{B}}(\vartheta_0) \right]}{\partial \vartheta_j} \right) \\ = \text{Tr} \left\{ \mathbf{V}_i \left[\mathbf{M}_X \boldsymbol{\Sigma}(\vartheta_0) \mathbf{M}_X \right]^+ \mathbf{V}_j \boldsymbol{\Sigma}^{-1}(\vartheta_0) \mathbf{X} \left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\vartheta_0)\mathbf{X} \right]^{-1} \mathbf{H}' (\mathbf{Z}\mathbf{Z}')^{-1} \mathbf{H} \right. \\ \left. \times \left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\vartheta_0)\mathbf{X} \right]^{-1} \mathbf{X}'\boldsymbol{\Sigma}^{-1}(\vartheta_0) \right\} = \{ \mathbf{U}_H(\vartheta_0) \}_{i,j}, \quad i, j = 1, \dots, p.$$

Proof The equality

$$\text{Var}_{\vartheta_0} \left[\text{vec}(\underline{\mathbf{v}}) \right] = \mathbf{M}_{Z'} \otimes \boldsymbol{\Sigma}(\vartheta_0) + \mathbf{P}_{Z'} \otimes \left\{ \boldsymbol{\Sigma}(\vartheta_0) - \mathbf{X} \left[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\vartheta_0)\mathbf{X} \right]^{-1} \mathbf{X}' \right\}$$

must be used. Further procedures are analogous as in the proof of Lemma 1. \square

Since

$$\begin{aligned} \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] &\approx \text{Tr} \left\{ \mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0) \right. \\ &\left. - \sum_{i=1}^p \text{Tr} \left\{ \mathbf{V}_i \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \underline{\mathbf{y}} \mathbf{Z}' (\mathbf{Z}\mathbf{Z}')^{-1} \mathbf{H} [\mathbf{X}' \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \mathbf{X}]^{-1} \mathbf{X}' \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \right\} \widehat{\delta\boldsymbol{\vartheta}}_i \right\}, \end{aligned}$$

and $\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0)$ and $\underline{\mathbf{y}}$ are non-correlated, it is valid that

$$\begin{aligned} E_{\boldsymbol{\vartheta}_0} \left\{ \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] \middle| \widehat{\delta\boldsymbol{\vartheta}} \right\} &\approx \text{Tr}(\mathbf{H}\mathbf{B}), \\ \text{Var}_{\boldsymbol{\vartheta}_0} \left\{ \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] \middle| \widehat{\delta\boldsymbol{\vartheta}} \right\} &\approx \\ &\approx \text{Tr} \left\{ \mathbf{H} [\mathbf{X}' \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \mathbf{X}]^{-1} \mathbf{H}' (\mathbf{Z}\mathbf{Z}')^{-1} \right\} + \widehat{\delta\boldsymbol{\vartheta}}' \mathbf{U}_H(\boldsymbol{\vartheta}_0) \widehat{\delta\boldsymbol{\vartheta}}. \end{aligned}$$

Thus the following statement is valid.

Theorem 3

$$\begin{aligned} \text{Var}_{\boldsymbol{\vartheta}_0} \left(E_{\boldsymbol{\vartheta}_0} \left\{ \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] \middle| \widehat{\delta\boldsymbol{\vartheta}} \right\} \right) &\approx 0, \\ E_{\boldsymbol{\vartheta}_0} \left(\text{Var}_{\boldsymbol{\vartheta}_0} \left\{ \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\boldsymbol{\vartheta}_0 + \widehat{\delta\boldsymbol{\vartheta}})] \middle| \widehat{\delta\boldsymbol{\vartheta}} \right\} \right) &\approx \text{Tr} \left\{ \mathbf{H} [\mathbf{X}' \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \mathbf{X}]^{-1} \mathbf{H}' (\mathbf{Z}\mathbf{Z}')^{-1} \right\} \\ &+ \delta\boldsymbol{\vartheta}' \mathbf{U}_H(\boldsymbol{\vartheta}_0) \delta\boldsymbol{\vartheta} + 2 \text{Tr} \left\{ \mathbf{U}_H(\boldsymbol{\vartheta}_0) \left[(r-m) \mathbf{S}_{\boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0)}^{-1} + m \mathbf{S}_{[M_X \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) M_X]^+}^{-1} \right]^{-1} \right\}. \end{aligned}$$

Remark 3 If the estimator of $\boldsymbol{\vartheta}$ is determined by the iteration, the estimator of $\text{Var}_{\boldsymbol{\vartheta}} \left\{ \text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})] \right\}$ can be given as

$$\begin{aligned} \text{Var}_{\boldsymbol{\vartheta}} \left\{ \widehat{\text{Tr} [\mathbf{H}\widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})]} \right\} &\approx \text{Tr} \left\{ \mathbf{H} [\mathbf{X}' \boldsymbol{\Sigma}^{-1}(\widehat{\boldsymbol{\vartheta}}) \mathbf{X}]^{-1} \mathbf{H}' (\mathbf{Z}\mathbf{Z}')^{-1} \right\} \\ &+ 2 \text{Tr} \left\{ \mathbf{U}_H(\widehat{\boldsymbol{\vartheta}}) \left[(r-m) \mathbf{S}_{\boldsymbol{\Sigma}^{-1}(\widehat{\boldsymbol{\vartheta}})}^{-1} + m \mathbf{S}_{[M_X \boldsymbol{\Sigma}(\widehat{\boldsymbol{\vartheta}}) M_X]^+}^{-1} \right]^{-1} \right\}. \end{aligned}$$

4 Numerical example

Let

$$\text{vec}(\underline{\mathbf{Y}}_{6,3}) \sim [(\mathbf{I} \otimes \mathbf{X}) \text{vec}(\mathbf{B}), \mathbf{I} \otimes \boldsymbol{\Sigma}]$$

where

$$\mathbf{X}_{6,2} = \begin{pmatrix} 1, & -3 \\ 1, & -2 \\ 1, & -1 \\ 1, & 1 \\ 1, & 2 \\ 1, & 3 \end{pmatrix}, \quad \mathbf{B}_{2,3} = \begin{pmatrix} 0, & 0.5, & 1 \\ 0.5, & 1, & 1.5 \end{pmatrix}$$

$$\boldsymbol{\Sigma} = \vartheta_1 \begin{pmatrix} \mathbf{I}_{3,3}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{0}_{3,3} \end{pmatrix} + \vartheta_2 \begin{pmatrix} \mathbf{0}_{3,3}, & \mathbf{0} \\ \mathbf{0}, & \mathbf{I}_{3,3} \end{pmatrix},$$

$$\mathbf{H} = \begin{pmatrix} 1, & 1 \\ 1, & 1 \\ 1, & 1 \end{pmatrix}, \quad \vartheta_1 = (0.1)^2, \quad \vartheta_2 = (0.3)^2, \quad \boldsymbol{\vartheta}_0 = \begin{pmatrix} \vartheta_1 \\ \vartheta_2 \end{pmatrix}.$$

Then

$$\sqrt{\text{Tr} \left\{ \mathbf{H} [\mathbf{X}' \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}) \mathbf{X}]^{-1} \mathbf{H}' \right\}} = 0.1946272,$$

$$\sqrt{\frac{1}{9999} \sum_{i=1}^{10000} \left\{ \text{Tr} [\mathbf{H} \widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})] - \frac{1}{10000} \text{Tr} [\mathbf{H} \mathbf{B}(\boldsymbol{\vartheta})] \right\}^2} = 0.2145846,$$

$$\sqrt{\text{Tr} \left\{ \mathbf{H} [\mathbf{X}' \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \mathbf{X}]^{-1} \mathbf{H}' \right\} + \frac{2}{3} \text{Tr} [\mathbf{Z}_H(\boldsymbol{\vartheta}_0) \mathbf{S}_{[M_X \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) M_X]^+}^{-1}]} = 0.2037746.$$

Since

$$\frac{0.2037746}{0.2145846} = 0.9496236,$$

the approximate standard deviation of the plug-in estimator attains in the mean 95 % of the actual value.

Let in the same case $\vartheta_1 = 1^2$ and $\vartheta_2 = 3^2$. Since

$$\sqrt{\text{Tr} \left\{ \mathbf{H} [\mathbf{X}' \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}) \mathbf{X}]^{-1} \mathbf{H}' \right\}} = 1.946272,$$

$$\sqrt{\frac{1}{9999} \sum_{i=1}^{10000} \left\{ \text{Tr} [\mathbf{H} \widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})] - \frac{1}{10000} \text{Tr} [\mathbf{H} \mathbf{B}(\boldsymbol{\vartheta})] \right\}^2} = 2.165647,$$

$$\sqrt{\text{Tr} \left\{ \mathbf{H} [\mathbf{X}' \boldsymbol{\Sigma}^{-1}(\boldsymbol{\vartheta}_0) \mathbf{X}]^{-1} \mathbf{H}' \right\} + \frac{2}{3} \text{Tr} [\mathbf{Z}_H(\boldsymbol{\vartheta}_0) \mathbf{S}_{[M_X \boldsymbol{\Sigma}(\boldsymbol{\vartheta}_0) M_X]^+}^{-1}]} = 2.037746.$$

Also in this case

$$\frac{2.037746}{2.165647} = 0.940941,$$

the approximate standard deviation of the plug-in estimator attains in mean 94 % of the actual value.

The probability density functions of the random variable $\text{Tr} [\mathbf{H} \widehat{\mathbf{B}}(\widehat{\boldsymbol{\vartheta}})]$ see for $\vartheta_1 = 0.1^2$, $\vartheta_2 = 0.3^2$ on Fig. 1, for the $\vartheta_1 = 1^2$, $\vartheta_2 = 3^2$ on Fig. 2.

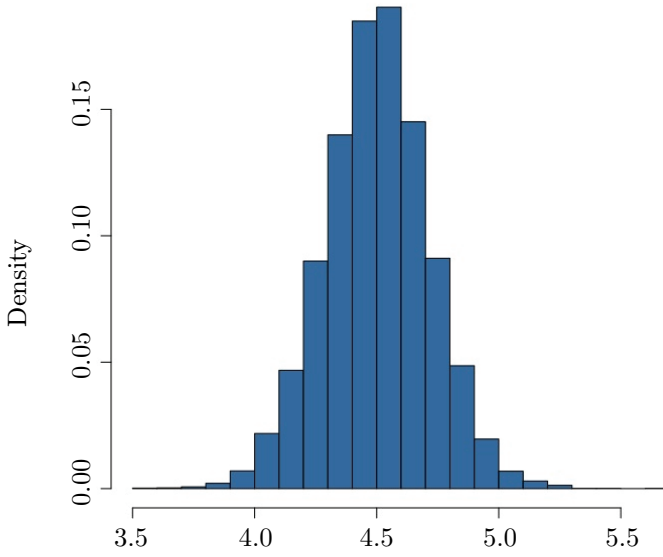


Fig. 1: $E\{\text{Tr}[\mathbf{H}\hat{\mathbf{B}}(\hat{\vartheta})]\} = 4.5$

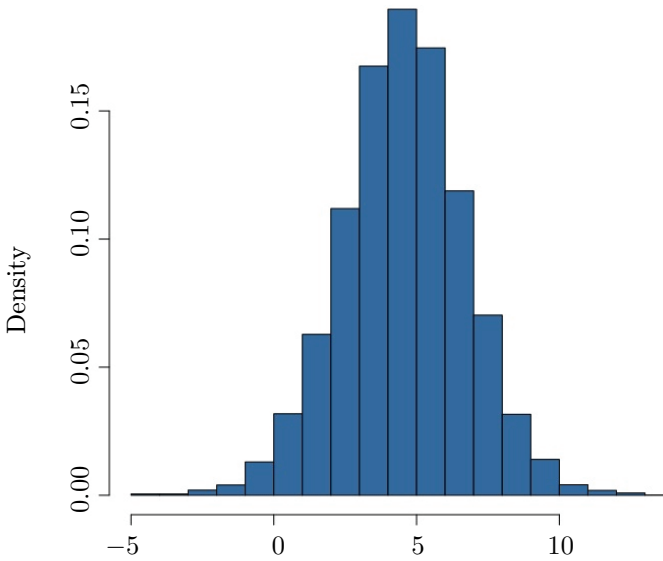
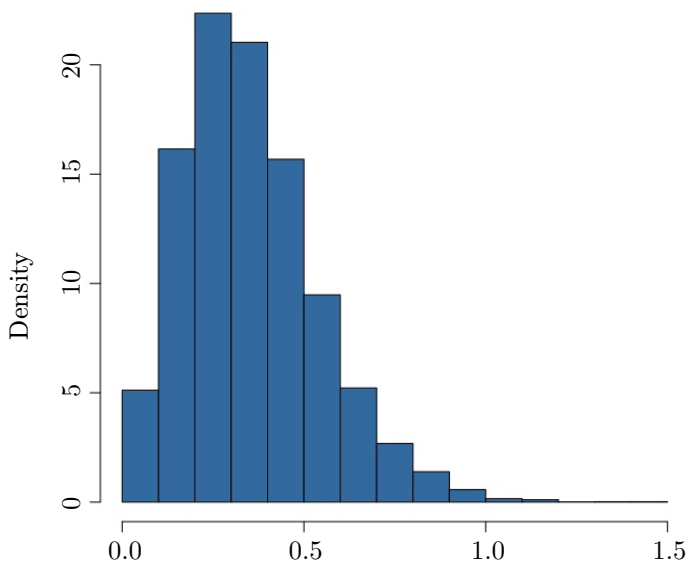
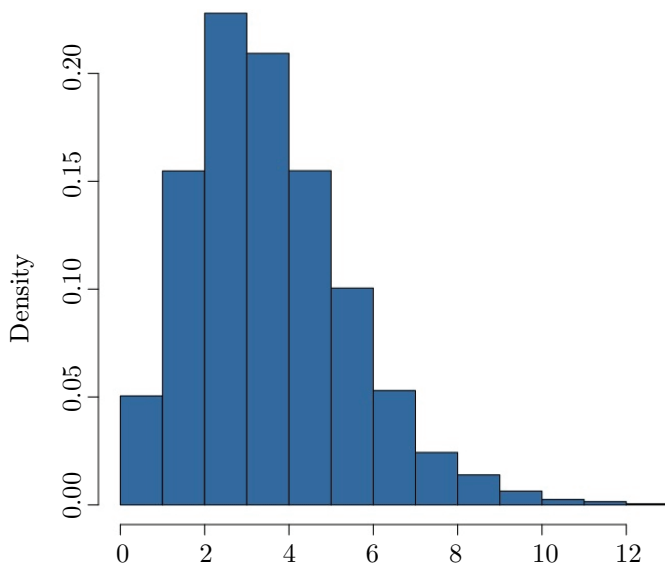


Fig. 2: $E\{\text{Tr}[\mathbf{H}\hat{\mathbf{B}}(\hat{\vartheta})]\} = 4.5$

The probability density of the random variable

$$\tau(\hat{\vartheta}) = \text{Tr}\left\{\mathbf{H}[\mathbf{X}'\boldsymbol{\Sigma}^{-1}(\hat{\vartheta})\mathbf{X}]^{-1}\mathbf{H}'\right\} + \frac{2}{3}\text{Tr}\left[\mathbf{Z}_H(\hat{\vartheta})\mathbf{S}_{[M_X\Sigma(\hat{\vartheta})M_X]^+}^{-1}\right]$$

see for $\vartheta_1 = 0.1^2$, $\vartheta_2 = 0.3^2$ on Fig. 3, for the $\vartheta_1 = 1^2$, $\vartheta_2 = 3^2$ on Fig. 4.

Fig. 3: $E\{\tau(\hat{\vartheta})\} = 0.03787975$ Fig. 4: $E\{\tau(\hat{\vartheta})\} = 3.787975$

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