Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **52**, 1 (2013) 35–48

Global Parametrization of Scalar Holomorphic Coadjoint Orbits of a Quasi-Hermitian Lie Group

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(Received January 8, 2013)

Abstract

Let G be a quasi-Hermitian Lie group with Lie algebra $\mathfrak{g}$ and K be a compactly embedded subgroup of G. Let ξ_0 be a regular element of $\mathfrak{g}^*$ which is fixed by K. We give an explicit G-equivariant diffeomorphism from a complex domain onto the coadjoint orbit $\mathcal{O}(\xi_0)$ of ξ_0. This generalizes a result of [B. Cahen, *Berezin quantization and holomorphic representations*, Rend. Sem. Mat. Univ. Padova, to appear] concerning the case where $\mathcal{O}(\xi_0)$ is associated with a unitary irreducible representation of G which is holomorphically induced from a unitary character of K. In particular, we consider the case $G = SU(p,q)$ and the case where G is the Jacobi group.

Key words: quasi-Hermitian Lie group, coadjoint orbit, stereographic projection, Berezin quantization, unitary holomorphic representation, unitary group, Jacobi group

2000 Mathematics Subject Classification: 32M10, 22E15, 22E10, 22E45, 32M05, 32M15, 81S10

1 Introduction

Let us first consider the following situation. Let $G = SU(1,1)$ and K be the torus of G consisting of matrices of the form $\mathrm{Diag}(e^{i\theta}, e^{-i\theta})$ where $\theta \in \mathbb{R}$. The Lie algebra $\mathfrak{g}$ of G has basis

$$u_1 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad u_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}.$$

Let (u_1^*, u_2^*, u_3^*) be the dual basis of $\mathfrak{g}^*$. For $r > 0$, let $\xi_0 = r u_3^*$. Then the orbit $\mathcal{O}(\xi_0)$ of ξ_0 for the coadjoint action of G is the upper sheet $x_3 > 0$ of the two-sheet hyperboloid $\{\xi = x_1 u_1^* + x_2 u_2^* + x_3 u_3^* \colon -x_1^2 - x_2^2 + x_3^2 = r^2\}$. Since the stabilizer of ξ_0 for the coadjoint action of G is K, we have $\mathcal{O}(\xi_0) \simeq G/K$. On the other hand, G/K is diffeomorphic to the unit disc $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$. Then, by composition, we get a global chart $\psi \colon \mathbb{D} \to \mathcal{O}(\xi_0)$. Explicitly, we have

$$\psi(z) := r \left(\frac{z + \bar{z}}{1 - z\bar{z}} u_1^* + \frac{z - \bar{z}}{i(1 - z\bar{z})} u_2^* + \frac{1 + z\bar{z}}{1 - z\bar{z}} u_3^* \right).$$

Note that ψ intertwines the natural action on G on $\mathbb{D}$ (by fractional linear transforms) and the coadjoint action of G on $\mathcal{O}(\xi_0)$. Note also that ψ^{-1} is an analog of the stereographic projection from the two-sphere $\mathbb{S}^2$ onto $\mathbb{C} \cup (\infty)$. Moreover, if we take $r = n/2$ where n is an integer ≥ 2 then $\mathcal{O}(\xi_0)$ is associated with a holomorphic discrete series representation π_n of G by the Kirillov–Kostant method of orbits [26], [27]. In that case, the differential $d\pi_n$ of π_n is related to ψ by the Berezin calculus S, that is, we have $S(d\pi_n(X))(z) = i\langle(\psi(z), X)$ for each $X \in \mathfrak{g}$ and each $z \in \mathbb{D}$ [12].

The goal of the present note is to extend the above considerations to a large setting. To this aim, we consider a quasi-Hermitian Lie group G and a compactly embedded subgroup $K \subset G$. In [20], we considered a unitary representation π of G which is holomorphically induced from a unitary character of K and we proved that the dequantization of $d\pi$ by means of the Berezin calculus provides an explicit diffeomorphism from a complex domain onto the coadjoint orbit of G associated with π (see also [16] and [18]). Here we show that, more generally, such a diffeomorphism can also be constructed for the coadjoint orbit $\mathcal{O}(\xi_0) := \mathrm{Ad}^*(G)\,\xi_0$ of an element $\xi_0 \in \mathfrak{g}^*$ which is fixed by K and assumed to be regular (in a sense defined below). We call such an orbit $\mathcal{O}(\xi_0)$ a *scalar* orbit.

Note that similar parametrizations for coadjoint orbits of compact Lie groups can be found in [30] and [8]. For unitary groups, explicit expressions for generalized stereographic projections are given in [30].

Parametrizations of coadjoint orbits have many applications in deformation theory, harmonic analysis and mathematical physics. Let us mention some of them:

1. Construction of covariant star-products on coadjoint orbits [1], [11], [22];
2. Construction of some quantization maps, as adapted Weyl correspondences and Stratonovich-Weyl correspondences [13], [19];
3. Geometric quantization of coadjoint orbits [3], [21];
4. Contractions and restrictions of unitary irreducible representations associated with integral coadjoint orbits [15], [17], [23], [2], [14].

This note is organized as follows. Section 2 is devoted to generalities about quasi-Hermitian Lie groups. In Section 3 and Section 4, we review some results from [20]. In Section 5, we give a G-equivariant parametrization of a scalar coadjoint orbit of a quasi-Hermitian Lie group G. In Section 6, we consider the case of the unitary group $SU(p,q)$ and, in Section 7, the case of the (generalized) Jacobi group.

2 Generalities

The material of this section and of the first part of Section 3 is taken from the excellent book of K.-H. Neeb, [28], Chapter VIII and Chapter XII (see also [29], Chapter II and, for the Hermitian case, [25], Chapter VIII).

Let $\mathfrak{g}$ be a real quasi-Hermitian Lie algebra [28, p. 241]. We assume that $\mathfrak{g}$ is not compact. Let $\mathfrak{g}^c$ be the complexification of $\mathfrak{g}$ and let $Z = X + iY \to Z^* = -X + iY$ be the corresponding involution. We fix a compactly embedded Cartan subalgebra $\mathfrak{h} \subset \mathfrak{k}$, [28, p. 241] and we denote by $\mathfrak{h}^c$ the corresponding Cartan subalgebra of $\mathfrak{g}^c$. We write $\Delta := \Delta(\mathfrak{g}^c, \mathfrak{h}^c)$ for the set of roots of $\mathfrak{g}^c$ relative to $\mathfrak{h}^c$ and $\mathfrak{g}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta} \mathfrak{g}_\alpha$ for the root space decomposition of $\mathfrak{g}^c$. Note that $\alpha(\mathfrak{h}) \subset i\mathbb{R}$ for each $\alpha \in \Delta$ [28, p. 233]. We write Δ_k, respectively Δ_p, for the set of compact, respectively non-compact, roots [28, p. 233–235]. Note that one has $\mathfrak{k}^c = \mathfrak{h}^c \oplus \sum_{\alpha\in\Delta_k} \mathfrak{g}_\alpha$ [28, p. 235]. We fix a positive adapted system Δ^+ [28, p. 236] and we set $\Delta_p^+ := \Delta^+ \cap \Delta_p$ and $\Delta_k^+ := \Delta^+ \cap \Delta_k$, see [28, p. 241].

Let G^c be a simply connected complex Lie group with Lie algebra $\mathfrak{g}^c$ and $G \subset G^c$, respectively, $K \subset G^c$, the analytic subgroup corresponding to $\mathfrak{g}$, respectively, $\mathfrak{k}$. We also set $K^c = \exp(\mathfrak{k}^c) \subset G^c$ as in [28, p. 506].

Let $\mathfrak{p}^+ = \sum_{\alpha\in\Delta_p^+} \mathfrak{g}_\alpha$ and $\mathfrak{p}^- = \sum_{\alpha\in\Delta_p^+} \mathfrak{g}_{-\alpha}$. Let P^+ and P^- be the analytic subgroups of G^c with Lie algebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$. Then G is a group of the Harish-Chandra type [28, p. 507], that is, the following properties are satisfied:

1. $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$ is a direct sum of vector spaces, $(\mathfrak{p}^+)^* = \mathfrak{p}^-$ and $[\mathfrak{k}^+, \mathfrak{p}^\pm] \subset \mathfrak{p}^\pm$;
2. The multiplication map $P^+K^cP^- \to G^c$, $(z,k,y) \to zky$ is a biholomorphic diffeomorphism onto its open image;
3. $G \subset P^+K^cP^-$ and $G \cap K^cP^- = K$.

Moreover, there exists an open connected subset $\mathcal{D} \subset \mathfrak{p}^+$ such that $GK^cP^- = \exp(\mathcal{D})K^cP^-$ [28, p. 497]. We denote by $\zeta\colon P^+K^cP^- \to P^+$, $\kappa\colon P^+K^cP^- \to K^c$ and $\eta\colon P^+K^cP^- \to P^-$ the projections onto P^+-, K^c- and P^--components. For $Z \in \mathfrak{p}^+$ and $g \in G^c$ with $g \exp Z \in P^+K^cP^-$, we define the element $g \cdot Z$ of $\mathfrak{p}^+$ by $g \cdot Z := \log \zeta(g \exp Z)$. Note that we have $\mathcal{D} = G \cdot 0$.

We also denote by $g \to g^*$ the involutive anti-automorphism of G^c which is obtained by exponentiating $X \to X^*$. We denote by $p_{\mathfrak{p}^+}$ the projection of $\mathfrak{g}^c$ onto $\mathfrak{p}^+$ associated with the direct decomposition $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$.

3 Holomorphic representations

In this section, we consider the case of a coadjoint orbit associated with a scalar holomorphic discrete series representation of G.

We fix a unitary character χ of K. We also denote by χ the extension of χ to K^c. We set $K_\chi(Z,W) = \chi(\kappa(\exp W^* \exp Z))^{-1}$ for $Z, W \in \mathcal{D}$ and $J_\chi(g,Z) = \chi(\kappa(g \exp Z))$ for $g \in G$ and $Z \in \mathcal{D}$. Let $\mathcal{H}_\chi$ be the Hilbert space of holomorphic functions on $\mathcal{D}$ such that

$$\|f\|_\chi^2 := \int_{\mathcal{D}} |f(Z)|^2 \, K_\chi(Z,Z)^{-1} d\mu(Z) < +\infty$$

Here μ denotes the G-invariant measure on $\mathcal{D}$, that is,

$$d\mu(Z) := \chi_0(\kappa(\exp Z^* \exp Z)) \, d\mu_L(Z)$$

where χ_0 is the character on K^c defined by $\chi_0(k) = \mathrm{Det}_{\mathfrak{p}^+}(\mathrm{Ad}\, k)$ and $d\mu_L(Z)$ is a Lebesgue measure on $\mathcal{D}$ [28, p. 538].

In this section, we assume that $\mathcal{H}_\chi \neq (0)$. Then $\mathcal{H}_\chi$ contains the polynomials [28, p. 546] and the formula

$$\pi_\chi(g)f(Z) = J_\chi(g^{-1}, Z)^{-1} \, f(g^{-1} \cdot Z)$$

defines a unitary representation of G on $\mathcal{H}_\chi$ which is a highest weight representation with highest weight $\lambda := d\chi|_{\mathfrak{h}^c}$ [28, p. 540].

We introduce the constant c_χ defined by

$$c_\chi^{-1} = \int_{\mathcal{D}} K_\chi(Z,Z)^{-1} \, d\mu(Z).$$

and we set $e_Z(W) := c_\chi K_\chi(W,Z)$. Then we have the reproducing property $f(Z) = \langle f, e_Z \rangle_\chi$ for each $f \in \mathcal{H}_\chi$ and each $Z \in \mathcal{D}$ [28, p. 540]. Here $\langle \cdot, \cdot \rangle_\chi$ denotes the inner product on $\mathcal{H}_\chi$.

The Berezin calculus on $\mathcal{D}$ is then defined as follows [4], [5], [21]. Consider an operator (not necessarily bounded) A on $\mathcal{H}_\chi$ whose domain contains e_Z for each $Z \in \mathcal{D}$. Then the Berezin symbol of A is the function $S_\chi(A)$ defined on $\mathcal{D}$ by

$$S_\chi(A)(Z) := \frac{\langle A\, e_Z, \, e_Z \rangle_\chi}{\langle e_Z, \, e_Z \rangle_\chi}.$$

It is known that each operator is determined by its Berezin symbol and that if an operator A has adjoint A^* then we have $S_\chi(A^*) = \overline{S_\chi(A)}$ [4], [21]. The Berezin calculus is G-equivariant with respect to π_χ, that is, we have the following property: for each operator A on $\mathcal{H}_\chi$ whose domain contains the coherent states e_Z for each $Z \in \mathcal{D}$ and each $g \in G$, the domain of $\pi_\chi(g^{-1})A\pi_\chi(g)$ also contains e_Z for each $Z \in \mathcal{D}$ and we have

$$S_\chi(\pi_\chi(g)^{-1} A \pi_\chi(g))(Z) = S_\chi(A)(g \cdot Z) \tag{3.1}$$

for each $g \in G$ and $Z \in \mathcal{D}$.

Now, we consider the linear form ξ on $\mathfrak{g}^c$ defined by $\xi = -i d\chi$ on $\mathfrak{k}^c$ and $\xi = 0$ on $\mathfrak{p}^{\pm}$. Then we have $\xi(\mathfrak{g}) \subset \mathbb{R}$ and the restriction ξ_0 of ξ to $\mathfrak{g}$ is an element of $\mathfrak{g}^*$. Let $\mathcal{O}(\xi_0)$ be the orbit of ξ_0 in $\mathfrak{g}^*$ for the coadjoint action of G. In [20], we proved the following proposition (see also [17]).

Proposition 3.1

1. *For each* $X \in \mathfrak{g}^c$ *and each* $Z \in \mathcal{D}$, *we have*
$$S(d\pi_\chi(X))(Z) = i\langle \psi(Z), X\rangle$$
where $\psi(Z) := \mathrm{Ad}^*\big(\exp(-Z^*)\,\zeta(\exp Z^* \exp Z)\big)\,\xi_0$.
2. *For each* $g \in G$ *and each* $Z \in \mathcal{D}$, *we have* $\psi(g\cdot Z) = \mathrm{Ad}^*(g)\,\psi(Z)$.
3. *The map* ψ *is a diffeomorphism from* $\mathcal{D}$ *onto* $\mathcal{O}(\xi_0)$.

Note that (2) immediately follows from the G-equivariance of the Berezin calculus. In the following section, we extend (2) and (3) to scalar coadjoint orbits.

4 Parametrization of scalar coadjoint orbits

If $\xi_0 \in \mathfrak{g}^*$ is associated with a unitary character of K as in Section 3 then we have $\mathrm{Ad}^*(k)\xi_0 = \xi_0$ for each $k \in K$ and, by Lemma 3.1 of [20], the Hermitian form $(Z, W) \to \langle \xi_0, [Z, W^*]\rangle$ is not isotropic. This leads us to consider the elements $\xi_0 \in \mathfrak{g}^*$ which are fixed by K and regular in the sense that the Hermitian form $(Z, W) \to \langle \xi_0, [Z, W^*]\rangle$ is not isotropic. Such elements ξ_0 are called *scalar* and we say that the coadjoint orbit $\mathcal{O}(\xi_0)$ of a scalar element ξ_0 is a *scalar* orbit.

Lemma 4.1 *Let* $\xi_0 \in \mathfrak{g}^*$ *fixed by* K. *Let us also denote by* ξ_0 *the linear extension of* ξ_0 *to* $\mathfrak{g}^c$.

1. *We have* $\xi_0|_{\mathfrak{p}^\pm} \equiv 0$;
2. *Let* $E_1, E_2, \ldots, E_m$ *be a basis of* $\mathfrak{p}^+$ *such that* $E_j \in \mathfrak{g}_{\alpha_j}$ *where* $\alpha_j \in \Delta_p^+$ *for* $j = 1, 2, \ldots, m$. *Then* ξ_0 *is regular hence scalar if and only if we have* $i\langle \xi_0, [E_j^*, E_j]\rangle > 0$ *for each* $j = 1, 2, \ldots, m$ *or* $i\langle \xi_0, [E_j^*, E_j]\rangle < 0$ *for each* $j = 1, 2, \ldots, m$.

Proof (1) If $\xi_0 \in \mathfrak{g}^*$ is fixed by K then one has $\mathrm{ad}^*\, U\,\xi_0 = 0$ for each $U \in \mathfrak{k}$ or, equivalently, $\langle \xi_0, [U, X]\rangle = 0$ for each $U \in \mathfrak{k}$ and $X \in \mathfrak{g}$. Then, taking $X = E_j$ where $j = 1, 2, \ldots, m$ and $U \in \mathfrak{g}_{\alpha_j}$ such that $\alpha_j(U) \neq 0$ we get $\langle \xi_0, E_j\rangle = 0$ for each $j = 1, 2, \ldots, m$ hence the result.

(2) Let $Z = \sum_{j=1}^m z_j E_j \in \mathfrak{p}^+$. Then, by using (1), we get
$$\langle \xi_0, [Z^*, Z]\rangle = \sum_{j=1}^m \langle \xi_0, [E_j^*, E_j]\rangle |z_j|^2$$
where $i[E_j^*, E_j] \in \mathfrak{h}$ for each j [28], p. 233. The result then follows. □

In the rest of this section, we fix a scalar element $\xi_0 \in \mathfrak{g}^*$. For $Z \in \mathcal{D}$, we set
$$\psi(Z) := \mathrm{Ad}^*\big(\exp(-Z^*)\,\zeta(\exp Z^* \exp Z)\big)\,\xi_0.$$

Proposition 4.2 *For each $g \in G$ and each $Z \in \mathcal{D}$, we have*

$$\psi(g \cdot Z) = \mathrm{Ad}^*(g)\,\psi(Z).$$

Proof Let $g \in G$ and $Z \in \mathcal{D}$. We write $g \exp Z = zky$ where $z \in P^+$, $k \in K^c$ and $y \in P^-$. Then, since $g^* = g^{-1}$, we have $\exp Z^* \exp Z = y^* k^* z^* zky$. This implies that

$$\zeta(\exp Z^* \exp Z) = y^* k^* \zeta(z^* z) {k^*}^{-1}.$$

Thus, noting that $z = \exp(g \cdot Z)$, we get

$$\begin{aligned} &\exp(-(g \cdot Z)^*)\, \zeta(\exp(g \cdot Z)^* \exp(g \cdot Z)) = {z^*}^{-1} \zeta(z^* z) \\ &= g \exp(-Z^*) y^* k^* \zeta(z^* z) = g \exp(-Z^*)\, \zeta(\exp Z^* \exp Z) k^*. \end{aligned}$$

Hence we obtain $\psi(g \cdot Z) = \mathrm{Ad}^*(g)\,\psi(Z)$. □

Corollary 4.3 *The stabilizer of ξ_0 for the coadjoint action of G is K.*

Proof First, we prove that for $Z \in \mathcal{D}$ the equality $\psi(Z) = \xi_0$ implies that $Z = 0$. Assume that $\psi(Z) = \xi_0$. Then we have

$$\mathrm{Ad}^*\big(\zeta(\exp Z^* \exp Z)\big)\, \xi_0 = \mathrm{Ad}^*(\exp Z)\, \xi_0$$

or, equivalently,

$$\langle \xi_0, \mathrm{Ad}(\zeta(\exp Z^* \exp Z)^{-1}) X \rangle = \langle \xi_0, \mathrm{Ad}(\exp(-Z^*)) X \rangle.$$

for each $X \in \mathfrak{g}^c$. Thus, taking $X = Z$ and using (1) of Lemma 4.1, we get $\langle \xi_0, [Z^*, Z] \rangle = 0$ hence $Z = 0$.

Now, consider $g \in G$ such that $\mathrm{Ad}^*(g)\xi_0 = \xi_0$. Then, by Proposition 4.2, we have $\psi(g \cdot 0) = \xi_0$ and, by the assertion already proved, we get $g \cdot 0 = \xi_0$. Hence we obtain $g \in K^c P^- \cap G = K$. □

Proposition 4.4 *The map ψ is a diffeomorphism from $\mathcal{D}$ onto $\mathcal{O}(\xi_0)$.*

Proof Let $Z \in \mathcal{D}$. There exists $g \in G$ such that $g \cdot 0 = Z$. Then, by Proposition 4.2, we have $\psi(Z) = \mathrm{Ad}^*(g)\xi_0$. This shows that ψ has values in $\mathcal{O}(\xi_0)$ and that ψ is surjective. Now, suppose that $\psi(Z) = \psi(Z')$ for some $Z, Z' \in \mathcal{D}$. Let $g, g' \in G$ such that $g \cdot 0 = Z$ and $g' \cdot 0 = Z'$. Then, by Proposition 4.2, we have $\mathrm{Ad}^*(g)\xi_0 = \mathrm{Ad}^*(g')\xi_0$. Thus, by Corollary 4.3, we get $g^{-1}g' \in K$ hence $Z = g \cdot 0 = g' \cdot 0 = Z'$. This proves that ψ is injective hence bijective.

Now, we show that ψ is regular. Using Proposition 4.2, we have just to verify that ψ is regular at $Z = 0$. By differentiating the multiplication map from $P^+ \times K^c \times P^-$ onto $P^+ K^c P^-$, we easily see that, for each $g \in G$ such that $g = zky$ with $z \in P^+$, $k \in K^c$ and $y \in P^-$ and each $X \in \mathfrak{g}^c$, we have

$$d\zeta_g(X^+(g)) = (\mathrm{Ad}(z)\, p_{\mathfrak{p}^+}(\mathrm{Ad}(z^{-1})\, X))^+(z).$$

Here, we have denoted by Y^+ the right-invariant vector field generated by Y. From this, it follows that, for each $Y \in \mathfrak{p}^+$ and each $X \in \mathfrak{g}^c$, we have

$$\langle (d\psi)_0(Y), X\rangle = \langle \xi_0, [X, Y - Y^*]\rangle. \tag{4.1}$$

Now, assume that $(d\psi)_0(Y) = 0$ for some $Y \in \mathfrak{p}^+$. By taking $X = Y$ in (4.1) we get $\langle \xi_0, [Y, Y^*]\rangle = 0$ hence $Y = 0$. □

Now, we construct a section of the action of G on $\mathcal{D}$, that is, a map $Z \to g_Z$ from $\mathcal{D}$ to G such that $g_Z \cdot 0 = Z$ for each $Z \in \mathcal{D}$ and we show that ψ can be recovered by using this section. Note that such sections are useful in practice, in particular to determine explicitly $\mathcal{D}$, see, for instance [28, p. 501].

Proposition 4.5 *Let $Z \in \mathcal{D}$. There exists an element k_Z in K^c such that $k_Z^* = k_Z$ and $k_Z^2 = \kappa(\exp Z^* \exp Z)^{-1}$. Each $g \in G$ such that $g \cdot 0 = Z$ is then of the form $g = \exp(-Z^*)\,\zeta(\exp Z^* \exp Z)k_Z^{-1}h$ where $h \in K$. Consequently, the map $Z \to g_Z := \exp(-Z^*)\,\zeta(\exp Z^* \exp Z)k_Z^{-1}$ is a section for the action of G on $\mathcal{D}$. In particular, by using the equality $\psi(Z) = \mathrm{Ad}^*(g_Z)\xi_0$, we recover the expression of ψ given above.*

Proof Let $Z \in \mathcal{D}$ and $g \in G$ such that $g \cdot 0 = Z$. Then we can write $g = (\exp Z)ky$ where $k \in K^c$ and $y \in P^-$. Thus we have

$$g^*g = y^*k^*(\exp Z^* \exp Z)ky = e.$$

Consequently, passing to the K^c-component, we get $k^*\kappa(\exp Z^* \exp Z)k = e$. Now, using the polar decomposition $K^c = \exp(i\mathfrak{k})K$ [28, p. 506], we can write $k = k_Z h$ where $k_Z \in \exp(i\mathfrak{k})$ and $h \in K$. Hence we obtain $k_Z^2 = \kappa(\exp Z^* \exp Z)^{-1}$. Moreover, passing similarly to the P^--component, we get $k^{-1}\eta(\exp Z^* \exp Z)ky = e$ hence $ky = \eta(\exp Z^* \exp Z)^{-1}k$. This gives

$$\begin{aligned} g &= \exp Z \eta(\exp Z^* \exp Z)^{-1}k \\ &= \exp(-Z^*)(\exp Z^* \exp Z)\eta(\exp Z^* \exp Z)^{-1}k_Z h \\ &= \exp(-Z^*)\,\zeta(\exp Z^* \exp Z)k_Z^{-1}h. \end{aligned}$$

This shows the second assertion of the proposition. Finally, writing

$$\begin{aligned} \psi(Z) = \mathrm{Ad}^*(g_Z)\xi_0 &= \mathrm{Ad}^*(\exp(-Z^*)\,\zeta(\exp Z^* \exp Z)k_Z^{-1})\xi_0 \\ &= \mathrm{Ad}^*(\exp(-Z^*)\,\zeta(\exp Z^* \exp Z))\xi_0, \end{aligned}$$

we recover the expression of ψ. □

5 Example 1: the unitary group $SU(p, q)$

In this section, we take $G = SU(p, q)$ and $K = S(U(p) \times U(q))$. Recall that K consists of the matrices

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}, \qquad A \in U(p),\ D \in U(q), \quad \mathrm{Det}(A)\,\mathrm{Det}(D) = 1.$$

For $X = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathfrak{g}^c$ we have $X^* = \left(\begin{smallmatrix} -A^\star & C^\star \\ B^\star & -D^\star \end{smallmatrix}\right)$ where $\star$ denotes conjugate-transposition.

Let $\mathfrak{h}$ be the abelian subalgebra of $\mathfrak{k}$ consisting of the matrices

$$\begin{pmatrix} iaI_p & 0 \\ 0 & ibI_q \end{pmatrix}, \qquad a, b \in \mathbb{R}, \quad pa + bq = 0.$$

Then $\mathfrak{h}^c$ consists of all matrices $X = \mathrm{Diag}(x_1, x_2, \ldots, x_{p+q})$, $x_k \in \mathbb{C}$, such that $\sum_{k=1}^{p+q} x_k = 0$. The set of roots of $\mathfrak{h}^c$ on $\mathfrak{g}^c$ is $\lambda_i - \lambda_j$ for $1 \le i \ne j \le p+q$ where $\lambda_i(X) = x_i$ for $X \in \mathfrak{h}^c$ as above. The set of compact roots is $\lambda_i - \lambda_j$ for $1 \le i \ne j \le p$ and $p+1 \le i \ne j \le p+q$. We take the set of positive roots Δ^+ to be $\lambda_i - \lambda_j$ for $1 \le i < j \le p+q$. Then we have

$$P^+ = \left\{ \begin{pmatrix} I_p & Z \\ 0 & I_q \end{pmatrix} : Z \in M_{pq}(\mathbb{C}) \right\}, \qquad P^- = \left\{ \begin{pmatrix} I_p & 0 \\ Y & I_q \end{pmatrix} : Y \in M_{qp}(\mathbb{C}) \right\}.$$

In the rest of this section, we identify $\mathfrak{p}^+$ to $M_{pq}(\mathbb{C})$ by means of the map $Z \to \left(\begin{smallmatrix} 0 & Z \\ 0 & 0 \end{smallmatrix}\right)$.

The $P^+K^cP^-$-decomposition of a matrix $g \in G^c$ is given by

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} I_p & BD^{-1} \\ 0 & I_q \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I_p & 0 \\ D^{-1}C & I_q \end{pmatrix}. \tag{5.1}$$

Note that a matrix $g \in G^c$ have such a decomposition if and only if $\mathrm{Det}(D) \ne 0$. In particular we verify that $G \subset P^+K^cP^-$. Moreover, the action of G^c on $\mathcal{D}$ is then given by

$$g \cdot Z = (AZ + B)(CZ + D)^{-1}, \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Note that $g \cdot 0 = BD^{-1} = Z$ satisfies $I_p - ZZ^\star > 0$ [28]. From this we see that

$$\mathcal{D} = \{ Z \in M_{pq}(\mathbb{C}) \colon I_p - ZZ^\star > 0 \}.$$

The Killing form β on $\mathfrak{g}^c$ is defined by $\beta(X,Y) := 2(p+q)\,\mathrm{Tr}(XY)$ [31, p. 295]. We identify G-equivariantly $\mathfrak{g}^*$ with $\mathfrak{g}$ by means of β. We easily verify that the set of all elements of $\mathfrak{g}$ fixed by K is $\mathfrak{h}$. Each $\xi_0 \in \mathfrak{h}$ can be written as

$$\xi_0 = i\lambda \begin{pmatrix} -qI_p & 0 \\ 0 & pI_q \end{pmatrix}$$

where $\lambda \in \mathbb{R}$. Then we have $\langle \xi_0, [Z^*, Z] \rangle = -2i\lambda(p+q)^2\,\mathrm{Tr}(ZZ^\star)$ for each $Z \in \mathcal{D}$. This shows that ξ_0 is regular if and only if $\lambda \ne 0$. In that case, we can compute the section $Z \to g_Z$ hence $\psi(Z)$ as follows. For $Z \in \mathcal{D}$, we have

$$\exp Z^* \exp Z = \begin{pmatrix} I_p & Z \\ -Z^\star & I_q - Z^\star Z \end{pmatrix}.$$

Then, by (5.1), we get

$$\kappa(\exp Z^* \exp Z) = \begin{pmatrix} (I_p - ZZ^\star)^{-1} & 0 \\ 0 & I_q - Z^\star Z \end{pmatrix},$$

$$\zeta(\exp Z^* \exp Z) = \begin{pmatrix} I_p & Z(I_q - Z^\star Z)^{-1} \\ 0 & I_q \end{pmatrix}$$

and we can take

$$k_Z = \begin{pmatrix} (I_p - ZZ^\star)^{1/2} & 0 \\ 0 & (I_q - Z^\star Z)^{-1/2} \end{pmatrix}.$$

Thus we have

$$g_Z = \exp(-Z^*)\zeta(\exp Z^* \exp Z)k_Z^{-1} = \begin{pmatrix} (I_p - ZZ^\star)^{-1/2} & Z(I_q - Z^\star Z)^{-1/2} \\ Z^\star(I_p - ZZ^\star)^{-1/2} & (I_q - Z^\star Z)^{-1/2} \end{pmatrix}.$$

Hence we obtain

$$\psi(Z) = i\lambda \begin{pmatrix} (I_p - ZZ^\star)^{-1}(-pZZ^\star - qI_p) & (p+q)Z(I_q - Z^\star Z)^{-1} \\ -(p+q)(I_q - Z^\star Z)^{-1}Z^\star & (pI_q + qZ^\star Z)(I_q - Z^\star Z)^{-1} \end{pmatrix}.$$

6 Example 2: the Jacobi group

The Jacobi group is the semi-direct product of the $(2n+1)$-dimensional real Heisenberg group by the symplectic group $Sp(n, \mathbb{R})$. This group plays an important role in different areas of Mathematics and Physics, see [10] and [6]. In particular, the Jacobi group appears as an important example of non-reductive Lie group of Harish-Chandra type [29], [28] and its holomorphic unitary representations were studied in [28], [9], [10], [6] and [7].

Consider the symplectic form ω on $\mathbb{C}^{2n} \times \mathbb{C}^{2n}$ defined by

$$\omega((z, w), (z', w')) = \frac{i}{2} \sum_{k=1}^{n} (z_k w'_k - z'_k w_k).$$

for $z, w, z', w' \in \mathbb{C}^n$. The $(2n+1)$-dimensional real Heisenberg group is

$$H := \{((z, \bar{z}), c) \colon z \in \mathbb{C}^n, c \in \mathbb{R}\}$$

endowed with the multiplication

$$((z, \bar{z}), c) \cdot ((z', \bar{z}'), c') = ((z + z', \bar{z} + \bar{z}'), c + c' + \tfrac{1}{2}\omega((z, \bar{z}), (z', \bar{z}'))). \tag{6.1}$$

Then the complexification H^c of H is

$$H^c := \{((z, w), c) \colon z, w \in \mathbb{C}^n, c \in \mathbb{C}\}$$

and the multiplication of H^c is obtained by replacing $(z, \bar{z})$ by (z, w) and $(z', \bar{z}')$ by (z', w') in (6.1). We denote by $\mathfrak{h}$ and $\mathfrak{h}^c$ the Lie algebras of H and H^c.

Now consider the group $S := Sp(n,\mathbb{C}) \cap SU(n,n) \simeq Sp(n,\mathbb{R})$ [28, p. 501], [24, p. 175]. Then S consists of all matrices

$$h = \begin{pmatrix} P & Q \\ \bar{Q} & \bar{P} \end{pmatrix}, \quad P, Q \in M_n(\mathbb{C}), \quad PP^\star - QQ^\star = I_n, \quad PQ^t = QP^t$$

and $S^c = Sp(n,\mathbb{C})$.

The group S acts on H by $h \cdot ((z,\bar{z}),c) = h(z,\bar{z}) = Pz + Q\bar{z}$ where the elements of $\mathbb{C}^n$ and $\mathbb{C}^n \times \mathbb{C}^n$ are considered as column vectors. Then we can form the semi-direct product $G := H \rtimes S$ called the Jacobi group. The elements of G can be written as $((z,\bar{z}),c,h)$ where $z \in \mathbb{C}^n$, $c \in \mathbb{R}$ and $h \in S$. The multiplication of G is thus given by

$$((z,\bar{z}),c,h)\cdot((z',\bar{z}'),c',h') = ((z,\bar{z}) + h(z',\bar{z}'), c + c' + \tfrac{1}{2}\omega((z,\bar{z}),h(z',\bar{z}')), hh').$$

The complexification G^c of G is then the semi-direct product $G^c = H^c \rtimes Sp(n,\mathbb{C})$ and the multiplication of G^c is obtained by replacing $\bar{z}$ and $\bar{z}'$ by w and w' in the preceding formula. We denote by $\mathfrak{s}$, $\mathfrak{s}^c$, $\mathfrak{g}$ and $\mathfrak{g}^c$ the Lie algebras of S, S^c, G and G^c. The Lie bracket of $\mathfrak{g}^c$ is given by

$$[((z,w),c,A),((z',w'),c',A')] = (A(z',w') - A'(z,w), \omega((z,w),(z',w')), [A,A']).$$

We easily verify that

$$\text{if } X = \left((z,w),c,\left(\begin{smallmatrix} A & B \\ C & -A^t \end{smallmatrix}\right)\right) \in \mathfrak{g}^c \text{ then } X^* = \left((-\bar{w},-\bar{z}),-\bar{c},\left(\begin{smallmatrix} \bar{A}^t & -\bar{C} \\ -\bar{B} & -\bar{A} \end{smallmatrix}\right)\right).$$

We take K to be the subgroup of G consisting of all elements $\left((0,0),c,\left(\begin{smallmatrix} P & 0 \\ 0 & \bar{P} \end{smallmatrix}\right)\right)$ where $c \in \mathbb{R}$ and $P \in U(n)$. Then the Lie algebra $\mathfrak{k}$ of K is a maximal compactly embedded subalgebra of $\mathfrak{g}$ and the subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ consisting of elements of the form $((0,0),c,A)$ where A is diagonal is a compactly embedded Cartan subalgebra of $\mathfrak{g}$ [28, p. 250]. Choosing an adapted positive system of non-compact positive roots relative to $\mathfrak{t}$ as in [28, p. 249], we get

$$\mathfrak{p}^+ = \left\{ a(z,Z) := \left((z,0),0,\begin{pmatrix} 0 & Z \\ 0 & 0 \end{pmatrix}\right) : z \in \mathbb{C}^n, Z \in M_n(\mathbb{C}), Z^t = Z \right\}$$

and

$$\mathfrak{p}^- = \left\{ \left((0,w),0,\begin{pmatrix} 0 & 0 \\ W & 0 \end{pmatrix}\right) : w \in \mathbb{C}^n, W \in M_n(\mathbb{C}), W^t = W \right\}.$$

Then we obtain

$$P^+ = \left\{ \left((z,0),0,\begin{pmatrix} I_n & Z \\ 0 & I_n \end{pmatrix}\right) : z \in \mathbb{C}^n, Z \in M_n(\mathbb{C}), Z^t = Z \right\}$$

and

$$P^- = \left\{ \left((0,w),0,\begin{pmatrix} I_n & 0 \\ W & I_n \end{pmatrix}\right) : w \in \mathbb{C}^n, W \in M_n(\mathbb{C}), W^t = W \right\}.$$

Thus we easily verify that $g = \big((z_0, w_0), c_0, \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\big) \in G^c$ has a $P^+K^cP^-$-decomposition

$$g = \left((z,0),0,\begin{pmatrix} I_n & Z \\ 0 & I_n \end{pmatrix}\right)\cdot\left((0,0),c,\begin{pmatrix} P & 0 \\ 0 & (P^t)^{-1} \end{pmatrix}\right)\cdot\left((0,w),0,\begin{pmatrix} I_n & 0 \\ W & I_n \end{pmatrix}\right)$$

if and only if $\mathrm{Det}(D) \neq 0$ and, in this case, we have $z = z_0 - BD^{-1}w_0$, $Z = BD^{-1}$, $w = D^{-1}w_0$, $W = D^{-1}C$, $P = A - BD^{-1}C = (D^t)^{-1}$ and $c = c_0 - (1/4)i(z_0 - BD^{-1}w_0)^t w_0$. From this, we deduce that the action of $g = \big((z_0, w_0), c_0, \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\big) \in G^c$ on $a(z, Z) \in \mathfrak{p}^+$ is given by $g \cdot a(z, Z) = a(z', Z')$ where $Z' = (AZ + B)(CZ + D)^{-1}$ and

$$z' = z_0 + Az - (AZ + B)(CZ + D)^{-1}(w_0 + Cz).$$

This implies that

$$\mathcal{D} = G \cdot 0 = \{a(z, Z) \in \mathfrak{p}^+ \colon I_n - Z\bar{Z} > 0\}.$$

Now we aim to compute the coadjoint action of G^c. This can be done as follows. First, we compute the adjoint action of G^c. Let $g = (v_0, c_0, h_0) \in G^c$ where $v_0 \in \mathbb{C}^{2n}$, $c_0 \in \mathbb{C}$ and $h_0 \in S^c = Sp(n, \mathbb{C})$ and $X = (w, c, U) \in \mathfrak{g}^c$ where $w \in \mathbb{C}^{2n}$, $c \in \mathbb{C}$ and $U \in \mathfrak{s}^c$. We set $\exp(tX) = (w(t), c(t), \exp(tU))$. Then, since the derivatives of $w(t)$ and $c(t)$ at $t = 0$ are w and c, we find that

$$\begin{aligned}\mathrm{Ad}(g)X &= \frac{d}{dt}(g \exp(tX) g^{-1})|_{t=0} \\ &= \big(h_0 w - (\mathrm{Ad}(h_0)U)v_0, c + \omega(v_0, h_0 w) - \tfrac{1}{2}\omega(v_0, (\mathrm{Ad}(h_0)U)v_0), \mathrm{Ad}(h_0)U\big).\end{aligned}$$

On the other hand, let us denote by $\xi = (u, d, \varphi)$, where $u \in \mathbb{C}^{2n}$, $d \in \mathbb{C}$ and $\varphi \in (\mathfrak{s}^c)^*$, the element of $(\mathfrak{g}^c)^*$ defined by

$$\langle \xi, (w, c, U)\rangle = \omega(u, w) + dc + \langle \varphi, U\rangle.$$

Moreover, for $u, v \in \mathbb{C}^{2n}$, we denote by $v \times u$ the element of $(\mathfrak{s}^c)^*$ defined by $\langle v \times u, U\rangle := \omega(u, Uv)$ for $U \in \mathfrak{s}^c$.

Let $\xi = (u, d, \varphi) \in (\mathfrak{g}^c)^*$ and $g = (v_0, c_0, h_0) \in G^c$. Then, by using the relation $\langle \mathrm{Ad}^*(g)\xi, X\rangle = \langle \xi, \mathrm{Ad}(g^{-1})X\rangle$ for $X \in \mathfrak{g}^c$, we obtain

$$\mathrm{Ad}^*(g)\xi = \big(h_0 u - d v_0, d, \mathrm{Ad}^*(h_0)\varphi + v_0 \times (h_0 u - \tfrac{d}{2} v_0)\big)$$

By restriction, we also get the formula for the coadjoint action of G. Now, we are in position to determine the scalar elements of $(\mathfrak{g}^c)^*$.

Proposition 6.1

1. *The elements ξ_0 of $\mathfrak{g}^*$ fixed by K are the elements of the form $(0, d, \varphi_\lambda)$ where $d, \lambda \in \mathbb{R}$ and $\varphi_\lambda \in \mathfrak{s}^*$ is defined by $\langle \varphi_\lambda, \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\rangle = i\lambda\,\mathrm{Tr}(A)$.*
2. *Let $\xi_0 = (0, d, \varphi_\lambda)$ as above. Then ξ_0 is regular hence scalar if and only if $\lambda d \neq 0$.*

Proof (1) Let $\xi_0 = ((u_0, \bar{u}_0), d, \varphi) \in \mathfrak{g}^*$ where $u_0 \in \mathbb{C}^n$, $d \in \mathbb{R}$ and $\varphi \in \mathfrak{s}^*$. Assume that ξ_0 is fixed by K. Then for each $k = \big((u_0, \bar{u}_0), c_0, \left(\begin{smallmatrix} P & 0 \\ 0 & \bar{P} \end{smallmatrix}\right)\big) \in K$ with $u_0 \in \mathbb{C}^n$, $c_0 \in \mathbb{R}$ and $P \in U(n)$, we have

$$\mathrm{Ad}^*(k)\xi_0 = \big((Pu_0, \bar{P}\bar{u}_0), d, \mathrm{Ad}^*\left(\begin{smallmatrix} P & 0 \\ 0 & \bar{P} \end{smallmatrix}\right)\varphi\big) = ((u_0, \bar{u}_0), d, \varphi).$$

This gives $Pu_0 = u_0$ for each $P \in U(n)$ hence $u_0 = 0$ and $\mathrm{Ad}^*(k_0)\varphi = \varphi$ for each k_0 in the subgroup K_0 of S consisting of the matrices of the form $\left(\begin{smallmatrix} P & 0 \\ 0 & \bar{P} \end{smallmatrix}\right)$ where $P \in U(n)$. Then, denoting by $\mathfrak{k}_0$ the Lie algebra of K_0, we have $\langle \varphi, [U, X]\rangle = 0$ for each $U \in \mathfrak{k}_0$ and each $X \in \mathfrak{s}$. This implies that φ is zero on $[\mathfrak{k}_0, \mathfrak{k}_0]$ and also on the elements of $\mathfrak{s}$ of the form $\left(\begin{smallmatrix} 0 & Q \\ \bar{Q} & 0 \end{smallmatrix}\right)$. Then φ is completely determined by its value on the element $\left(\begin{smallmatrix} iI_n & 0 \\ 0 & -iI_n \end{smallmatrix}\right)$ which generates the center of $\mathfrak{k}_0$, hence the result.

(2) Let ξ_0 as above. Then we have $\langle \xi_0, [a(z, Z)^*, a(z, Z)]\rangle = d|z|^2 + i\lambda \,\mathrm{Tr}(Z\bar{Z})$. The result follows. $\square$

In the rest of this section, we fix a scalar element $\xi_0 = (0, d, \varphi_\lambda)$ of $\mathfrak{g}^*$ as above and we compute $\psi(a(z, Z))$ for $a(z, Z) \in \mathcal{D}$. In order to make the expression of $\psi(a(z, Z))$ more explicit, we introduce the following notation. For $\varphi \in \mathfrak{s}^*$, let $\theta(\varphi)$ the unique element of $\mathfrak{s}$ such that $\langle \varphi, X\rangle = \mathrm{Tr}(\theta(\varphi)X)$ for each $X \in \mathfrak{s}$. In particular, one has $\theta(\varphi_\lambda) = \frac{\lambda}{2}\left(\begin{smallmatrix} iI_n & 0 \\ 0 & -iI_n \end{smallmatrix}\right)$. Moreover, for $u = (x, \bar{x}) \in \mathbb{C}^{2n}$ and $u = (y, \bar{y}) \in \mathbb{C}^{2n}$ we have

$$\theta(v \times u) = \frac{1}{2}\begin{pmatrix} -iy\bar{x}^t & iyx^t \\ -i\bar{y}\bar{x}^t & i\bar{y}x^t \end{pmatrix}.$$

Note also that θ intertwines Ad^* and Ad.

Proposition 6.2 *The map $\psi\colon \mathcal{D} \to \mathcal{O}(\xi_0)$ is given by*

$$\psi(a(y, Z)) = \big(-d(y_1, \bar{y}_1), d, \varphi(y, Z)\big)$$

where $y_1 = (I_n - Z\bar{Z})^{-1}(y + Z\bar{y})$ and

$$\varphi(y, Z) := \mathrm{Ad}^*\begin{pmatrix} (I_n - Z\bar{Z})^{-1/2} & (I_n - Z\bar{Z})^{-1/2}Z \\ (I_n - \bar{Z}Z)^{-1/2}\bar{Z} & (I_n - \bar{Z}Z)^{-1/2} \end{pmatrix}\varphi_\lambda - \frac{d}{2}(y_1, \bar{y}_1) \times (y_1, \bar{y}_1).$$

Moreover, we have

$$\theta(\varphi(y, Z)) = -\frac{d}{4}\begin{pmatrix} -iy_1\bar{y}_1^t & iy_1y_1^t \\ -i\bar{y}_1\bar{y}_1^t & i\bar{y}_1y_1^t \end{pmatrix} + \frac{\lambda}{2}i$$
$$\times\begin{pmatrix} (I_n + Z\bar{Z})(I_n - Z\bar{Z})^{-1/2}(I_n - \bar{Z}Z)^{-1/2} & -2Z(I_n - \bar{Z}Z)^{-1/2}(I_n - Z\bar{Z})^{-1/2} \\ 2\bar{Z}(I_n - Z\bar{Z})^{-1/2}(I_n - \bar{Z}Z)^{-1/2} & -(I_n + \bar{Z}Z)(I_n - \bar{Z}Z)^{-1/2}(I_n - Z\bar{Z})^{-1/2} \end{pmatrix}.$$

Proof For $(y, Z) \in \mathbb{C}^n \times M_n(\mathbb{C})$ such that $a(y, Z) \in \mathcal{D}$ we set

$$g(y, Z) := \left((y_1, \bar{y}_1), 0, \begin{pmatrix} (I_n - Z\bar{Z})^{-1/2} & (I_n - Z\bar{Z})^{-1/2}Z \\ (I_n - \bar{Z}Z)^{-1/2}\bar{Z} & (I_n - \bar{Z}Z)^{-1/2} \end{pmatrix}\right) \in G$$

where $y_1 = (I_n - Z\bar{Z})^{-1}(y + Z\bar{y})$. Then the map $a(y, Z) \to g(y, Z)$ is a section for the action of G on $\mathcal{D}$ and we have $\psi(a(y, Z)) = \mathrm{Ad}^*(g(y, Z))\xi_0$ (in fact, we use here this section since the expression of the section given by Proposition 4.5 is too complicated in this case). Thus, by using the formula for the coadjoint action of G and the above considerations on θ, we easily obtain the desired result. □

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