Fekete–Szegő Problem for a New Class of Analytic Functions Defined by Using a Generalized Differential Operator

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Abstract

In this paper, we obtain Fekete–Szegő inequalities for a generalized class of analytic functions \( f(z) \in A \) for which \( 1 + b \left( \frac{D_n^{\alpha,\beta,\lambda,\delta} f(z)}{D_n^{\alpha,\beta,\lambda,\delta} f(z)} - 1 \right)^{1-n} \) \((\alpha, \beta, \lambda, \delta \geq 0; \beta > \alpha; \lambda > \delta; b \in \mathbb{C}^*; n \in \mathbb{N}_0; z \in U)\) lies in a region starlike with respect to 1 and is symmetric with respect to the real axis.

Key words: analytic, subordination, Fekete–Szegő problem

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1 Introduction

Let \( A \) denote the class of functions \( f(z) \) of the form:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in U),
\]

which are analytic in the open unit disc \( U = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \} \). Further let \( S \) denote the family of functions of the form (1.1) which are univalent in \( U \).
A classical theorem of Fekete–Szegő [7] states that, for \( f(z) \in S \) given by (1.1) that

\[
|a_3 - ma_2^2| \leq \begin{cases} 
3 - 4\mu, & \text{if } \mu \leq 0, \\
1 + 2 \exp \left( \frac{-2\mu}{1-\mu} \right), & \text{if } 0 \leq \mu \leq 1, \\
4\mu - 3, & \text{if } \mu \geq 1.
\end{cases}
\]

(1.2)

The result is sharp.

Given two functions \( f(z) \) and \( g(z) \), which are analytic in \( U \) with \( f(0) = g(0) \), the function \( f(z) \) is said to be subordinate to \( g(z) \) in \( U \) if there exists a function \( w(z) \), analytic in \( U \), such that \( w(0) = 0 \) and \( |w(z)| < 1 \) (\( z \in U \)) and \( f(z) = g(w(z)) \) (\( z \in U \)). We denote this subordination by \( f(z) \prec g(z) \) in \( U \) (see [13]).

Let \( \varphi(z) \) be an analytic function with positive real part on \( U \), which satisfies \( \varphi(0) = 1 \) and \( \varphi'(0) > 0 \), and which maps the unit disc \( U \) onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let \( S^*(\varphi) \) be the class of functions \( f(z) \in S \) for which

\[
zf'(z) \prec \varphi(z) \quad (z \in U),
\]

(1.3)

and \( C(\varphi) \) be the class of functions \( f(z) \in S \) for which

\[
1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z) \quad (z \in U).
\]

(1.4)

The classes of \( S^*(\varphi) \) and \( C(\varphi) \) were introduced and studied by Ma and Minda [12]. The familiar class \( S^*(\alpha) \) of starlike functions of order \( \alpha \) and the class \( C(\alpha) \) of convex functions of order \( \alpha \) (\( 0 \leq \alpha < 1 \)) are the special cases of \( S^*(\varphi) \) and \( C(\varphi) \), respectively, when

\[
\varphi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (0 \leq \alpha < 1).
\]

Ma and Minda [12] have obtained the Fekete–Szegő problem for the functions in the class \( C(\varphi) \). For a function \( f(z) \in S \), Ramadan and Darus [18] introduced the generalized differential operator \( D_{\alpha,\beta,\lambda,\delta}^n \) as following:

\[
D_{\alpha,\beta,\lambda,\delta}^0 f(z) = f(z),
\]

\[
D_{\alpha,\beta,\lambda,\delta}^1 f(z) = [1 - (\lambda - \delta) (\beta - \alpha)] f(z) + (\lambda - \delta) (\beta - \alpha) zf'(z)
\]

\[
= z + \sum_{k=2}^{\infty} [(\lambda - \delta) (\beta - \alpha) (k - 1) + 1] a_k z^k,
\]

\[
D_{\alpha,\beta,\lambda,\delta}^n f(z) = D_{\alpha,\beta,\lambda,\delta}^{n-1} \left( D_{\alpha,\beta,\lambda,\delta}^1 f(z) \right),
\]

(1.5)

\[
D_{\alpha,\beta,\lambda,\delta}^n f(z) = z + \sum_{k=2}^{\infty} [(\lambda - \delta) (\beta - \alpha) (k - 1) + 1] a_k z^k,
\]

\( (\alpha, \beta, \lambda, \delta \geq 0; \; \delta \geq 0; \; \beta > \alpha; \; \lambda > \delta; \; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \; \mathbb{N} = \{1, 2, 3, \ldots \} ) \).
Remark 1 (i) Taking $\alpha = 0$, then operator $D_{0,\beta,\lambda,\delta}^n = D_{\beta,\lambda,\delta}^n$, was introduced and studied by Darus and Ibrahim [6];
(ii) Taking $\alpha = \delta = 0$ and $\beta = 1$, then operator $D_{0,1,\lambda,0}^n = D_{\lambda}^n$, was introduced and studied by Al-Oboudi [1];
(iii) Taking $\alpha = \delta = 0$ and $\lambda = \beta = 1$, then operator $D_{0,1,1,0}^n = D^n$, was introduced and studied by Salagean [20].

Using the generalized operator $D_{\alpha,\beta,\lambda,\delta}^n$ we introduce a new class of analytic functions as following:

Definition 1 For $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, the class $G_{\alpha,\beta,\lambda,\delta}^{n,b}(\varphi)$ consists of all functions $f(z) \in \mathcal{A}$ satisfying the following subordination:

$$1 + \frac{1}{b} \left( \frac{z \left( D_{\alpha,\beta,\lambda,\delta}^n f(z) \right)'}{D_{\alpha,\beta,\lambda,\delta}^n f(z)} - 1 \right) \prec \varphi(z), \quad (1.6)$$

$(\alpha, \beta, \lambda, \delta \geq 0; \beta > \alpha; \lambda > \delta; n \in \mathbb{N}_0; z \in U).$

Specializing the parameters $\alpha, \beta, \lambda, \delta, n, b$ and $\varphi(z)$, we obtain the following subclasses studied by various authors:

(i) $G_{\alpha,\beta,\lambda,\delta}^{n,1}(\varphi) = M_{\alpha,\beta,\lambda,\delta}^n(\varphi)$ (see Ramadan and Darus [18]);
(ii) $G_{0,1,1,0}^{n,b}(\varphi) = H_{n,b}(\varphi)$ (see Aouf and Silverman [4]);
(iii) $G_{0,1,1,0}^{b,b}(\varphi) = S_b^*(\varphi)$ and $G_{0,1,1,0}^{1,b}(\varphi) = C_b(\varphi)$
    (see Ravichandran et al. [19]);
(iv) $G_{0,1,1,0}^{n,b} \left( \frac{1+z}{1-z} \right) = S^n(b)$ (see Aouf et al. [2]);
(v) $G_{0,1,1,0}^{b,b} \left( \frac{1+z}{1-z} \right) = S(b)$ (see Nasr and Aouf [17] see also Aouf et al. [3]);
(vi) $G_{0,1,1,0}^{l,b} \left( \frac{1+z}{1-z} \right) = C(b)$ (see Nasr and Aouf [14] see also Aouf et al. [3]);
(vii) $G_{0,1,1,0}^{b,(1-\rho) \cos \eta e^{-i\theta}} \left( \frac{1+z}{1-z} \right) = S^n(\rho) \left( |\eta| < \frac{\pi}{2}, 0 \leq \rho < 1 \right)$
    (see Libera [10] see also Keogh and Merkes [9]);
(viii) $G_{0,1,1,0}^{l,(1-\rho) \cos \eta e^{-i\theta}} \left( \frac{1+z}{1-z} \right) = C^n(\rho) \left( |\eta| < \frac{\pi}{2}, 0 \leq \rho < 1 \right)$ (see Chichra [5]).

Also we note that for additional choices of parameters we have the following new subclasses of $\mathcal{A}$:
(i) \[ G^n_{\alpha,\beta,\lambda,\delta}\left(\frac{1+Az}{1+Bz}\right) = S^n_{\alpha,\beta,\lambda,\delta}(A,B) \]
\[ = \left\{ f(z) \in A : 1 + \frac{1}{b}\left(\frac{z(D^n_{\alpha,\beta,\lambda,\delta}f(z))'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right) < \frac{1+Az}{1+Bz} \right\} \]
\[ (-1 \leq B < A \leq 1; \alpha, \beta, \lambda, \delta \geq 0; \beta > \alpha; \lambda > \delta; n \in \mathbb{N}_0; z \in U) \}; \\

(ii) \[ G^n_{\alpha,\beta,\lambda,\delta}\left(\frac{1+(1-2\rho)z}{1-z}\right) = S^n_{\alpha,\beta,\lambda,\delta}(\rho) \]
\[ = \left\{ f(z) \in A : \text{Re}\left\{ 1 + \frac{1}{b}\left(\frac{z(D^n_{\alpha,\beta,\lambda,\delta}f(z))'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - 1\right) \right\} > \rho \right\} \]
\[ (\alpha, \beta, \lambda, \delta \geq 0; \beta > \alpha; \lambda > \delta; 0 \leq \rho < 1; n \in \mathbb{N}_0; z \in U) \}; \\

(iii) \[ G^n_{\alpha,\beta,\lambda,\delta}\left(\frac{1-(1-\rho)\cos\psi e^{-i\eta}}{1-z}\right) = S^n_{\rho,\eta}(\varphi) \]
\[ = \left\{ f(z) \in A : \frac{e^{i\eta}z(D^n_{\alpha,\beta,\lambda,\delta}f(z))'}{D^n_{\alpha,\beta,\lambda,\delta}f(z)} - \rho \cos \eta - i \sin \eta \prec \varphi(z) \right\} \]
\[ (|\eta| < \frac{\pi}{2}; \alpha, \beta, \lambda, \delta \geq 0; \beta > \alpha; \lambda > \delta; 0 \leq \rho < 1; n \in \mathbb{N}_0; z \in U) \}. \\

In this paper, we obtain the Fekete–Szegö inequalities for functions in the class \( G^n_{\alpha,\beta,\lambda,\delta}(\varphi) \).

2 Fekete–Szegö problem

Unless otherwise mentioned, we assume in the reminder of this paper that \( \alpha, \beta, \lambda, \delta \geq 0, \beta > \alpha, \lambda > \delta, b \in \mathbb{C}^* \) and \( z \in U \).

To prove our results, we shall need the following lemmas:

Lemma 1 [12] If \( p(z) = 1 + c_1z + c_2z^2 + \ldots \) (\( z \in U \)) is a function with positive real part in \( U \) and \( \mu \) is a complex number, then
\[ |c_2 - \mu c_1^2| \leq 2 \max\{1; |2\mu - 1|\}. \] (2.1)
The result is sharp for the functions given by

\[ p(z) = \frac{1 + z^2}{1 - z^2} \quad \text{and} \quad p(z) = \frac{1 + z}{1 - z} \quad (z \in U). \]  

(2.2)

**Lemma 2** [12] If \( p_1(z) = 1 + c_1z + c_2z^2 + \ldots \) is a function with positive real part in \( U \), then

\[ |c_2 - \nu c_1^2| \leq \begin{cases} 
-4\nu + 2, & \text{if } \nu \leq 0, \\
2, & \text{if } 0 \leq \nu \leq 1, \\
4\nu - 2, & \text{if } \nu \geq 1.
\end{cases} \]

When \( \nu < 0 \) or \( \nu > 1 \), the equality holds if and only if

\[ p_1(z) = \frac{1 + z}{1 - z} \]

or one of its rotations. If \( 0 < \nu < 1 \), then the equality holds if and only if

\[ p_1(z) = \frac{1 + z^2}{1 - z^2} \]

or one of its rotations. If \( \nu = 0 \), the equality holds if and only if

\[ p_1(z) = \left( \frac{1}{2} + \frac{1}{2} \right) \frac{1 + z}{1 - z} + \left( \frac{1}{2} - \frac{1}{2} \gamma \right) \frac{1 - z}{1 + z} \quad (0 \leq \gamma \leq 1), \]

or one of its rotations. If \( \nu = 1 \), the equality holds if and only if

\[ \frac{1}{p_1(z)} = \left( \frac{1}{2} + \frac{1}{2} \gamma \right) \frac{1 + z}{1 - z} + \left( \frac{1}{2} - \frac{1}{2} \gamma \right) \frac{1 - z}{1 + z} \quad (0 \leq \gamma \leq 1). \]

Also the above upper bound is sharp and it can be improved as follows when \( 0 < \nu < 1 \):

\[ |c_2 - \nu c_1^2| + \nu |c_1|^2 \leq 2 \quad (0 < \nu < \frac{1}{2}), \]

and

\[ |c_2 - \nu c_1^2| + (1 - \nu) |c_1|^2 \leq 2 \quad \left( \frac{1}{2} < \nu < 1 \right). \]

Using Lemma 1, we have the following theorem:

**Theorem 1** Let \( \varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots \), where \( \varphi(z) \in A \) and \( \varphi'(0) > 0 \). If \( f(z) \) given by (1.1) belongs to the class \( G_{\alpha, \beta, \lambda, \delta}(\varphi) \) and if \( \mu \) is a complex number, then

\[ |a_3 - \mu a_2^2| \leq \frac{|b|B_1}{2[2(\lambda - \delta)(\beta - \alpha) + 1]^n} \times \max \left\{ 1, \left| \frac{B_2}{B_1} + \left( 1 - \frac{2[2(\lambda - \delta)(\beta - \alpha) + 1]^n}{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n}} \right) \mu \right| bB_1 \right\}. \]

(2.3)

The result is sharp.
Definethefunction
\[ p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1 z + c_2 z^2 + \ldots \] (2.5)

Since \( w(z) \) is a Schwarz function, we see that \( \text{Re} \{ p_1(z) \} > 0 \) and \( p_1(0) = 1 \).

Define the function \( p(z) \) by:
\[
p(z) = 1 + \left( \frac{z}{1 - w(z)} \right) = 1 + b_1 z + b_2 z^2 + \ldots \] (2.6)

In view of the equations (2.4), (2.5) and (2.6), we have
\[
p(z) = \varphi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = \varphi \left( \frac{c_1 z + c_2 z^2 + \ldots}{2 + c_1 z + c_2 z^2 + \ldots} \right)
\]
\[
= \varphi \left( \frac{1}{2} c_1 z + \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \ldots \right)
\]
\[
= 1 + \frac{1}{2} B_1 c_1 z + \left[ \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2 \right] z^2 + \ldots \] (2.7)

Thus
\[
b_1 = \frac{1}{2} B_1 c_1 \quad \text{and} \quad b_2 = \frac{1}{2} B_1 \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} B_2 c_1^2. \] (2.8)

Since
\[
1 + \frac{1}{b} \left( \frac{z(D^n_{\alpha,\beta,\lambda,\delta} f(z))'}{D^n_{\alpha,\beta,\lambda,\delta} f(z)} - 1 \right) = 1 + \left\{ \frac{1}{b} \left[ (\lambda - \delta) (\beta - \alpha) + 1 \right] a_2 \right\} z
\]
\[
+ \left( \frac{1}{b} \left( 2 \left[ 2 (\lambda - \delta) (\beta - \alpha) + 1 \right] a_3 - 2 \left( \lambda - \delta \right) (\beta - \alpha) + 1 \right] a_2 \right) z^2 + \ldots
\]

Then from (2.6) and (2.8), we obtain
\[
a_2 = \frac{b B_1 c_1}{2 [(\lambda - \delta) (\beta - \alpha) + 1]^n}, \] (2.9)

and
\[
a_3 = \frac{b B_1 c_2}{4 [2 (\lambda - \delta) (\beta - \alpha) + 1]^n} + \frac{c_1^2}{8 [2 (\lambda - \delta) (\beta - \alpha) + 1]^n} \left[ b^2 B_1^2 - b (B_1 - B_2) \right]. \] (2.10)
Fekete–Szegő problem for a new class of analytic functions... Therefore, we have
\[ a_3 - \mu a_2^2 = \frac{b B_1}{4 \left[ 2 (\lambda - \delta) (\beta - \alpha) + 1 \right]^n} \left[ c_2 - \nu c_1^2 \right], \quad (2.11) \]
where
\[ \nu = \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + \left( \frac{2 \left[ 2 (\lambda - \delta) (\beta - \alpha) + 1 \right]^n}{\left[ (\lambda - \delta) (\beta - \alpha) + 1 \right]^{2n}} \mu - 1 \right) b B_1 \right]. \quad (2.12) \]

Our result now follows by an application of Lemma 1. The result is sharp for the function \( f(z) \) given by
\[ 1 + \frac{1}{b} \left( \frac{z \left( D^n_{\alpha,\beta,\lambda,\delta} f(z) \right)'}{D^n_{\alpha,\beta,\lambda,\delta} f(z)} - 1 \right) = \varphi(z^2), \quad (2.13) \]
or
\[ 1 + \frac{1}{b} \left( \frac{z \left( D^n_{\alpha,\beta,\lambda,\delta} f(z) \right)'}{D^n_{\alpha,\beta,\lambda,\delta} f(z)} - 1 \right) = \varphi(z). \quad (2.14) \]

This completes the proof of Theorem 1. \( \Box \)

**Remark 2** (i) Taking \( n = 0 \) in Theorem 1, we improve the result obtained by Ravichandran et al. [19, Theorem 4.1];
(ii) Taking \( \alpha = \delta = 0, \beta = \lambda = 1, b = (1 - \rho) \cos \eta e^{-i \eta} (|\eta| < \frac{\pi}{2}, 0 \leq \rho < 1) \) and \( \varphi(z) = \frac{1 + z}{1 - z} \) (equivalently \( B_1 = B_2 = 2 \)) in Theorem 1, we obtain the result obtained by Goyal and Kumar [8, Corollary 2.10];
(iii) Taking \( b = (1 - \rho) \cos \eta e^{-i \eta} (|\eta| < \frac{\pi}{2}, 0 \leq \rho < 1), n = 0 \) and \( \varphi(z) = \frac{1 + z}{1 - z} \) in Theorem 1, we obtain the result obtained by Keogh and Merkes [9, Thm 1];
(iv) Taking \( \alpha = \delta = 0 \) and \( \beta = \lambda = 1 \) in Theorem 1, we obtain the result obtained by Aouf and Silverman [4, Theorem 1].

Also by specializing the parameters in Theorem 1, we obtain the following new sharp results.

Putting \( b = 1 \) in Theorem 1, we obtain the following corollary:

**Corollary 1** If \( f(z) \) given by (1.1) belongs to the class \( M^n_{\alpha,\beta,\lambda,\delta}(\varphi) \), then for any complex number \( \mu \), we have
\[ |a_3 - \mu a_2^2| \leq \frac{B_1}{2 \left[ 2 (\lambda - \delta) (\beta - \alpha) + 1 \right]^n} \times \max \left\{ 1, \left| \frac{B_2}{B_1} + \left( 1 - \frac{2 \left[ 2 (\lambda - \delta) (\beta - \alpha) + 1 \right]^n}{\left[ (\lambda - \delta) (\beta - \alpha) + 1 \right]^{2n}} \mu \right) B_1 \right| \right\}. \quad (2.15) \]
The result is sharp.
Corollary 2 If $f(z)$ given by (1.1) belongs to the class $S^{a,b}_{\alpha,\beta,\lambda,\delta}(A, B)$, then for any complex number $\mu$, we have

$$|a_3 - \mu a_2^2| \leq \frac{(A - B) |b|}{2[2(\lambda - \delta) (\beta - \alpha) + 1]^n} \times \max \left\{ 1, \left| 1 - \frac{2[2(\lambda - \delta) (\beta - \alpha) + 1]^n}{[2(\lambda - \delta) (\beta - \alpha) + 1]^{2n\mu}} \right| (A - B) b - B \right\}.$$  \hspace{1cm} (2.16)

The result is sharp.

Putting $\varphi(z) = \frac{1 + (1 - 2\rho)z}{1 - z}$ ($0 \leq \rho < 1$) in Theorem 1, we obtain the following corollary:

Corollary 3 If $f(z)$ given by (1.1) belongs to the class $S^{n,b}_{\alpha,\beta,\lambda,\delta}(\rho)$, then for any complex number $\mu$, we have

$$|a_3 - \mu a_2^2| \leq \frac{(1 - \rho) |b|}{2(\lambda - \delta) (\beta - \alpha) + 1]^n} \times \max \left\{ 1, \left| 2 \left( 1 - \frac{2[2(\lambda - \delta) (\beta - \alpha) + 1]^n}{[2(\lambda - \delta) (\beta - \alpha) + 1]^{2n\mu}} \right) (1 - \rho) b + 1 \right| \right\}.$$  \hspace{1cm} (2.17)

The result is sharp.

Putting $b = (1 - \rho) \cos \eta e^{-i\theta}$ ($|\eta| < \frac{\pi}{2}$, $0 \leq \rho < 1$) in Theorem 1, we obtain the following corollary:

Corollary 4 If $f(z)$ given by (1.1) belongs to the class $S^{a,\rho,\eta}_{\alpha,\beta,\lambda,\delta}(\varphi)$, then for any complex number $\mu$, we have

$$|a_3 - \mu a_2^2| \leq \frac{B_1 (1 - \rho) \cos \eta}{2[2(\lambda - \delta) (\beta - \alpha) + 1]^n} \times \max \left\{ 1, \left| \frac{B_2}{B_1} e^{i\eta} \left( 1 - \frac{2[2(\lambda - \delta) (\beta - \alpha) + 1]^n}{[2(\lambda - \delta) (\beta - \alpha) + 1]^{2n\mu}} \right) (1 - \rho) B_1 \cos \eta \right| \right\}.$$  \hspace{1cm} (2.18)

The result is sharp.

Putting $\alpha = \delta = 0$, $\beta = \lambda = 1$ and $\varphi(z) = \frac{1 + \eta x}{1 - x}$ in Theorem 1, we obtain the result of Aouf et al. [2, Theorem 3, with $m = 1$]:

Corollary 5 If $f(z)$ given by (1.1) belongs to the class $S^{a}(b)$, then for any complex number $\mu$, we have

$$|a_3 - \mu a_2^2| \leq \frac{|b|}{3^a} \max \{ 1, |1 + 2 \left( \frac{3}{4} \right)^n \mu b| \}.$$  \hspace{1cm} (2.19)

The result is sharp.
Putting $n = 0$ and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 1, we obtain the result of and Nasr and Aouf [17, Theorem 2] see also Nasr and Aouf [16, Theorem 1, with $m = 1$]:

**Corollary 6** If $f(z)$ given by (1.1) belongs to the class $S(b)$, then for any complex number $\mu$, we have

$$|a_3 - \mu a_2^2| \leq |b| \max \{1, |1 + 2(1 - 2\mu)b|\}.$$ \hspace{1cm} (2.20)

The result is sharp.

Putting $\alpha = \delta = 0$, $\beta = \lambda = 1$, $n = 0$, $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 1, we obtain the result of Keogh and Merkes [9, Theorem 1]:

**Corollary 8** If $f(z)$ given by (1.1) belongs to the class $S^\eta(\rho)$, then for any complex number $\mu$, we have

$$|a_3 - \mu a_2^2| \leq (1 - \rho)\cos \eta \max \{1, \left|2(2\mu - 1)(1 - \rho)\cos \eta - e^{i\eta}\right|\}.$$ \hspace{1cm} (2.22)

The result is sharp.

Using Lemma 2, we have the following theorem:
Theorem 2 Let \( \varphi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots \) (\( b > 0; B_i > 0; i \in \mathbb{N} \)).

Also let 
\[
\sigma_1 = \frac{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n} (B_2 - B_1 + bB_1^2)}{2 [2(\lambda - \delta)(\beta - \alpha) + 1]^n bB_1^2},
\]
and 
\[
\sigma_2 = \frac{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n} (B_2 + B_1 + bB_1^2)}{2 [2(\lambda - \delta)(\beta - \alpha) + 1]^n bB_1^2}.
\]

If \( f(z) \) is given by (1.1) belongs to the class \( G_{a,b}^{n,b}(\varphi) \), then we have the following sharp results:

(i) If \( \mu \leq \sigma_1 \), then
\[
|a_3 - \mu a_2^2| \leq \frac{b}{2 [2(\lambda - \delta)(\beta - \alpha) + 1]^n}
\times \left\{ B_2 - \left( \frac{2 [2(\lambda - \delta)(\beta - \alpha) + 1]^n}{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n} \mu - 1} \right) bB_1^2 \right\}. \tag{2.24}
\]

(ii) If \( \sigma_1 \leq \mu \leq \sigma_2 \), then
\[
|a_3 - \mu a_2^2| \leq \frac{bB_1}{2 [2(\lambda - \delta)(\beta - \alpha) + 1]^n}. \tag{2.25}
\]

(iii) If \( \mu \geq \sigma_2 \), then
\[
|a_3 - \mu a_2^2| \leq \frac{b}{2 [2(\lambda - \delta)(\beta - \alpha) + 1]^n}
\times \left\{ -B_2 + \left( \frac{2 [2(\lambda - \delta)(\beta - \alpha) + 1]^n}{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n} \mu - 1} \right) bB_1^2 \right\}. \tag{2.26}
\]

Proof For \( f(z) \in G_{a,b}^{n,b}(\varphi) \), \( p(z) \) given by (2.6) and \( p_1(z) \) given by (2.5), then \( a_2 \) and \( a_3 \) are given as same as in Theorem 1. Also
\[
a_3 - \mu a_2^2 = \frac{bB_1}{4 [2(\lambda - \delta)(\beta - \alpha) + 1]^n} \left[ c_2 - \nu c_1^2 \right], \tag{2.27}
\]
where
\[
\nu = \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + \left( \frac{2 [2(\lambda - \delta)(\beta - \alpha) + 1]^n}{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n} \mu - 1} \right) bB_1 \right]. \tag{2.28}
\]

First, if \( \mu \leq \sigma_1 \), then we have \( \nu \leq 0 \), then by applying Lemma 2 to equality (2.27), we have
\[
|a_3 - \mu a_2^2| \leq \frac{b}{2 [2(\lambda - \delta)(\beta - \alpha) + 1]^n}
\times \left\{ B_2 - \left( \frac{2 [2(\lambda - \delta)(\beta - \alpha) + 1]^n}{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n} \mu - 1} \right) bB_1^2 \right\},
\]

which is evidently inequality (2.24) of Theorem 2.
If \( \mu = \sigma_1 \), then we have \( \nu = 0 \), therefore equality holds if and only if
\[
p_1(z) = \left( \frac{1 + \gamma}{2} \right) \frac{1 + z}{1 - z} + \left( \frac{1 - \gamma}{2} \right) \frac{1 - z}{1 + z} \quad (0 \leq \gamma \leq 1; \ z \in U).
\]
Next, if \( \sigma_1 \leq \mu \leq \sigma_2 \), we note that
\[
\max \left\{ \frac{1}{2} \left[ 1 - \frac{B_2}{B_1} + \left( \frac{2[2(\lambda - \delta)(\beta - \alpha) + 1]^n}{[(\lambda - \delta)(\beta - \alpha) + 1]^{2m}} \mu - 1 \right) bB_1 \right] \right\} \leq 1, \quad (2.29)
\]
then applying Lemma 2 to equality (2.27), we have
\[
|a_3 - \mu a_2^2| \leq \frac{bB_1}{2[2(\lambda - \delta)(\beta - \alpha) + 1]^n},
\]
which is evidently inequality (2.25) of Theorem 2.
If \( \sigma_1 < \mu < \sigma_2 \), then we have
\[
p_1(z) = \frac{1 + z^2}{1 - z^2}.
\]
Finally, if \( \mu \geq \sigma_2 \), then we have \( \nu \geq 1 \), therefore by applying Lemma 2 to (2.27), we have
\[
\leq \frac{b}{2[2(\lambda - \delta)(\beta - \alpha) + 1]^n} \left\{ -B_2 + \left( \frac{2[2(\lambda - \delta)(\beta - \alpha) + 1]^n}{[(\lambda - \delta)(\beta - \alpha) + 1]^{2m}} \mu - 1 \right) bB_1 \right\},
\]
which is evidently inequality (2.26) of Theorem 2.
If \( \mu = \sigma_2 \), then we have \( \nu = 1 \), therefore by applying Lemma 2 to (2.27), we have
\[
\frac{1}{p_1(z)} = \left( \frac{1 + \gamma}{2} \right) \frac{1 + z}{1 - z} + \left( \frac{1 - \gamma}{2} \right) \frac{1 - z}{1 + z} \quad (0 \leq \gamma \leq 1; \ z \in U).
\]
To show that the bounds are sharp, we define the functions \( K_\varphi^s(s \geq 2) \) by
\[
1 + \frac{1}{b} \left( \frac{z(D_{\alpha,\beta,\lambda,\delta}^a K_\varphi(z))'}{D_{\alpha,\beta,\lambda,\delta}^a K_\varphi(z)} - 1 \right) = \varphi(z^{s-1}), \quad K_\varphi(0) = 0 = K_\varphi^s(0) - 1, \quad (2.30)
\]
and the functions \( F_t \) and \( G_t \) \( (0 \leq t \leq 1) \) by
\[
1 + \frac{1}{b} \left( \frac{z(D_{\alpha,\beta,\lambda,\delta}^a F_t(z))'}{D_{\alpha,\beta,\lambda,\delta}^a F_t(z)} - 1 \right) = \varphi \left( \frac{z(t + t)}{1 + tz} \right), \quad F_t(0) = 0 = F_t^t(0) - 1, \quad (2.31)
\]
and

\[ 1 + \frac{1}{b} \left( \frac{z(D^n_{\alpha,\beta,\lambda,\gamma} G_t(z))'}{D^n_{\alpha,\beta,\lambda,\gamma} G_t(z)} - 1 \right) = \varphi \left( - \frac{z(z + t)}{1 + tz} \right), \quad G_t(0) = 0 = G_t'(0) - 1. \quad (2.32) \]

Clearly the functions \( K_{\varphi}, F_t \) and \( G_t \in C^{n,b}_{\alpha,\beta,\lambda,\gamma}(\varphi) \). Also we write \( K_{\varphi} = K_{\varphi}^2 \).

If \( \mu < \sigma_1 \) or \( \mu > \sigma_2 \), then the equality holds if and only if \( f \) is \( K_{\varphi}^2 \) or one of its rotations. When \( \sigma_1 < \mu < \sigma_2 \), then the equality holds if \( f \) is \( K_{\varphi}^3 \) or one of its rotations. If \( \mu = \sigma_1 \), then the equality holds if and only if \( f \) is \( G_t \) or one of its rotations. If \( \mu = \sigma_2 \), then the equality holds if and only if \( f \) is \( G_t \) or one of its rotations.

**Remark 3**

(i) Taking \( b = 1 \) in Theorem 2, we improve the result obtained by Ramadan and Darus [18, Theorem 1];

(ii) Taking \( \alpha = \delta = 0 \) and \( \beta = \lambda = 1 \) in Theorem 2, we obtain the result obtained by Goyal and Kumar [8, Corollary 2.7] and Aouf and Silverman [4, Theorem 2].

Also, using Lemma 2 we have the following theorem:

**Theorem 3** For \( \varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots \) \((b > 0; B_i > 0; i \in \mathbb{N})\) and \( f(z) \) given by (1.1) belongs to the class \( C^{n,b}_{\alpha,\beta,\lambda,\gamma}(\varphi) \) and \( \sigma_1 \leq \mu \leq \sigma_2 \), then in view of Lemma 2, Theorem 2 can be improved. Let

\[ \sigma_3 = \frac{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n}}{2 \left[ 2(\lambda - \delta)(\beta - \alpha) + 1 \right]^{2n} bB_1^2}. \]

(i) If \( \sigma_1 \leq \mu \leq \sigma_3 \), then

\[
|a_3 - \mu a_2^2| + \frac{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n}}{2 \left[ 2(\lambda - \delta)(\beta - \alpha) + 1 \right]^{2n} bB_1^2} \left\{ 1 - \frac{B_2}{B_1} + \left( \frac{2[(\lambda - \delta)(\beta - \alpha) + 1]^{2n}}{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n}} - 1 \right) bB_1 \right\} |a_2|^2
\]

\[ \leq \frac{bB_1^2}{2 \left[ 2(\lambda - \delta)(\beta - \alpha) + 1 \right]^{2n}}. \quad (2.33) \]

(ii) If \( \sigma_3 \leq \mu \leq \sigma_2 \), then

\[
|a_3 - \mu a_2^2| + \frac{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n}}{2 \left[ 2(\lambda - \delta)(\beta - \alpha) + 1 \right]^{2n} bB_1^2} \left\{ 1 + \frac{B_2}{B_1} - \left( \frac{2[(\lambda - \delta)(\beta - \alpha) + 1]^{2n}}{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n}} - 1 \right) bB_1 \right\} |a_2|^2
\]

\[ \leq \frac{bB_1^2}{2 \left[ 2(\lambda - \delta)(\beta - \alpha) + 1 \right]^{2n}}. \quad (2.34) \]
Proof For the values of $\sigma_1 \leq \mu \leq \sigma_3$, we have

\[
|a_3 - \mu a_2^2| + (\mu - \sigma_1)|a_2|^2 = \frac{bB_1}{4[2(\lambda - \delta)(\beta - \alpha) + 1]^n}|c_2 - \nu c_1^2|
\]

\[
+ \left( \mu - \frac{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n} (B_2 - B_1 + bB_1^2)}{2[2(\lambda - \delta)(\beta - \alpha) + 1]^n bB_1^2} \right) \frac{b^2B_1^2}{4[(\lambda - \delta)(\beta - \alpha) + 1]^{2n}} |c_1|^2
\]

\[
= \frac{bB_1}{2[2(\lambda - \delta)(\beta - \alpha) + 1]^n} \left\{ \frac{1}{2} \left( |c_2 - \nu c_1^2| + \nu |c_1|^2 \right) \right\}. \tag{2.35}
\]

Now apply Lemma 2 to equality (2.35), then we have

\[
|a_3 - \mu a_2^2| + (\mu - \sigma_1)|a_2|^2 \leq \frac{bB_1}{2[2(\lambda - \delta)(\beta - \alpha) + 1]^n},
\]

which is evidently inequality (2.33) of Theorem 3.

Next, for the values of $\sigma_3 \leq \mu \leq \sigma_2$, we have

\[
|a_3 - \mu a_2^2| + (\sigma_2 - \mu)|a_2|^2 = \frac{bB_1}{4[2(\lambda - \delta)(\beta - \alpha) + 1]^n}|c_2 - \nu c_1^2|
\]

\[
+ \left( \frac{[(\lambda - \delta)(\beta - \alpha) + 1]^{2n} (B_2 + B_1 - bB_1^2)}{2[2(\lambda - \delta)(\beta - \alpha) + 1]^n bB_1^2} - \mu \right) \frac{b^2B_1^2}{4[(\lambda - \delta)(\beta - \alpha) + 1]^{2n}} |c_1|^2
\]

\[
= \frac{bB_1}{2[2(\lambda - \delta)(\beta - \alpha) + 1]^n} \left\{ \frac{1}{2} \left( |c_2 - \nu c_1^2| + (1 - \nu) |c_1|^2 \right) \right\}. \tag{2.36}
\]

Now apply Lemma 2 to equality (2.36), then we have

\[
|a_3 - \mu a_2^2| + (\sigma_2 - \mu)|a_2|^2 \leq \frac{bB_1}{2[2(\lambda - \delta)(\beta - \alpha) + 1]^n},
\]

which is evidently inequality (2.34). This completes the proof of Theorem 3. □

Remark 4 (i) taking $\alpha = \delta = 0$ and $\beta = \lambda = 1$ in Theorem 3, we improve the result obtained by Goyal and Kumar [8, Remark 2.8];

(ii) taking $b = 1$ in Theorem 3, we improve the result obtained by Ramadan and Darus [18, Remark 2].

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References


