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Interior and Closure Operators on Commutative Bounded Residuated Lattices*

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Abstract

Commutative bounded integral residuated lattices form a large class of algebras containing some classes of algebras behind many valued and fuzzy logics. In the paper we introduce and investigate additive closure and multiplicative interior operators on this class of algebras.

Key words: residuated lattice, bounded integral residuated lattice, interior operator, closure operator

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1 Introduction

Commutative bounded integral residuated lattices form a large class of algebras containing some classes of algebras behind many valued and fuzzy logics, such as MV-algebras [2], BL-algebras [9], MTL-algebras [7] and commutative $R\ell$-monoids [12], [6]. Moreover, Heyting algebras [1] which are algebras of the intuitionistic logic can be also viewed as commutative bounded integral lattices.

Topological Boolean algebras, i.e. closure or interior algebras [15], are generalizations of topological spaces defined by means of topological closure and interior operators. In [13] closure and interior MV-algebras as generalizations

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of topological Boolean algebras were introduced by means of so-called additive closure and multiplicative interior operators. It is known that every MV-algebra M contains the greatest Boolean subalgebra $B(M)$ of all complemented elements. By [13], the restriction of any additive closure operator on M onto $B(M)$ is a topological closure operator on $B(M)$. Moreover, if M is a complete MV-algebra, then every topological closure operator on $B(M)$ can be extended to an additive closure operator on M. Since the addition and multiplication of MV-algebras are mutually dual operations, analogous properties are also true for multiplicative interior operators on M and $B(M)$.

The notions of additive closure and multiplicative interior operators (ac- and mi- operators, for short) were generalized in [14] to commutative residuated ℓ-monoids (= commutative $R\ell$-monoids), i.e. commutative bounded integral residuated lattices satisfying divisibility [11], [8]. But the dual operation to multiplication in such residuated lattices does not exist in general. Hence, connections between mi- and ac-operators are more complicated than those in the case of MV-algebras.

In the paper we introduce and investigate analogous operators on arbitrary commutative bounded integral residuated lattices. We describe connections between mi-operators and ac-operators in this general setting. Moreover, we generalize the notions of mi- and ac-operators to so-called weak mi-operators and strong ac-operators and show that there is an antitone Galois connection between them. Furthermore, we describe, for residuated lattices with Glivenko property, connections between mi- and ac- operators on them and on the residuated lattices of their regular elements.

2 Preliminaries

A *commutative bounded integral residuated lattice* is an algebra

$$M = (M; \odot, \vee, \wedge, \rightarrow, 0, 1)$$

of type $(2, 2, 2, 2, 0, 0)$ satisfying the following conditions:

(i) $(M; \odot, 1)$ is a commutative monoid,

(ii) $(M; \vee, \wedge, 0, 1)$ is a bounded lattice,

(iii) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ for all $x, y, z \in M$.

In what follows, by a *residuated lattice* we will mean *a commutative bounded integral residuated lattice.*

For any residuated lattice M we define a unary operation (negation) $^-$ on M such that $x^- := x \rightarrow 0$.

Recall that algebras of logics mentioned in Introduction are characterized in the class of residuated lattices as follows:

A residuated lattice M is

(a) an MTL-algebra if M satisfies the identity of pre-linearity

(iv) $(x \rightarrow y) \vee (y \rightarrow x) = 1$;

(b) involutive if M satisfies the identity of double negation

(v) $x^{--} = x$;

(c) an $R\ell$-monoid (or a bounded commutative GBL-algebra) if M satisfies the identity of divisibility

(vi) $(x \to y) \odot x = x \wedge y$;

(d) a BL-algebra if M satisfies both (iv) and (vi);
(e) an MV-algebra if M is an involutive BL-algebra;
(f) a Heyting algebra if the operations "$\odot$" and "$\wedge$" coincide.

Proposition 2.1 [4, 11] *Let M be a residuated lattice. Then for any $x, y, z \in M$ we have:*

(i) $x \le y \Longrightarrow y^- \le x^-$,

(ii) $x \odot y \le x \wedge y$,

(iii) $(x \to y) \odot x \le y$,

(iv) $x \le x^{--}$,

(v) $x^{---} = x^-$,

(vi) $x \to (y \to z) = y \to (x \to z)$,

(vii) $x \to (y \to z) = (x \odot y) \to z$,

(viii) $x \le y \Longrightarrow z \to x \le z \to y$,

(ix) $x \le y \Longrightarrow y \to z \le x \to z$,

(x) $y \to z \le (x \to y) \to (x \to z)$,

(xi) $x \to y \le (y \to z) \to (x \to z)$.

(xii) $x^{--} \to y^{--} = x \to y^{--}$,

(xiii) $(x \to y^{--})^{--} = x \to y^{--}$,

(xiv) $(x \odot y)^- = y \to x^- = x \to y^- = x^{--} \to y^- = y^{--} \to x^-$,

(xv) $(x \odot y)^{--} \ge x^{--} \odot y^{--}$.

Let M be a residuated lattice. We define a binary operation $\oplus$ on M as follows:

$$x \oplus y = (x^- \odot y^-)^-.$$

Lemma 2.2 [4] *Let M be a residuated lattice. For any $x, y \in M$ we have*

(i) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,

(ii) $x \oplus y \ge x^{--} \vee y^{--} \ge x \vee y$,

(iii) $x \oplus 0 = x^{--}$,

(iv) $(x \oplus y)^{--} = x^{--} \oplus y^{--} = x \oplus y$,

(v) $x \odot x^- = 0, \quad x \oplus x^- = 1$.

We call a residuated lattice M *normal* if it satisfies the identity

$$(x \odot y)^{--} = x^{--} \odot y^{--}.$$

For example, every involutive residuated lattice, every Heyting algebra and every BL-algebra is normal [5] (note that the name "normal" is sometimes used for non-commutative residuated lattices where all filters are normal, see [10]).

Similarly as in [14] for residuated ℓ-monoids we can prove the following identities.

Lemma 2.3 *Let M be a normal residuated lattice. Then for any $x, y \in M$*

(i) $(x \oplus y)^- = x^- \odot y^-$,

(ii) $(x \odot y)^- = x^- \oplus y^-$.

Proof (i) Since M is normal, we have

$$(x \oplus y)^- = (x^- \odot y^-)^{--} = x^{---} \odot y^{---} = x^- \odot y^-.$$

(ii) By Lemma 2.2 (iv), we have

$$\begin{aligned} x^- \oplus y^- &= (x^- \oplus y^-)^{--} = ((x^{--} \odot y^{--})^-)^{--} \\ &= (x^{--} \odot y^{--})^- = (x \odot y)^{---} = (x \odot y)^-. \end{aligned}$$

□

3 Connections between interior and closure operators

Definition 3.1 Let M be a residuated lattice. A mapping $f\colon M \to M$ is called a *multiplicative interior operator (mi-operator)* on M if for any $x, y \in M$

(1) $f(x \odot y) = f(x) \odot f(y)$,

(2) $f(x) \leq x$,

(3) $f(f(x)) = f(x)$,

(4) $f(1) = 1$.

(5) $x \leq y \Rightarrow f(x) \leq f(y)$.

Remark 3.2 If M is an $R\ell$-monoid, i.e. a residuated lattice satisfying

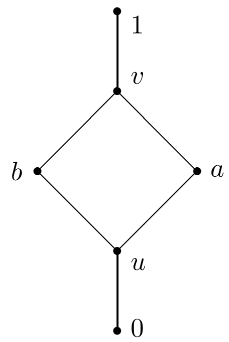
$$x \odot (x \to y) = x \wedge y$$

for any $x, y \in M$, then it can be shown [14] that the property 5 from the definition follows from properties 1–4.

Example 3.3 Let $M_1 = \{0, u, a, b, v, 1\}$. We define the operations $\odot$ and $\to$ on M_1 as follows:

$\odot$	0	u	a	b	v	1
0	0	0	0	0	0	0
u	0	0	0	0	0	u
a	0	0	a	0	a	a
b	0	0	0	b	b	b
v	0	0	a	b	v	v
1	0	u	a	b	v	1

$\to$	0	u	a	b	v	1
0	1	1	1	1	1	1
u	v	1	1	1	1	1
a	b	b	1	b	1	1
b	a	a	a	1	1	1
v	u	u	a	b	1	1
1	0	u	a	b	v	1



Then M_1 is an involutive normal residuated lattice in which pre-linearity and divisibility are not satisfied since we have $(a \to b) \vee (b \to a) = b \vee a \neq 1$, and $v \odot (v \to u) = v \odot u = 0 \neq u = v \wedge u$. However, we get $x^{--} = x$ for all $x \in M_1$.

Let $f_1\colon M_1 \to M_1$ be the mapping such that $f_1(0) = 0$, $f_1(u) = u$, $f_1(a) = a$, $f_1(b) = 0$, $f_1(v) = v$, $f_1(1) = 1$. Then the mapping f_1 satisfies the conditions 1–4 from the definition of an mi-operator, but the mapping f_1 is not monotone since $u < b$, whereas $f_1(u) \nleq f_1(b)$.

Example 3.4 Let M_1 be the residuated lattice from Example 3.3. Let us consider the mapping $f_2\colon M_1 \to M_1$ such that $f_2(0) = f_2(u) = f_2(a) = f_2(b) = 0$, $f_2(v) = v$, $f_2(1) = 1$. Then f_2 is an mi-operator on M_1.

Lemma 3.5 *Let f be an mi-operator on a residuated lattice M. Then for any $x, y \in M$*

$$f(x \to y) \leq f(x) \to f(y).$$

Proof Let $x, y \in M$. Then $(x \to y) \odot x \leq y$ and we have $f(x \to y) \odot f(x) = f((x \to y) \odot x) \leq f(y)$. Thus $f(x \to y) \leq f(x) \to f(y)$. □

Let $f\colon M \to M$ be a mapping on a residuated lattice M. We define a mapping $f^-\colon M \to M$ such that

$$f^-(x) = (f(x^-))^-,$$

for any $x \in M$.

Proposition 3.6 *If $f: M \to M$ is a monotone mapping on a residuated lattice M, then the mapping f^- is monotone, too.*

Proof Let $x, y \in M$ be such that $x \leq y$. Then by Proposition 2.1 $y^- \leq x^-$, so $f(y^-) \leq f(x^-)$. Therefore $(f(x^-))^- \leq (f(y^-))^-$ or equivalently $f^-(x) \leq f^-(y)$. □

Proposition 3.7 *Let M be a residuated lattice. If f is an mi-operator on M and $x, y \in M$, then*

(i) $x \leq f^-(x)$,

(ii) $f^-(f^-(x)) = f^-(x)$,

(iii) $f^-(0) = 0$,

(iv) $x \leq y \Longrightarrow f^-(x) \leq f^-(y)$.

Proof (i) If $x \in M$, then $f^-(x) = (f(x^-))^- \geq x^{--} \geq x$.

(ii) For any $x \in M$ we have $f^-(f^-(x)) = f^-((f(x^-))^-) = (f(f(x^-))^{--})^-$ and $f(x^-) \leq (f(x^-))^{--}$ by Proposition 2.1. Since f is monotone $f(f(x^-)) = f(x^-) \leq f((f(x^-))^{--})$, thus $(f(x^-))^- \geq (f((f(x^-))^{--}))^-$, and $f^-(x) \geq f^-(f^-(x))$. By (i) we also have $f^-(x) \leq f^-(f^-(x))$. Thus $f^-(f^-(x)) = f^-(x)$.

(iii) $f^-(0) = (f(0^-))^- = (f(1))^- = 1^- = 0$.

(iv) It follows from Proposition 3.6. □

Proposition 3.8 *Let M be a normal residuated lattice and f be an mi-operator on M. Then the mapping f^- satisfies the identity*

$$f^-(x \oplus y) = f^-(x) \oplus f^-(y).$$

Proof Let $x, y \in M$. Then $f^-(x) \oplus f^-(y) = ((f^-(x))^- \odot (f^-(y))^-)^- = ((f(x^-))^{--} \odot (f(y^-))^{--})^- = (f(x^-) \odot f(y^-))^{---} = (f(x^-) \odot f(y^-))^- = (f(x^- \odot y^-))^- = (f((x \oplus y)^-))^- = f^-(x \oplus y)$. □

Definition 3.9 Let M be a residuated lattice. A mapping $g: M \to M$ is called an *additive closure operator (ac-operator)* on M if for any $x, y \in M$

(1) $g(x \oplus y) = g(x) \oplus g(y)$,

(2) $x \leq g(x)$,

(3) $g(g(x)) = g(x)$,

(4) $g(0) = 0$,

(5) $x \leq y \Rightarrow g(x) \leq g(y)$.

Proposition 3.10 *If M is a normal residuated lattice and f is an mi-operator on M, then the mapping f^- is an ac-operator on M.*

Proof It follows from Propositions 3.7 and 3.8. □

Lemma 3.11 *If M is a residuated lattice and g is an ac-operator on M, then g satisfies the identity*

$$g(x^{--}) = (g(x))^{--}.$$

Proof By Lemma 2.2 (iii), we have $g(x^{--}) = g(x \oplus 0) = g(x) \oplus g(0) = g(x) \oplus 0 = (g(x))^{--}$. □

Proposition 3.12 *Let M be a normal residuated lattice and g be an ac-operator on M. Then we have for any $x, y \in M$*

(i) $g^-(x \odot y) = g^-(x) \odot g^-(y)$,

(ii) $g^-(x) \leq x^{--}$,

(iii) $g^-(g^-(x)) = g^-(x)$,

(iv) $g^-(1) = 1$,

(v) $x \leq y \Longrightarrow g^-(x) \leq g^-(y)$.

Proof (i) Let $x, y \in M$. Then we have

$$g^-(x \odot y) = (g((x \odot y)^-))^-,$$

and by Lemma 2.3 we get

$$(g((x \odot y)^-))^- = (g(x^-) \oplus g(y^-))^- = (g(x^-))^- \odot (g(y^-))^- = g^-(x) \odot g^-(y).$$

(ii) Since $x^- \leq g(x^-)$, we have $(g(x^-))^- = g^-(x) \leq x^{--}$.

(iii) By Lemma 3.11,

$$g^-(g^-(x)) = (g((g(x^-))^{--}))^- = (g(g(x^-)))^{---} = (g(x^-))^- = g^-(x).$$

(iv) $g^-(1) = (g(1^-))^- = (g(0))^- = 0^- = 1$.

(v) For any $x, y \in M$ such that $x \leq y$ we have $y^- \leq x^-$, thus $g(y^-) \leq g(x^-)$ and $g^-(x) = (g(x^-))^- \leq (g(y^-))^- = g^-(y)$. □

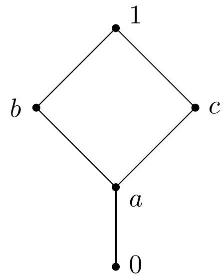
Remark 3.13 If g is an ac-operator on a normal residuated lattice M, then g^- need not be an mi-operator, i.e. condition 2 from the definition of an mi-operator need not be satisfied on M as we can see in the following example.

Example 3.14 Let $M_2 = \{0, a, b, c, 1\}$. Let the operations $\odot$ and $\to$ be defined on M_2 as follows:

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	a	a	a
b	0	a	b	a	b
c	0	a	a	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	c	1	c	1
c	0	b	b	1	1
1	0	a	b	c	1

$\oplus$	0	a	b	c	1
0	0	1	1	1	1
a	1	1	1	1	1
b	1	1	1	1	1
c	1	1	1	1	1
1	1	1	1	1	1



Then $M_2 = (M_2; \odot, \vee, \wedge, \rightarrow, 0, 1)$ is a residuated lattice which is both a BL-algebra and a Heyting algebra with the derived operation $\oplus$:

Let $g\colon M_2 \rightarrow M_2$ be the mapping such that $g(0) = 0, g(a) = g(b) = b, g(c) = 1, g(1) = 1$. Then we can easily verify that g is an ac-operator on M_2. However, the inequality $g^-(x) \leq x$ does not hold for all $x \in M_2$, since, for instance, $g^-(a) = (g(a^-))^- = (g(0))^- = 0^- = 1 \not\leq a$.

Recall that a residuated lattice M is called *involutive* if it satisfies $x^{--} = x$ for any $x \in M$.

Remark 3.15 It is obvious that any involutive residuated lattice is normal. Hence by Proposition 3.10, if f is an mi-operator on such a residuated lattice M, then f^- is an ac-operator on M. Furthermore, if g is an ac-operator on an involutive residuated lattice M, then by Proposition 3.12, g^- is an mi-operator on M. Moreover, $f \mapsto f^-$ and $g \mapsto g^-$ are one-to-one correspondences between mi-operators and ac-operators on an involutive residuated lattice.

Remark 3.16 The situation for normal residuated lattices which are not involutive is more complicated. Namely, although f^- is still an ac-operator for any mi-operator f on a residuated lattice M, for ac-operator g on M, g^- need not be an mi-operator. Furthermore, if f is an mi-operator on M, then by the proof of Proposition 3.7 (i), f^- satisfies in fact a condition that is stronger than axiom 2 in the definition of an ac-operator on M. Therefore, we will introduce now the notions of wmi- and sac- operators on normal residuated lattices.

Definition 3.17 Let M be a residuated lattice and $f\colon M \rightarrow M$. Then f is called a *weak mi-operator (a wmi-operator)* on M if it satisfies conditions 1 and 3–5 of the definition of an mi-operator and for any $x \in M$

2a $\quad f(x) \leq x^{--}$.

Definition 3.18 Let M be a normal residuated lattice and $g\colon M \rightarrow M$. Then g is called a *strong ac-operator (an sac-operator)* on M if it satisfies conditions 1 and 3–5 of the definition of an ac-operator and for any $x \in M$

2b $\quad x^{--} \leq g(x)$.

Remark 3.19 We have that if f is an mi-operator, then f^- is an sac-operator and if g is an ac-operator, then g^- is a wmi-operator.

Now we will describe connections among mi-, ac-, wmi- and sac-operators on normal residuated lattices.

Proposition 3.20 *Let M be a normal residuated lattice.*

(i) If f is a wmi-operator on M, then f^- is an sac-operator on M.

(ii) If g is an sac-operator on M, then g^- is a wmi-operator on M.

Proof (i) It suffices to prove condition 2b. If $x \in M$, then by 2a, $f(x^-) \leq x^{---} = x^-$, hence $(f(x^-))^- = f^-(x) \geq x^{--}$.

(ii) Analogously we will only verify condition 2a. If $x \in M$, then $x^- = (x^-)^{--} \leq g(x^-)$, thus $x^{--} \geq (g(x^-))^- = g^-(x)$. □

If M is a normal residuated lattice, denote by $wmi(M)$ the set of wmi-operators on M and by $sac(M)$ the set of sac-operators on M. Suppose that $wmi(M)$ and $sac(M)$ are pointwise ordered.

Let $\alpha\colon wmi(M) \to sac(M)$ be the mapping such that $\alpha(f) = f^-$, for any $f \in wmi(M)$, and $\beta\colon sac(M) \to wmi(M)$ be the mapping such that $\beta(g) = g^-$, for any $g \in sac(M)$.

Theorem 3.21 *If M is a normal residuated lattice, then α and β form an antitone Galois connection, i.e. $f \leq \beta(g)$ if and only if $g \leq \alpha(f)$, for any $f \in wmi(M)$ and $g \in sac(M)$.*

Proof Let $f \in wmi(M)$, $g \in sac(M)$ and $f \leq \beta(g) = g^-$. Then $f(x) \leq g^-(x) = (g(x^-))^-$, thus $f(x)^- \geq (g(x^-))^{--}$, for any $x \in M$. Therefore

$$(f(x^-))^- \geq (g(x^{--}))^{--} \geq (g(x))^{--} \geq g(x),$$

thus $\alpha(f)(x) \geq g(x)$, for any $x \in M$. That means $g \leq \alpha(f)$.

Conversely, let $g \leq \alpha(f)$. Then $f^-(x) \geq g(x)$, i.e. $(f(x^-))^- \geq g(x)$, and so $(f(x^-))^{--} \leq (g(x))^-$, for any $x \in M$. Hence

$$(f(x^{--}))^{--} \leq (g(x^-))^- = g^-(x), \quad \text{and} \quad (f(x^{--}))^{--} \geq (f(x))^{--} \geq f(x).$$

That means $\beta(g)(x) = g^-(x) \geq (f(x^{--}))^{--} \geq f(x)$, for any $x \in M$, and thus $f \leq \beta(g)$. □

The following theorem is now an immediate consequence.

Theorem 3.22 *Let M be a normal residuated lattice.*

(i) If f is an mi-operator on M and $h = (f^-)^-$ is the corresponding wmi-operator on M, then the induced sac-operators f^- and h^- are the same.

(ii) If g is an ac-operator on M and $k = (g^-)^-$ is the corresponding sac-operator on M, then the induced wmi-operators g^- and k^- are the same.

4 Operators on residuated lattices with Glivenko property

Definition 4.1 Let M be a residuated lattice. A nonempty subset F of M is called a *filter* of M if the following conditions hold

(1) $x, y \in F \Rightarrow x \odot y \in F$,

(2) $x \in F$, $y \in M$, $x \leq y \Rightarrow y \in F$.

A subset D of M is called a *deductive system* of M if

(3) $1 \in D$,

(4) $x, x \to y \in D \Rightarrow y \in D$.

It is known that a nonempty subset of M is a filter of M if and only if it is a deductive system of M.

By [11], filters of commutative residuated lattices are in a one-to-one correspondence with their congruences. If F is a filter of a commutative residuated lattice M, then for the corresponding congruence Θ_F we have:

$$\begin{aligned}\langle x, y\rangle \in \Theta_F &\iff (x \to y) \wedge (y \to x) \in F \iff (x \to y) \odot (y \to x) \in F \\ &\iff x \to y, y \to x \in F,\end{aligned}$$

for each $x, y \in M$. In such a case, $F = \{x \in M : \langle x, 1\rangle \in \Theta_F\}$. For any filter F of M we put $M/F := M/\Theta_F$.

If M is a residuated lattice, denote $D(M) = \{x \in M : x^{--} = 1\}$ the set of *dense elements* in M.

We say that a residuated lattice M has *Glivenko property* [3] if for any $x, y \in M$

$$(x \to y)^{--} = x \to y^{--}.$$

Proposition 4.2 [3] *A residuated lattice M has Glivenko property if and only if M satisfies the identity*

$$(x^{--} \to x)^{--} = 1.$$

An element x of a residuated lattice M is called *regular* if $x^{--} = x$. Denote by $\mathrm{Reg}(M)$ the set of all regular elements in M. If $x, y \in \mathrm{Reg}(M)$, put $x \vee_* y := (x \vee y)^{--}$, $x \wedge_* y := (x \wedge y)^{--}$, $x \odot_* y := (x \odot y)^{--}$ and $x \oplus_* y = (x \oplus y)^{--}$.

Theorem 4.3 [3] *For any residuated lattice M the following conditions are equivalent:*

(i) *M has Glivenko property;*

(ii) $(\mathrm{Reg}(M); \vee_*, \wedge_*, \odot_*, \to, 0, 1)$ *is an involutive residuated lattice and the mapping* ${}^{--} : M \to \mathrm{Reg}(M)$ *such that* ${}^{--} : x \mapsto x^{--}$ *is a surjective homomorphism of residuated lattices.*

Remark 4.4 If M is a normal residuated lattice and $x, y \in \mathrm{Reg}(M)$, then $x \odot_* y = (x \odot y)^{--} = x^{--} \odot y^{--} = x \odot y$. For an arbitrary residuated lattice we have $x \oplus_* y = x \oplus y$.

Proposition 4.5 *A residuated lattice M has Glivenko property if and only if $(x \to y)^{--} = x^{--} \to y^{--}$ for any $x, y \in M$.*

Proof It follows from Proposition 2.1 (xii). □

Remark 4.6 Every $R\ell$-monoid has Glivenko property because by [12] it satisfies the identity $(x \to y)^{--} = x^{--} \to y^{--}$.

Proposition 4.7 *If M is a residuated lattice, then $D(M)$ is a filter of M.*

Proof Let $x, y \in D(M)$, i.e. $x^{--} = 1 = y^{--}$. Then by Proposition 2.1, $(x \odot y)^{--} \geq x^{--} \odot y^{--} = 1$, hence $(x \odot y)^{--} = 1$, and so $x \odot y \in D(M)$.
If $x \in D(M), z \in M$ and $x \leq z$, then obviously $z \in D(M)$. □

The following assertions concerning connections between $D(M)$ and $\mathrm{Reg}(M)$ are consequences of Theorem 4.3.

Theorem 4.8 *If M is a residuated lattice with Glivenko property, then for any $x, y \in M$ we have $\langle x, y \rangle \in \Theta_{D(M)}$ if and only if $x^{--} = y^{--}$. Moreover, the quotient residuated lattice $M/D(M)$ is involutive.*

Proof Let $x, y \in M$. Then

$$\langle x, y \rangle \in \Theta_{D(M)} \iff x \to y, y \to x \in D(M)$$
$$\iff (x \to y)^{--} = 1 = (y \to x)^{--} \iff x^{--} \to y^{--} = 1 = y^{--} \to x^{--}$$
$$\iff x^{--} \leq y^{--}, y^{--} \leq x^{--} \iff x^{--} = y^{--}.$$

Therefore, $(x/D(M))^{--} = x^{--}/D(M) = x/D(M)$. □

Theorem 4.9 *If M is a residuated lattice with Glivenko property, then the residuated lattices* $\mathrm{Reg}(M)$ *and $M/D(M)$ are isomorphic.*

Remark 4.10 It is obvious that the mappings $\varphi\colon \mathrm{Reg}(M) \to M/D(M)$ and $\psi\colon M/D(M) \to Reg(M)$ such that $\varphi\colon x \mapsto x/D(M)$ and $\psi\colon y/D(M) \mapsto y^{--}$ are mutually inverse isomorphisms between $\mathrm{Reg}(M)$ and $M/D(M)$.

Theorem 4.11 *Let M be a normal residuated lattice with Glivenko property, f an mi-operator (resp. an ac-operator) on M and $f^*\colon M/D(M) \to M/D(M)$ the mapping such that $f^*(x/D(M)) = f(x^{--})/D(M)$. Then f^* is an mi-operator (resp. an ac-operator) on $M/D(M)$.*

Proof Let f be an mi-operator on M and $x, y \in M$ be elements such that $x/D(M) = y/D(M)$. Then

$$f^*(x/D(M)) = f(x^{--})/D(M) = f(y^{--})/D(M) = f^*(y)/D(M).$$

Therefore f^* is defined correctly. We will verify that it is an mi-operator.

$$(1)\quad f^*(x/D(M)) \odot f^*(y/D(M)) = f(x^{--})/D(M) \odot f(y^{--})/D(M)$$
$$= (f(x^{--} \odot y^{--}))/D(M) = f((x \odot y)^{--})/D(M) = f^*((x \odot y)/D(M))$$
$$= f^*((x/D(M)) \odot (y/D(M))).$$

$$(2)\quad f^*(x/D(M)) = f(x^{--})/D(M) \leq x^{--}/D(M) = x/D(M).$$

$$(3)\quad f^*(f^*(x/D(M))) = f^*(f(x^{--})/D(M)) = f((f(x^{--}))^{--})/D(M)$$
$$\leq (f(x^{--}))^{--}/D(M) = f(x^{--})/D(M) = f^*(x/D(M)).$$

Conversely,

$$(f(x^{--}))^{--}/D(M) \geq f(x^{--})/D(M)$$
$$\Longrightarrow f((f(x^{--}))^{--})/D(M) \geq f(f(x^{--}))/D(M) = f(x^{--})/D(M)$$
$$\Longrightarrow f^*(f^*(x/D(M))) \geq f^*(x/D(M)).$$

Hence, $f^*(f^*(x/D(M))) = f^*(x/D(M))$.

$$(4)\quad f^*(1/D(M)) = f(1^{--})/D(M) = f(1)/D(M) = 1/D(M).$$

$$(5)\quad x/D(M) \leq y/D(M) \Longrightarrow x^{--}/D(M) \leq y^{--}/D(M)$$
$$\Longrightarrow f(x^{--})/D(M) \leq f(y^{--})/D(M) \Longrightarrow f^*(x/D(M)) \leq f^*(y/D(M)).$$

Similarly for ac-operators on M. □

Theorem 4.12 *If M is a normal residuated lattice with Glivenko property and f is an mi-operator (resp. an ac-operator) on M, then the mapping $f^\#$ such that $f^\#(x) = f(x)^{--}$ for any $x \in \mathrm{Reg}(M)$ is an mi-operator (resp. an ac-operator) on the residuated lattice* $\mathrm{Reg}(M)$.

Proof If $x \in \mathrm{Reg}(M)$, then also $f(x)^{--} \in \mathrm{Reg}(M)$. The assertion is hence a direct consequence of the preceeding theorem because the mapping ψ from Remark 4.10 is an isomorphism of residuated lattices. □

Theorem 4.13 *Let M be a normal residuated lattice with Glivenko property. If $g\colon \mathrm{Reg}(M) \to \mathrm{Reg}(M)$ is an mi-operator on the involutive residuated lattice $\mathrm{Reg}(M)$, then the mapping $g^+\colon M \to M$ such that $g^+(x) := g(x^{--})$ for any $x \in M$, is a wmi-operator on M.*

Proof Let g be an mi-operator on $\mathrm{Reg}(M)$ and $g^+(x) = g(x^{--})$ for any $x \in M$.

(1) $g^+(x \odot y) = g((x \odot y)^{--}) = g(x^{--} \odot y^{--}) = g(x^{--} \odot_* y^{--}) = g(x^{--}) \odot_* g(y^{--}) = g(x^{--}) \odot g(y^{--}) = g^+(x) \odot g^+(y)$.

(2) $g^+(x) = g(x^{--}) \leq x^{--}$.

(3) $g^+(g^+(x)) = g((g^+(x))^{--}) = g((g(x^{--}))^{--}) = g(g(x^{--})) = g(x^{--}) = g^+(x)$.

(4) $g^+(1) = g(1^{--}) = g(1) = 1$.

(5) $x \leq y \Rightarrow g^+(x) = g(x^{--}) \leq g(y^{--}) = g^+(y)$.

Hence g is an mi-operator on M. □

Theorem 4.14 *Let M be a residuated lattice with Glivenko property. If $h\colon \mathrm{Reg}(M) \to \mathrm{Reg}(M)$ is an ac-operator on $\mathrm{Reg}(M)$, then the mapping*

$$\hat{h}(x) = h(x^{--})$$

for any $x \in M$, is an sac-operator on M.

Proof

$$\begin{aligned} 1.\quad \hat{h}(x \oplus y) &= h((x \oplus y)^{--}) = h(x^{--} \oplus y^{--}) = h(x^{--} \oplus_* y^{--}) \\ &= h(x^{--}) \oplus_* h(y^{--}) = h(x^{--}) \oplus h(y^{--}) = \hat{h}(x) \oplus \hat{h}(y). \end{aligned}$$

2. $\hat{h}(x) = h(x^{--}) \geq x^{--}$.

3.–5. Similarly as in the proof of Theorem 4.13. □

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