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On the Existence of Oscillatory Solutions of the Second Order Nonlinear ODE*

Martin ROHLEDER

Department of Mathematical Analysis and Applications of Mathematics
Faculty of Science, Palacký University
17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: martin.rohleder01@upol.cz

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Abstract

The paper investigates the singular initial problem

$$(p(t)u'(t))' + q(t)f(u(t)) = 0,\ u(0) = u_0,\ u'(0) = 0$$

on the half-line $[0,\infty)$. Here $u_0 \in [L_0, L]$, where L_0, 0 and L are zeros of f, which is locally Lipschitz continuous on $\mathbb{R}$. Function p is continuous on $[0,\infty)$, has a positive continuous derivative on $(0,\infty)$ and $p(0) = 0$. Function q is continuous on $[0,\infty)$ and positive on $(0,\infty)$. For specific values u_0 we prove the existence and uniqueness of damped solutions of this problem. With additional conditions for f, p and q it is shown that the problem has for each specified u_0 a unique oscillatory solution with decreasing amplitudes.

Key words: singular ordinary differential equation of the second order, time singularities, unbounded domain, asymptotic properties, damped solutions, oscillatory solutions

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1 Introduction

We investigate a singular boundary value problem motivated by some models used in hydrodynamics. In [3] it is shown that the study of the behavior of

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non-homogeneous fluids in the Cahn–Hilliard theory can lead to the system of PDE's

$$\varrho_t + \operatorname{div}(\varrho\vec{v}) = 0, \qquad \frac{d\vec{v}}{dt} + \nabla(\mu(\varrho) - \gamma\Delta\varrho) = 0. \tag{1}$$

Here ϱ is the density and $\vec{v}$ is the velocity of the fluid, $\mu(\varrho)$ is its chemical potential, γ is a constant. If we suppose that a motion of the fluid is zero, then system (1) is reduced to the equation

$$\gamma\Delta\varrho = \mu(\varrho) - \mu_0, \tag{2}$$

where γ and μ_0 are suitable constants. When we search a solution with the spherical symmetry, then equation (2) is reduced to the ODE

$$\gamma\left(\varrho'' + \frac{2}{r}\varrho'\right) = \mu(\varrho) - \mu(\varrho_\ell), \quad r \in (0,\infty). \tag{3}$$

Equation (3) with the boundary conditions

$$\varrho'(0) = 0, \quad \lim_{r\to\infty} \varrho(r) =: \varrho_\ell > 0 \tag{4}$$

describe the formation of microscopic bubbles in a fluid, in particular, vapor inside liquid. In the simplest model of non-homogeneous fluid, problem (3), (4) is reduced to the form

$$(t^2u')' = 4\lambda^2 t^2(u+1)u(u-\xi), \tag{5}$$

$$u'(0) = 0, \qquad u(\infty) = \xi, \tag{6}$$

where $\lambda \in (0,\infty)$ and $\xi \in (0,1)$ are parameters. If there exists an increasing solution of problem (5), (6) with just one zero, many important physical properties of the bubbles depend on it. In particular, the gas density inside the bubble, the bubble radius and the surface tension. For a numerical investigation of the problem see [3], [6] and [9].

The equation $(p(t)u'(t))' = p(t)f(u(t))$ is a generalization of equation (5) and it has been studied in [14]–[18], where all types of its possible solutions have been described with conditions which guarantee their existence and specify their asymptotic behavior.

In this paper we continue the generalization and study the problem

$$(p(t)u'(t))' + q(t)f(u(t)) = 0, \quad t \in (0,\infty), \tag{7}$$

$$u(0) = u_0, \quad u'(0) = 0, \tag{8}$$

where $u_0 \in [L_0, L]$.

In the whole paper we will assume that f, p and q satisfy the following conditions.

$$L_0 < 0 < L, \quad f(L_0) = f(0) = f(L) = 0, \tag{9}$$

$$f \in Lip_{\mathrm{loc}}(\mathbb{R}), \tag{10}$$

$$xf(x) > 0 \quad \text{for } x \in (L_0, L) \setminus \{0\}, \tag{11}$$

$$F(L_0) > F(L), \quad \text{where } F(x) = \int_0^x f(z)\mathrm{d}z, \tag{12}$$

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p(0) = 0, \tag{13}$$

$$p'(t) > 0 \quad \text{for } t \in (0,\infty), \qquad \lim_{t\to\infty} \frac{p'(t)}{p(t)} = 0, \tag{14}$$

$$q \in C[0,\infty), \quad q > 0 \text{ on } (0,\infty). \tag{15}$$

Definition 1.1 A function $u \in C^1[0,\infty) \cap C^2(0,\infty)$ satisfying equation (7) on $(0,\infty)$ and fulfilling conditions (8) is called a solution of problem (7), (8).

Definition 1.2 A solution u of problem (7), (8) is called damped, if $\sup\{u(t) \colon t \in [0,\infty)\} < L$, where L is of (9).

The aim of our paper is the investigation of damped solutions of problem (7), (8). The main results are contained in Theorem 2.6, where the existence of damped solutions is presented, and in Theorem 3.2, Theorem 4.2 and Theorem 4.6 describing asymptotic properties of damped solutions of problem (7), (8). In particular, Theorem 3.2 is devoted to nonoscillatory solutions, while Theorems 4.2 and 4.6 provide conditions which guarantee that damped solutions of equation (7) are oscillatory.

In the literature, the permanent attention has been devoted to oscillatory solutions of second order nonlinear differential equations. See e.g. the monograph [5] containing a lot of nice results about asymptotic properties of solutions. Let us mention also the papers [13] and [19] which investigate Emden–Fowler equations. Further extensions of these results have been reached for more general equations, see e.g. [1], [2], [4], [7], [8], [20]. Nonlinearities in equations in the cited papers have similar globally monotonous behavior. We want to emphasize that, in contrast to these papers, the nonlinearity f in our equation (7) needs not be globally monotonous. Moreover, we deal with solutions of (7) starting at a singular point $t = 0$, and we provide an interval for starting values u_0 giving oscillatory solutions, see Theorem 4.2 and Theorem 4.6. Therefore theorems from the papers cited above cannot be applied to singular problem (7), (8) satisfying assumptions (9)–(15). For example, the same equation (7) is studied in [2] but in the regular setting, that is function p in equation (7) must be strictly positive on $[0,\infty)$. One of basic assumptions in [2] is convergence or divergence of the integral

$$I_p = \int_0^\infty \frac{1}{p(t)}\,\mathrm{d}t.$$

In our paper, a typical choice in equation (7) is $p(t) = t^\alpha$, $\alpha > 0$. Then clearly $I_p = \infty$ and therefore theorems in [2], which require $I_p < \infty$, cannot be applied. Other important assumption in [2] concerns function f in equation (7) and has the form

$$\liminf_{|x|\to\infty} |f(x)| > 0. \tag{16}$$

In our paper, function f has three zeros $L_0 < 0 < L$ and an arbitrary behavior for $x < L_0$ and $x > L$. Consequently, (16) need not be fulfilled and theorems of [2] requiring (16) cannot be applied here, as well.

2 Existence of damped solutions

In order to prove our first existence result we provide some lemmas.

Lemma 2.1 *Let u be a solution of equation* (7).
a) Assume that there exists $t_1 \ge 0$ such that $u(t_1) \in (0, L)$ and $u'(t_1) = 0$. Then

$$u(t) \ge 0 \Rightarrow u'(t) < 0 \quad \text{for } t \in (t_1, \theta_1], \tag{17}$$

where θ_1 is the first zero of u on (t_1, ∞). If such θ_1 does not exist, then (17) *is valid for $t \in (t_1, \infty)$.*
b) Assume that there exists $t_2 \ge 0$ such that $u(t_2) \in (L_0, 0)$ and $u'(t_2) = 0$. Then

$$u(t) \le 0 \Rightarrow u'(t) > 0 \quad \text{for } t \in (t_2, \theta_2], \tag{18}$$

where θ_2 is the first zero of u on (t_2, ∞). If such θ_2 does not exist, then (18) *is valid for $t \in (t_2, \infty)$.*

Proof a) First, let us show that $u(t) < L$ on $[t_1, \infty)$. Assume on the contrary that there exists $t_2 > t_1$ such that

$$u(t) \in (0, L) \text{ for } t \in (t_1, t_2) \quad \text{and} \quad u(t_2) = L. \tag{19}$$

Then, by (7), (11) and (15), $(pu')'(t) < 0$ for $t \in (t_1, t_2)$ and pu' is decreasing on (t_1, t_2). Since $(pu')(t_1) = 0$, we get $(pu')(t) < 0$ for $t \in (t_1, t_2)$, and, due to (13) and (14), we have $u'(t) < 0$ for $t \in (t_1, t_2)$, that is u is decreasing on (t_1, t_2), contrary to (19). We have proved that $u(t) < L$ on $[t_1, \infty)$.

Let $u(t) > 0$ on $[t_1, \infty)$. Working on (t_1, ∞) instead of (t_1, t_2), we get as before that $u'(t) < 0$ for $t \in (t_1, \infty)$.

Finally, assume that there exists $\theta_1 > t_1$ such that $u(t) > 0$ on $[t_1, \theta_1)$ and $u(\theta_1) = 0$. Then we obtain as before that $u'(t) < 0$ on (t_1, θ_1). Further, since pu' is negative and decreasing on (t_1, θ_1), we obtain

$$(pu')(s) < (pu')(t) < 0 \quad \text{for } t_1 < t < s < \theta_1.$$

Letting $s \to \theta_1$, we get $(pu')(\theta_1) < 0$ and hence (17) is valid.

b) We argue similarly as in a). □

Since $F(0) = 0$, $F(L_0) > F(L) > 0$ and F is continuous on $\mathbb{R}$, we obtain, by (12),

$$\exists\, \bar{B} \in (L_0, 0)\colon F(\bar{B}) = F(L). \tag{20}$$

Define auxiliary functions

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \geq L_0, \\ 0 & \text{for } x < L_0, \end{cases} \qquad \tilde{F}(x) = \int_0^x \tilde{f}(z)\,\mathrm{d}z, \quad x \in \mathbb{R}$$

and consider the equation

$$(p(t)u'(t))' + q(t)\tilde{f}(u(t)) = 0. \tag{21}$$

We see that $\tilde{f}$ satisfies conditions (9)–(12) and so Lemma 2.1 is valid for equation (21), as well.

Lemma 2.2 *Assume that*

$$pq \textit{ is nondecreasing on } [0, \infty). \tag{22}$$

Let u be a solution of problem (21), (8) *for $u_0 \in (L_0, L)$. Assume that there exist $a \geq 0$, $\theta > a$ such that*

$$u(a) \in (0, L), \quad u'(a) = 0, \tag{23}$$

$$u(\theta) = 0, \quad u(t) > 0 \quad \textit{for } t \in [a, \theta). \tag{24}$$

Then u has one of the following properties:

$$u'(t) < 0 \ \ \textit{for } t \in (a, \infty), \quad \lim_{t\to\infty} u(t) \in (\bar{B}, 0), \tag{25}$$

$$\exists b \in (\theta, \infty)\colon u(b) \in (\bar{B}, 0),\ u'(b) = 0,\ u'(t) < 0 \ \ \textit{for } t \in (a, b). \tag{26}$$

Proof By (17), $u'(t) < 0$ holds for $t \in (a, \theta]$. Let us suppose that neither (25) nor (26) occur. Assume that there exists $b_1 > \theta$ satisfying $u(b_1) = \bar{B}$, $u' < 0$ on (a, b_1). Multiplying equation (21) by pu', integrating this over (a, b_1) and using the Mean value theorem, we get $\xi_1 \in [a, \theta]$, $\xi_2 \in [\theta, b_1]$ such that

$$\int_a^{b_1} (p(t)u'(t))'p(t)u'(t)\,\mathrm{d}t =$$

$$= -\int_a^{\theta} p(t)q(t)\tilde{f}(u(t))u'(t)\,\mathrm{d}t - \int_{\theta}^{b_1} p(t)q(t)\tilde{f}(u(t))u'(t)\,\mathrm{d}t$$

$$= -(pq)(\xi_1)\int_a^{\theta} \tilde{f}(u(t))u'(t)\,\mathrm{d}t - (pq)(\xi_2)\int_{\theta}^{b_1} \tilde{f}(u(t))u'(t)\,\mathrm{d}t.$$

Hence, by (22),

$$0 \leq \frac{(p(b_1)u'(b_1))^2}{2}$$

$$= (pq)(\xi_1)\Big(\tilde{F}(u(a)) - \tilde{F}(u(\theta))\Big) + (pq)(\xi_2)\Big(\tilde{F}(u(\theta)) - \tilde{F}(u(b_1))\Big)$$

$$\leq (pq)(\xi_2)\left(\tilde{F}(u(a)) - \tilde{F}(\bar{B})\right).$$

Therefore $\tilde{F}(u(a)) \geq \tilde{F}(\bar{B})$. By virtue of (12) we get

$$F(\bar{B}) = \tilde{F}(\bar{B}) \leq \tilde{F}(u(a)) = F(u(a)) < F(L)$$

which contradicts (20). Therefore, if there exists $b > \theta$ such that $u'(b) = 0$ and $u'(t) < 0$ for $t \in (a, b)$, then $u(b) \in (\bar{B}, 0)$ and (26) holds.

Let $u'(t) < 0$ for $t \in (a, \infty)$. Due to above arguments, $\lim_{t\to\infty} u(t) \geq \bar{B}$. Assume that $\lim_{t\to\infty} u(t) = \bar{B}$. Then, by $0 < u(a) < L$, we obtain

$$\tilde{F}(u(a)) < \tilde{F}(L) = \tilde{F}(\bar{B}) = \lim_{t\to\infty} \tilde{F}(u(t)).$$

Thus there exists $s^* > a$ such that $\tilde{F}(u(a)) < \tilde{F}(u(s^*)),\ u(s^*) < 0$. From the Mean value theorem we have $\xi_1 \in [a, \theta]$, $\xi_2 \in [\theta, s^*]$ such that

$$0 < \frac{(p(s^*)u'(s^*))^2}{2} =$$

$$= (pq)(\xi_1)\Big(\tilde{F}(u(a)) - \tilde{F}(u(\theta))\Big) + (pq)(\xi_2)\Big(\tilde{F}(u(\theta)) - \tilde{F}(u(s^*))\Big)$$

$$\leq (pq)(\xi_2)\left(\tilde{F}(u(a)) - \tilde{F}(u(s^*))\right) < 0,$$

a contradiction. Hence $\lim_{t\to\infty} u(t) \in (\bar{B}, 0)$ and (25) holds. □

We can prove the following lemma analogously.

Lemma 2.3 *Let* (22) *be satisfied and let u be a solution of problem* (21), (8) *for $u_0 \in (L_0, L)$. Assume that there exist $b \geq 0$, $\xi > b$ such that*

$$u(b) \in (\bar{B}, 0), \quad u'(b) = 0, \tag{27}$$

$$u(\xi) = 0, \quad u(t) < 0 \quad \textit{for } t \in [b, \xi). \tag{28}$$

Then u has one of the following properties:

$$u'(t) > 0 \ \ \textit{for } t \in (b, \infty), \quad \lim_{t\to\infty} u(t) \in (0, L), \tag{29}$$

$$\exists c \in (\xi, \infty)\colon u(c) \in (0, L),\ u'(c) = 0,\ u'(t) > 0 \quad \textit{for } t \in (b, c). \tag{30}$$

Lemma 2.4 *Assume that*

$$\exists\, C_L \in (0, \infty)\colon\ -C_L \leq f(x) \leq 0 \quad \textit{for } x \geq L. \tag{31}$$

Let u be a solution of problem (21), (8) *for $u_0 \in (L_0, \bar{B}]$. Assume that there exist $\theta > 0$, $a > \theta$ such that*

$$u(\theta) = 0, \quad u(t) < 0 \quad \textit{for } t \in [0, \theta), \tag{32}$$

$$u'(a) = 0, \quad u'(t) > 0 \quad \textit{for } t \in (\theta, a). \tag{33}$$

Then

$$u(a) \in (0, L), \quad u'(t) > 0 \quad \textit{for } t \in (0, a). \tag{34}$$

Proof By (18) and (33) we have $u'(t) > 0$ on $(0, a)$ and hence

$$(pu')(t) > 0 \quad \text{for } t \in (0, a). \tag{35}$$

Let us assume that $u(a) \geq L$. We derive a contradiction. By (10), the final value problem (21), $u(a) = L$, $u'(a) = 0$ has the unique solution $u_L \equiv L$ on $[0, a]$. Since our solution u is not constant on $[0, a]$, we have $u(a) > L$. Therefore there is $a_0 \in (\theta, a)$ such that $u(t) > L$ for $t \in (a_0, a]$ and, by (31), $\tilde{f}(u(t)) \leq 0$ for $t \in (a_0, a]$. Integrating equation (21) on $[a_0, a]$, we get

$$(pu')(a) - (pu')(a_0) = -\int_{a_0}^{a} q(s)\tilde{f}(u(s))\,\mathrm{d}s \geq 0.$$

Consequently $(pu')(a_0) \leq 0$, contrary to (35). □

Theorem 2.5 (Existence and uniqueness) *Assume* (22), (31) *and*

$$\lim_{t\to 0^+} \frac{1}{p(t)} \int_0^t q(s)\,\mathrm{d}s = 0. \tag{36}$$

Then, for each $u_0 \in [L_0, L]$*, problem* (7), (8) *has a unique solution* u*. This solution* u *satisfies*

$$u(t) > \bar{B} \quad \textit{for } u_0 \in (\bar{B}, L],\ t \in [0, \infty), \tag{37}$$

$$u(t) \geq u_0 \quad \textit{for } u_0 \in [L_0, \bar{B}],\ t \in [0, \infty). \tag{38}$$

Proof Let $u_0 \in [L_0, L]$.

Step 1 (Existence and uniqueness of a solution of problem (21), (8))
Equation (21) has an equivalent form

$$u(t) = u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\mathrm{d}\tau\,\mathrm{d}s, \quad t \in [0, \infty).$$

By (10) and (31)

$$\exists M > 0\colon \left|\tilde{f}(x)\right| \leq M, \quad x \in \mathbb{R}. \tag{39}$$

Put $\Lambda := \max\{|L_0|, L\}$. By (10) there exists $K > 0$ such that

$$|f(x) - f(y)| \leq K|x - y|, \quad \forall x, y \in [-\Lambda - 1, \Lambda + 1]. \tag{40}$$

Put $\varphi(t) = \frac{1}{p(t)} \int_0^t q(s)\mathrm{d}s$ for $t > 0$. Then conditions (15) and (36) give

$$0 < \varphi(t) < \infty \quad \text{for } t \in (0, \infty), \qquad \lim_{t\to 0^+} \varphi(t) = 0. \tag{41}$$

Therefore we can find $\eta \in (0, \infty)$ such that

$$\int_0^{\eta} \varphi(t)\mathrm{d}t \leq \min\left\{\frac{1}{2K}, \frac{1}{M}\right\}. \tag{42}$$

Consider the Banach space $C[0,\eta]$ with the maximum norm and define an operator $\mathcal{F}\colon C[0,\eta] \to C[0,\eta]$ by

$$(\mathcal{F}u)(t) = u_0 - \int_0^t \frac{1}{p(s)} \int_0^s q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau\mathrm{d}s.$$

From (39) and (42) it follows that

$$\|\mathcal{F}u\|_{C[0,\eta]} \le \Lambda + M \int_0^\eta \varphi(s)\,\mathrm{d}s \le \Lambda + 1, \quad \forall u \in C[0,\eta],$$

hence $\mathcal{F}$ maps the ball $\mathcal{B}(0,\Lambda+1) = \big\{u \in C[0,\eta]\colon \|u\|_{C[0,\eta]} \le \Lambda + 1\big\}$ on itself. Choose arbitrary $u_1, u_2 \in \mathcal{B}(0,\Lambda+1)$. Then, by (40) and (42), we obtain

$$\begin{aligned}\|\mathcal{F}u_1 - \mathcal{F}u_2\|_{C[0,\eta]} &\le \int_0^\eta \frac{1}{p(s)} \int_0^s q(\tau)\Big|\tilde{f}(u_1(\tau)) - \tilde{f}(u_2(\tau))\Big|\,\mathrm{d}\tau\mathrm{d}s \\ &\le K\|u_1 - u_2\|_{C[0,\eta]} \int_0^\eta \varphi(s)\,\mathrm{d}s \le \frac{1}{2}\|u_1 - u_2\|_{C[0,\eta]},\end{aligned}$$

thus $\mathcal{F}$ is a contraction on $\mathcal{B}(0,\Lambda+1)$. The Banach fixed point theorem yields a unique fixed point u of $\mathcal{F}$ in $\mathcal{B}(0,\Lambda+1)$. Therefore

$$u(0) = u_0, \qquad u'(t) = -\frac{1}{p(t)} \int_0^t q(\tau)\tilde{f}(u(\tau))\,\mathrm{d}\tau, \quad t \in (0,\eta]. \tag{43}$$

Since $|u'(t)| \le M\varphi(t)$, it holds $\lim_{t\to 0^+} u'(t) = 0$. From (43) it follows

$$(p(t)u'(t))' = -q(t)\tilde{f}(u(t)), \quad t \in (0,\eta],$$

thus the fixed point u is a solution of problem (21), (8) on $[0,\eta]$. According to (39), $\tilde{f}(u(t))$ is bounded on $[0,\infty)$ and hence, by [5], Theorem 11.5, u can be extended to $[0,\infty)$. Since $\tilde{f} \in Lip_{\mathrm{loc}}(\mathbb{R})$, this extension is unique.

Step 2 (Estimates of solutions of problem (21), (8))
Let $u_0 \in (0,L)$. If $u > 0$ on $(0,\infty)$, then (37) holds. Assume that there exists $\theta_1 > 0$ such that $u(\theta_1) = 0$, $u(t) > 0$ for $t \in [0,\theta_1)$. Using Lemma 2.2, where $a = 0$ and $\theta = \theta_1$, we obtain that u satisfies either (25) or (26). Condition (25) gives (37). Let condition (26) be valid, that is

$$\exists b \in (\theta_1,\infty)\colon u(b) \in (\bar{B},0), \quad u'(b) = 0, \quad u'(t) < 0 \text{ on } (0,b).$$

If $u < 0$ on (b,∞), then, by (18), u is increasing on (b,∞) and (37) is valid. Assume that there exists $\theta_2 > b$ such that $u(\theta_2) = 0$, $u(t) < 0$ for $t \in [b,\theta_2)$. Using Lemma 2.3, where $\xi = \theta_2$, we get that u satisfies either (29) or (30). Condition (29) gives (37). Let condition (30) be valid. Then we use previous arguments.

Let $u_0 \in (\bar{B},0)$. If $u < 0$ on $(0,\infty)$, then, by (18), u is increasing on $(0,\infty)$ and (37) is valid. Assume that there exists $\theta_3 > 0$ such that $u(\theta_3) = 0$, $u(t) < 0$ for $t \in [0,\theta_3)$. Using Lemma 2.3, where $b = 0$ and $\xi = \theta_3$, we obtain that u

satisfies either (29) or (30). Condition (29) gives (37). Let condition (30) be valid, that is

$$\exists c \in (\theta_3, \infty)\colon u(c) \in (0, L), \quad u'(c) = 0, \quad u'(t) > 0 \text{ on } (0, c).$$

If $u > 0$ on (c, ∞), then (37) holds. Assume that there exists $\theta_4 > c$ such that $u(\theta_4) = 0$, $u(t) > 0$ for $t \in [c, \theta_4)$. Using Lemma 2.2, where $a = c$ and $\theta = \theta_4$, we get that u satisfies either (25) or (26). Condition (25) gives (37). Let condition (26) be valid. Then we use previous arguments.

Let $u_0 \in (L_0, \bar{B}]$. If $u < 0$ on $(0, \infty)$, then u is increasing on $(0, \infty)$ and (38) is valid. Assume that there exists $\theta_5 > 0$ such that $u(\theta_5) = 0$, $u(t) < 0$ for $t \in [0, \theta_5)$. If $u > 0$ on (θ_5, ∞), then (38) holds. Assume that there exists $d > \theta_5$ such that $u'(d) = 0$, $u'(t) > 0$ for $t \in (\theta_5, d)$. Using Lemma 2.4, where $\theta = \theta_5$ and $a = d$, we have that (34) is valid. Now we have analogous situation as in case $u_0 \in (0, L)$, so we argue similarly.

Step 3 (Existence and uniqueness of a solution of problem (7), (8))
We have proved that estimates (37) and (38) are valid. By virtue of definition of $\tilde{f}$, the solution u of problem (21), (8) satisfies equation (7) on $(0, \infty)$.

Suppose that there exists another solution $\tilde{u}$ of problem (7), (8). We can prove as in Step 2 that $\tilde{u}$ fulfils (37) and (38). It means that $\tilde{u}$ satisfies equation (21) on $(0, \infty)$, too. Therefore, by Step 1, $u \equiv \tilde{u}$. □

For close existence results, see also Chapters 13 and 14 in [11] or Chapter 8 in [12].

Now we can prove the main result of this section.

Theorem 2.6 (Existence of damped solutions) *Let* (22) *and* (36) *be satisfied. Then, for each* $u_0 \in (\bar{B}, L)$*, problem* (7), (8) *has a unique solution. This solution is damped and fulfils* (37).

Proof Define a function

$$f^*(x) = \begin{cases} f(x) & \text{for } x \le L, \\ 0 & \text{for } x > L, \end{cases}$$

and consider the equation

$$(p(t)u'(t))' + q(t)f^*(u(t)) = 0. \tag{44}$$

Then f^* satisfies condition (31) with $f = f^*$. Due to Theorem 2.5 there exists a unique solution u of problem (44), (8), where $u_0 \in (\bar{B}, L)$. In addition u fulfils (37).

(i) If $u_0 = 0$, then $u \equiv 0$. It is clear that u is damped.

(ii) Let $u_0 \in (0, L)$. If $u > 0$ on $(0, \infty)$, then Lemma 2.1 yields $u' < 0$ on $(0, \infty)$ and hence u is damped. Let $\theta > 0$ be the first zero of u. By (17), $u' < 0$ on $(0, \theta]$. If $u < 0$ on (θ, ∞), then u is damped. Let $\xi > \theta$ be the second zero of u. Then there is $b \in (\theta, \xi)$ such that $u'(b) = 0$. Due to (37),

$u(b) \in (\bar{B}, 0)$. By (18), $u' > 0$ on $(b, \xi]$. If $u' > 0$ on (b, ∞), then, by Lemma 2.3, $\lim_{t\to\infty} u(t) \in (0, L)$ and so u is damped. Let there exist $c > \xi$ such that $u'(c) = 0$. Then Lemma 2.3 gives $u(c) \in (0, L)$ and we can continue as before working with $u(c)$ instead of u_0.

(iii) Let $u_0 \in (\bar{B}, 0)$. Working with u_0 in place of $u(b)$, we can use the arguments of part (ii). □

3 Properties of damped solutions

Since we will work with damped solutions of problem (7), (8), we will assume that all assumptions of Theorem 2.6 are fulfilled. To sum it up we will assume in the whole section that conditions (9)–(15), (22) and (36) hold.

Definition 3.1 Let u be a solution of problem (7), (8). Then u is called oscillatory, if the set of isolated zeros of u is unbounded. Otherwise, u is called nonoscillatory. Further, u is called eventually positive (eventually negative), if there exists $t_0 > 0$ such that $u(t) > 0$ ($u(t) < 0$) for $t \in (t_0, \infty)$.

Clearly (cf. (10)), each nonoscillatory solution of (7), (8) is either eventually positive or eventually negative. Papers [14]–[18] provide examples which demonstrate that equation (7) can have both oscillatory damped solutions and nonoscillatory ones.

In our study of damped solutions we will distinguish two cases according to the convergence or divergence of the integral $\int_1^\infty \frac{1}{p(s)}\mathrm{d}s$.

CASE I Let us assume that the function p in equation (7) fulfils

$$\int_1^\infty \frac{1}{p(s)}\,\mathrm{d}s < \infty. \tag{45}$$

A simple example of such function is $p(t) = t^\alpha$, $\alpha > 1$. Condition (45) with $p' > 0$ (cf. (14)) give

$$\lim_{t\to\infty} \frac{1}{p(t)} = 0. \tag{46}$$

In order to prove the existence of oscillatory solutions in this case we will need the following theorem about asymptotic behavior of nonoscillatory solutions.

Theorem 3.2 *Assume* (45) *and*

$$\liminf_{t\to\infty} \frac{1}{p(t)} \int_1^t q(s)\,\mathrm{d}s > 0. \tag{47}$$

Let u be a damped solution of problem (7), (8) *with $u_0 \in (L_0, 0) \cup (0, L)$ which is nonoscillatory. Then*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} u'(t) = 0. \tag{48}$$

Proof Assume that u is nonoscillatory damped solution of problem (7), (8) with $u_0 \in (L_0, 0) \cup (0, L)$. Then u is either eventually positive or eventually negative.

Step 1 Let u be eventually positive, i.e. there exists $t_0 \geq 1$ such that $u(t) > 0$ for $t \in [t_0, \infty)$.

Assume that $u' > 0$ on $[t_0, \infty)$. Then u is increasing on (t_0, ∞), $u(t_0) > 0$ and $\lim_{t\to\infty} u(t) =: \ell_0 \in (u(t_0), L)$. Denote $m_0 = \min\{f(x) \colon x \in [u(t_0), \ell_0]\} > 0$. By virtue of (7) we get

$$(p(t)u'(t))' = -q(t)f(u(t)) \leq -q(t)m_0, \quad t \in [t_0, \infty).$$

Integrating it over (t_0, t), we obtain

$$p(t)u'(t) - p(t_0)u'(t_0) \leq -m_0 \left(\int_1^t q(s)\,\mathrm{d}s - \int_1^{t_0} q(s)\,\mathrm{d}s \right), \quad t \in [t_0, \infty).$$

This together with (46) and (47) yield

$$0 < u'(t) \leq \frac{1}{p(t)} \left(p(t_0)u'(t_0) + m_0 \int_1^{t_0} q(s)\,\mathrm{d}s \right) - m_0 \frac{1}{p(t)} \int_1^t q(s)\,\mathrm{d}s, \ t \in [t_0, \infty),$$

$$0 \leq \liminf_{t\to\infty} u'(t) \leq -m_0 \liminf_{t\to\infty} \frac{1}{p(t)} \int_1^t q(s)\,\mathrm{d}s < 0,$$

which is a contradiction. Thus there exists $t_1 \geq t_0$ such that

$$u'(t_1) \leq 0. \tag{49}$$

From (7), (11) and (15) it follows

$$(pu')' < 0 \quad \text{on } [t_1, \infty). \tag{50}$$

Therefore $pu' < 0$ on (t_1, ∞) and

$$u'(t) < 0, \quad t \in (t_1, \infty). \tag{51}$$

Consequently u is decreasing on (t_1, ∞) and there exists

$$\lim_{t\to\infty} u(t) =: \ell_1 \in [0, u(t_1)).$$

Assume that $\ell_1 \in (0, u(t_1))$. Denote $m_1 = \min\{f(x) \colon x \in [\ell_1, u(t_1)]\} > 0$. By virtue of (7) we get

$$(p(t)u'(t))' = -q(t)f(u(t)) \leq -q(t)m_1, \quad t \in [t_1, \infty).$$

Integrating it over (t_1, t), we obtain

$$p(t)u'(t) - p(t_1)u'(t_1) \leq -m_1 \left(\int_1^t q(s)\,\mathrm{d}s - \int_1^{t_1} q(s)\,\mathrm{d}s \right), \quad t \in [t_1, \infty).$$

This together with (49) yield

$$-u'(t) \geq -m_1 \frac{1}{p(t)} \int_1^{t_1} q(s)\,\mathrm{d}s + m_1 \frac{1}{p(t)} \int_1^{t} q(s)\,\mathrm{d}s, \quad t \in [t_1, \infty).$$

Letting $t \to \infty$ and using (46) and (47), we get

$$\liminf_{t\to\infty}(-u'(t)) \geq m_1 \liminf_{t\to\infty} \frac{1}{p(t)} \int_1^t q(s)\,\mathrm{d}s > 0.$$

Therefore there exist $t_2 > t_1$ and $c > 0$ such that

$$-u'(t) \geq m_1 c > 0 \quad \text{for } t \geq t_2.$$

Integrating it over (t_2, t), we obtain

$$u(t) \leq u(t_2) - m_1 c(t - t_2) \quad \text{for } t \geq t_2.$$

Letting $t \to \infty$, we have $\lim_{t\to\infty} u(t) = -\infty$ which contradicts $\ell_1 \in (0, u(t_1))$. Therefore $\ell_1 = 0$. We have proved

$$\lim_{t\to\infty} u(t) = 0. \tag{52}$$

Now, we will prove that $\lim_{t\to\infty} u'(t) = 0$. Assume that $\liminf_{t\to\infty} u'(t) < 0$. Then there exist a sequence $\{t_n\}$ and $\varepsilon > 0$ such that

$$\lim_{n\to\infty} t_n = \infty, \quad \lim_{n\to\infty} u'(t_n) = -\varepsilon.$$

Having in mind (cf. (50)) that pu' is decreasing on $[t_1, \infty)$, we can find $n_0 \in \mathbb{N}$ such that

$$u'(t_n) \leq -\frac{\varepsilon}{2}, \quad p(t)u'(t) < p(t_n)u'(t_n), \quad t > t_n,\ n \geq n_0,$$

and hence

$$u'(t) < -\frac{\varepsilon}{2} p(t_n) \frac{1}{p(t)}, \quad t > t_n,\ n \geq n_0.$$

Integrating it over (t_n, t), we get

$$u(t) - u(t_n) < -\frac{\varepsilon}{2} p(t_n) \int_{t_n}^t \frac{1}{p(s)}\,\mathrm{d}s, \quad t > t_n,\ n \geq n_0.$$

Then, by (52) and (45), we obtain for $t \to \infty$

$$-u(t_n) \leq -\frac{\varepsilon}{2} p(t_n) \int_{t_n}^\infty \frac{1}{p(s)}\,\mathrm{d}s, \quad n \geq n_0.$$

Using l'Hospital's rule, (52), (45) and (14), we get

$$0 = -\lim_{n\to\infty} u(t_n) \leq -\frac{\varepsilon}{2} \lim_{n\to\infty} p(t_n) \int_{t_n}^\infty \frac{1}{p(s)}\,\mathrm{d}s = -\frac{\varepsilon}{2} \lim_{t\to\infty} p(t) \int_t^\infty \frac{1}{p(s)}\mathrm{d}s$$
$$= -\frac{\varepsilon}{2} \lim_{t\to\infty} \frac{p(t)}{p'(t)} = -\infty,$$

a contradiction. Therefore $\liminf_{t\to\infty} u'(t) = 0$ which together with (51) yield $\lim_{t\to\infty} u'(t) = 0$.

Step 2 Let u be eventually negative, i.e. there exists $t_0 \ge 1$ such that $u(t) < 0$ for $t \in [t_0, \infty)$. Assume that $u' < 0$ on $[t_0, \infty)$. Then u is decreasing on (t_0, ∞), $u(t_0) < 0$ and $\lim_{t\to\infty} u(t) =: \ell_2 \ge \min\{\bar{B}, u_0\}$, by Theorem 2.5. Hence $\ell_2 \in (L_0, u(t_0))$ and if we denote $m_2 = \min\{|f(x)|\colon x \in [\ell_2, u(t_0)]\}$, we get $m_2 > 0$. Therefore, analogously as in Step 1, we can derive that $\ell_2 = 0$ and $\lim_{t\to\infty} u'(t) = 0$, as well. □

The next lemmas discuss conditions leading to oscillatory solutions.

Lemma 3.3 *Assume* (45),

$$\liminf_{t\to\infty} \frac{q(t)}{p(t)} > 0, \tag{53}$$

$$\lim_{x\to 0^+} \frac{f(x)}{x} > 0, \tag{54}$$

$$p \in C^2(0,\infty), \quad \limsup_{t\to\infty} \left| \frac{p''(t)}{p'(t)} \right| < \infty. \tag{55}$$

Let u be a solution of problem (7), (8) *with $u_0 \in (0, L)$. Then there exists $\delta_1 > 0$ such that*

$$u(\delta_1) = 0, \quad u'(t) < 0 \quad \text{for } t \in (0, \delta_1]. \tag{56}$$

Proof Let us show that condition (53) implies that (47) holds. According to (53) there exist $c > 0$ and $t_1 > 0$ such that $q(t) > cp(t)$ for $t > t_1$. Consequently, by (14), (46) and l'Hospital's rule,

$$\lim_{t\to\infty} \frac{1}{p(t)} \int_1^t q(s)\,\mathrm{d}s \ge c \lim_{t\to\infty} \frac{1}{p(t)} \int_1^t p(s)\,\mathrm{d}s = c \lim_{t\to\infty} \frac{p(t)}{p'(t)} = \infty$$

which yields (47).

Now, suppose that such δ_1 satisfying (56) does not exist. Then u is positive on $[0, \infty)$. In addition, u is damped due to Theorem 2.6. Therefore, by Theorem 3.2, u satisfies (48).

We define a function $v(t) = \sqrt{p(t)}u(t)$, $t \in [0, \infty)$ and use the arguments of the proof of Lemma 2.7 in [14]. □

For negative starting values u_0 we can prove a dual lemma by similar arguments.

Lemma 3.4 *Assume* (45), (53), (55) *and*

$$\lim_{x\to 0^-} \frac{f(x)}{x} > 0. \tag{57}$$

Let u be a solution of problem (7), (8) *with $u_0 \in (L_0, 0)$. Then there exists $\theta_1 > 0$ such that*

$$u(\theta_1) = 0, \quad u'(t) > 0 \quad \text{for } t \in (0, \theta_1]. \tag{58}$$

If we argue as in the proofs of Lemma 3.3 and Lemma 3.4 working with a_1, A_1 and b_1, B_1 in place of 0, u_0, we get the next lemma.

Lemma 3.5 *Let* (45), (53)–(55), (57) *be satisfied and let* u *be a solution of problem* (7), (8) *with* $u_0 \in (L_0, 0) \cup (0, L)$.

I. Assume that there exist $b_1 > 0$ *and* $B_1 \in (L_0, 0)$ *such that*

$$u(b_1) = B_1, \quad u'(b_1) = 0. \tag{59}$$

Then there exists $\theta > b_1$ *such that*

$$u(\theta) = 0, \quad u'(t) > 0 \quad \text{for } t \in (b_1, \theta]. \tag{60}$$

II. Assume that there exist $a_1 > 0$ *and* $A_1 \in (0, L)$ *such that*

$$u(a_1) = A_1, \quad u'(a_1) = 0. \tag{61}$$

Then there exists $\delta > a_1$ *such that*

$$u(\delta) = 0, \quad u'(t) < 0 \quad \text{for } t \in (a_1, \delta]. \tag{62}$$

4 Main results

The goal of this section is to give sufficient conditions for the existence of oscillatory solutions of problem (7), (8).

Definition 4.1 Let u be an oscillatory solution of problem (7), (8). Denote $\{a_n\}$ ($\{b_n\}$) sequences of local maxima (minima) of u. Assume that either $a_n < b_n < a_{n+1} < b_{n+1}$, $n \in \mathbb{N}$, or $b_n < a_n < b_{n+1} < a_{n+1}$, $n \in \mathbb{N}$. Then the numbers $u(a_n) - u(b_n)$, $n \in \mathbb{N}$, will be called amplitudes of u.

Theorem 4.2 (Existence of oscillatory solutions I) *Assume* (9)–(15), (22), (36), (45), (53)–(55) *and* (57). *Then, for each* $u_0 \in (\bar{B}, 0) \cup (0, L)$, *problem* (7), (8) *has a unique solution* u. *The solution* u *is oscillatory. If moreover* q *fulfils*

$$q \in C^1(0, \infty), \qquad (pq)' > 0 \text{ on } (0, \infty), \tag{63}$$

then u *has decreasing amplitudes.*

Proof Let $u_0 \in (0, L)$. By Theorem 2.6 there exists a unique damped solution u of problem (7), (8). By (37) we can find $L_1 \in (0, L)$ such that $\bar{B} < u(t) \le L_1$ for $t \in [0, \infty)$. The following part of the proof (Step 1 and Step 2) has the same ideas as the proof of Theorem 2.10 in [14]. We write it here for the completeness.

Step 1 Lemma 3.3 yields $\delta_1 > 0$ satisfying (56). Therefore there exists a maximal interval (δ_1, b_1) such that $u' < 0$. If $b_1 = \infty$, then u is eventually negative and decreasing. On the other hand, by Theorem 3.2, u satisfies (48). But this is not possible. Hence $b_1 < \infty$ and there exists $B_1 \in (\bar{B}, 0)$ such that (59) holds. Lemma 3.5 yields $\theta_1 > b_1$ satisfying (60) with $\theta = \theta_1$. Thus u has just one negative local minimum $B_1 = u(b_1)$ between its first zero δ_1 and second zero θ_1.

Step 2 Due to (60) there exists a maximal interval (θ_1, a_1), where $u' > 0$. If $a_1 = \infty$, then u is eventually positive and increasing. On the other hand, by Theorem 3.2, u satisfies (48). We get a contradiction. Therefore $a_1 < \infty$ and there exists $A_1 \in (0, L)$ such that (61) holds. Lemma 3.5 yields $\delta_2 > a_1$ satisfying (62) with $\delta = \delta_2$. Hence u has just one positive local maximum $A_1 = u(a_1)$ between its second zero θ_1 and third zero δ_2.

Step 3 We can continue as in Step 1 and Step 2 and get the sequence

$$0 < \delta_1 < b_1 < \theta_1 < a_1 < \ldots < \delta_n < b_n < \theta_n < a_n < \ldots,$$

where $B_n = u(b_n)$ is a strict unique negative local minimum of u in (δ_n, θ_n) and $A_n = u(a_n)$ is a strict unique positive local maximum of u in (θ_n, δ_{n+1}), $n \in \mathbb{N}$. $\{\delta_n\}_{n=1}^\infty$ and $\{\theta_n\}_{n=1}^\infty$ are unbounded sequences of zeros of u. Therefore u is oscillatory.

Step 4 We will discuss the sequence $\{A_n - B_n\}_{n=1}^\infty$. Equation (7) has an equivalent form

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = -\frac{q(t)}{p(t)} f(u(t)), \quad t \in (0, \infty). \tag{64}$$

Multiplying equation (64) by $\frac{u'p}{q}$, we obtain

$$\frac{p(t)}{q(t)} u''(t)u'(t) + f(u(t))u'(t) = -\frac{p'(t)}{q(t)} u'^2(t), \quad t \in (0, \infty). \tag{65}$$

We define a Lyapunov function V_u by

$$V_u(t) = \frac{p(t)}{q(t)} \frac{u'^2(t)}{2} + F(u(t)), \quad t \in [0, \infty).$$

Then $F(u(t)) > 0$ for $t \in (0, \infty)$, $t \neq \delta_n, \theta_n$, $n \in \mathbb{N}$. Since $u'(\delta_n) \neq 0$, $u'(\theta_n) \neq 0$, $n \in \mathbb{N}$, we get, due to (13)–(15), that $V_u(t) > 0$ for $t \in (0, \infty)$. Further, by (65),

$$\begin{aligned}\frac{dV_u(t)}{dt} &= \left(\frac{p(t)}{q(t)}\right)' \frac{u'^2(t)}{2} + \frac{p(t)}{q(t)} u'(t)u''(t) + f(u(t))u'(t) \\ &= \left(\frac{p(t)}{q(t)}\right)' \frac{u'^2(t)}{2} - \frac{p'(t)}{q(t)} u'^2(t) = -\frac{u'^2(t)}{2} \frac{(p(t)q(t))'}{q^2(t)} \quad \text{for } t \in (0, \infty).\end{aligned}$$

Due to Step 3, we have $u'^2(t) > 0$ for $t \in (0, \infty)$, $t \neq a_n$, b_n, $n \in \mathbb{N}$. Consequently, by (63),

$$\frac{dV_u(t)}{dt} < 0 \quad \text{for } t \in (0, \infty),\ t \neq a_n, b_n,\ n \in \mathbb{N}.$$

Hence V_u is decreasing on $(0, \infty)$ and there exists

$$c_u := \lim_{t \to \infty} V_u(t) \geq 0.$$

So sequences $\{V_u(a_n)\}_{n=1}^\infty = \{F(A_n)\}_{n=1}^\infty$ and $\{V_u(b_n)\}_{n=1}^\infty = \{F(B_n)\}_{n=1}^\infty$ are also decreasing and

$$\lim_{n\to\infty} F(A_n) = \lim_{n\to\infty} F(B_n) = c_u.$$

According to (12), F is increasing on $(0, L)$ and decreasing on $(L_0, 0)$. Therefore the sequence $\{A_n\}$ is decreasing and the sequence $\{B_n\}$ is increasing. Consequently the sequence of amplitudes $\{A_n - B_n\}_{n=1}^\infty$ is decreasing.

For $u_0 \in (\bar{B}, 0)$ we proceed analogously. □

Remark 4.3 1) Let us choose $u_0 \in (L_0, \bar{B}]$ and consider the corresponding solution u of problem (7), (8). Note that u need not be damped in this case. But if u is damped, then u is oscillatory and its amplitudes are decreasing. This can be proved similarly as Theorem 4.2.

2) If we replace assumption $(pq)' > 0$ on $(0,\infty)$ by $(pq)' \geq 0$ on $(0,\infty)$ in Theorem 4.2, then, for $u_0 \in (\bar{B}, 0) \cup (0, L)$, problem (7), (8) has a unique solution. This solution is oscillatory and its amplitudes are nonincreasing.

Example 4.4 Consider the equation

$$u'' + \frac{2}{t}u' + tu(1-u)(u+2) = 0. \tag{66}$$

By virtue of (64) we see that (66) is equation (7) with

$$f(x) = x(1-x)(x+2), \quad p(t) = t^2, \quad q(t) = t^3.$$

Here

$$L_0 = -2, \quad L = 1, \quad F(x) = -\frac{x^4}{4} - \frac{x^3}{3} + x^2, \quad F(-2) = \frac{8}{3}, \quad F(1) = \frac{5}{12},$$
$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0}(1-x)(x+2) = 2.$$

We see that f satisfies conditions (9)–(12), (54), (57). We can easily check that p and q satisfy (13)–(15), (36), (45), (53), (55) and (63). Note that (63) implies (22).

Let us find $\bar{B}$ satisfying (20). Compute

$$\frac{F(x) - F(\bar{B})}{(x-1)^2} = \frac{-\frac{x^4}{4} - \frac{x^3}{3} + x^2 - \frac{5}{12}}{(x-1)^2} = -\frac{1}{4}x^2 - \frac{5}{6}x - \frac{5}{12}$$
$$= -\frac{1}{4}\left(x + \frac{\sqrt{10}+5}{3}\right)\left(x - \frac{\sqrt{10}-5}{3}\right).$$

So polynomial $F(x) - F(\bar{B})$ has roots

$$x_1 = 1, \quad x_2 = -\frac{\sqrt{10}+5}{3} \approx -2.72, \quad x_3 = \frac{\sqrt{10}-5}{3} \approx -0.61 \in (-2, 0).$$

Therefore $\bar{B} = \frac{\sqrt{10}-5}{3}$. We have satisfied all assumptions of Theorem 4.2, thus, for each $u_0 \in \left(\frac{\sqrt{10}-5}{3}, 0\right) \cup (0,1)$, problem (66), (8) has a unique solution and this solution is oscillatory with decreasing amplitudes.

Example 4.5 Let us consider problem (7), (8), where

$$f(x) = \begin{cases} x(1-x)(x+2) & \text{for } x \le 0, \\ \frac{5}{7}x(1-x)(x+3) & \text{for } x > 0, \end{cases} \qquad p(t) = t^\alpha, \quad q(t) = t^\beta, \quad \alpha > 1,\ \beta \ge \alpha.$$

Here

$$L_0 = -2, \quad L = 1, \quad F(-2) = \frac{8}{3}, \quad F(1) = \frac{5}{12}, \quad \bar{B} = \frac{\sqrt{10}-5}{3},$$
$$\lim_{x\to 0^-} \frac{f(x)}{x} = 2, \quad \lim_{x\to 0+} \frac{f(x)}{x} = \frac{15}{7}.$$

We can check that, just as in the previous example, all assumptions of Theorem 4.2 are satisfied.

CASE II Let us assume that the function p in equation (7) fulfils

$$\int_1^\infty \frac{1}{p(s)} \mathrm{d}s = \infty. \tag{67}$$

Simple examples of functions p satisfying (67) are $p(t) = t^\alpha$, $\alpha \in (0,1]$, $p(t) = \arctan t$.

Let us put

$$\psi(t) = \int_1^t \frac{1}{p(s)} \mathrm{d}s, \quad t \in [1,\infty). \tag{68}$$

Since $p > 0$ on $(0,\infty)$ (cf. (13), (14)), ψ is increasing on $(1,\infty)$. By (67), we get

$$\lim_{t\to\infty} \psi(t) = \infty. \tag{69}$$

Consequently there exists $\psi^{-1}\colon [0,\infty) \to [1,\infty)$. Since $p' > 0$ on $(0,\infty)$ (cf. (14)), there exists

$$\lim_{t\to\infty} \frac{1}{p(t)} \in [0,\infty). \tag{70}$$

By the substitution

$$x = \psi(t), \quad v(x) = v(\psi(t)) = u(t), \quad t \in [1,\infty), \tag{71}$$

equation (7) overcomes to the equation

$$\frac{\mathrm{d}^2 v(x)}{\mathrm{d}x^2} + p(\psi^{-1}(x))q(\psi^{-1}(x))f(v(x)) = 0. \tag{72}$$

Using (67)–(72), we will prove the next theorem.

Theorem 4.6 (Existence of oscillatory solutions II) *Assume* (9)–(15), (22), (36), (67) *and*

$$\lim_{t\to\infty}\frac{1}{p(t)}\int_1^t q(s)\mathrm{d}s=\infty. \tag{73}$$

Then, for each $u_0\in(\bar{B},0)\cup(0,L)$*, problem* (7)*,* (8) *has a unique solution* u*. The solution* u *is oscillatory. If moreover* q *satisfies* (63)*, then* u *has decreasing amplitudes.*

Note that Remark 4.3 holds for Theorem 4.6 too.

Proof Let us choose $u_0\in(\bar{B},0)\cup(0,L)$. By Theorem 2.6 there exists a unique solution u of problem (7), (8) which fulfils

$$\bar{B}<u(t)\text{ for }t\in[0,\infty),\qquad \sup\{u(t)\colon t\in[0,\infty)\}<L. \tag{74}$$

Assume that u is not oscillatory. Then u is either eventually positive or eventually negative.

Step 1 Let u be eventually positive, i.e. there exists $t_0\ge 1$ such that $u(t)>0$ for $t\in[t_0,\infty)$.

Assume that $u'>0$ on (t_0,∞). Then u is increasing on (t_0,∞) and, by (74), we have $u(t_0)>0$, $\lim_{t\to\infty}u(t)=:\ell_0\in(u(t_0),L)$. Denote $m_0=\min\{f(x)\colon x\in[u(t_0),\ell_0]\}>0$. By virtue of (7) we get

$$\big(p(t)u'(t)\big)'=-q(t)f(u(t))\le -q(t)m_0,\quad t\in[t_0,\infty).$$

Integrating it over (t_0,t), we obtain

$$p(t)u'(t)-p(t_0)u'(t_0)\le -m_0\left(\int_1^t q(s)\mathrm{d}s-\int_1^{t_0}q(s)\mathrm{d}s\right),\quad t\in[t_0,\infty).$$

Therefore

$$0<u'(t)\le\frac{1}{p(t)}\left(p(t_0)u'(t_0)+m_0\int_1^{t_0}q(s)\,\mathrm{d}s\right)-\frac{m_0}{p(t)}\int_1^t q(s)\,\mathrm{d}s,\quad t\in[t_0,\infty).$$

Put $K=p(t_0)u'(t_0)+m_0\int_1^{t_0}q(s)\mathrm{d}s$. Then $K\in(0,\infty)$ and, by (70), we have $\lim_{t\to\infty}\frac{K}{p(t)}\in[0,\infty)$. Hence, letting $t\to\infty$ in the inequality

$$0<u'(t)\le\frac{K}{p(t)}-\frac{m_0}{p(t)}\int_1^t q(s)\,\mathrm{d}s,$$

we obtain, by (73),

$$0\le\lim_{t\to\infty}\frac{K}{p(t)}-m_0\lim_{t\to\infty}\frac{1}{p(t)}\int_1^t q(s)\,\mathrm{d}s=-\infty,$$

a contradiction. Thus there exists $t_1\ge t_0$ such that

$$u'(t_1)\le 0. \tag{75}$$

From (7), (11) and (15) it follows $(pu')' < 0$ on $[t_1, \infty)$. Therefore $pu' < 0$ and $u' < 0$ on (t_1, ∞). Consequently u is decreasing on (t_1, ∞) and there exists $\lim_{t\to\infty} u(t) =: \ell_1 \in [0, u(t_1))$.

Assume that $\ell_1 \in (0, u(t_1))$. Denote $m_1 = \min\{f(x) \colon x \in [\ell_1, u(t_1)]\} > 0$. By virtue of (7) we get

$$(p(t)u'(t))' = -q(t)f(u(t)) \le -q(t)m_1, \quad t \in [t_1, \infty).$$

Integrating it over (t_1, t), we obtain

$$p(t)u'(t) - p(t_1)u'(t_1) \le -m_1 \left(\int_1^t q(s)\,\mathrm{d}s - \int_1^{t_1} q(s)\,\mathrm{d}s \right), \quad t \in [t_1, \infty).$$

This together with (75) yield

$$u'(t) \le m_1 \frac{1}{p(t)} \int_1^{t_1} q(s)\,\mathrm{d}s - m_1 \frac{1}{p(t)} \int_1^{t} q(s)\,\mathrm{d}s, \quad t \in [t_1, \infty).$$

Letting $t \to \infty$ and using (70) and (73), we get $\lim_{t\to\infty} u'(t) = -\infty$. This contradicts $\ell_1 \in (0, u(t_1))$. Thus $\ell_1 = 0$.

Now, consider substitution (71) and for t_1 of (75) denote $x_1 = \psi(t_1)$. We have $v(\psi(t)) = u(t) > 0$ for $t \ge t_1$. Hence

$$v(x) > 0 \ \text{ for } x \ge x_1, \quad \frac{\mathrm{d}v}{\mathrm{d}x}(x_1) = u'(t_1)p(t_1) \le 0.$$

Since v is a solution of equation (72), we get, by (11), (13)–(15), that $\frac{\mathrm{d}^2 v}{\mathrm{d}x^2}(x) < 0$ for $x \in [x_1, \infty)$. Therefore $\lim_{x\to\infty} \frac{\mathrm{d}v}{\mathrm{d}x}(x) < 0$ which contradicts $v > 0$ on $[x_1, \infty)$.

We proved that u cannot be eventually positive.

Step 2 Let u be eventually negative, i.e. there exists $t_0 \ge 1$ such that $u(t) < 0$ for $t \in [t_0, \infty)$.

Assume that $u' < 0$ on (t_0, ∞). Then u is decreasing on (t_0, ∞), $u(t_0) < 0$ and $\lim_{t\to\infty} u(t) =: \ell_2 \ge \min\{\bar{B}, u_0\}$, by Theorem 2.5. Since $\ell_2 \in (L_0, u(t_0))$, using the same arguments as in Step 1, we can derive that u cannot be eventually negative.

Step 3 Step 1 and Step 2 imply that u is oscillatory. Let $u_0 \in (0, L)$. Then, by Lemma 2.1, we have the sequence

$$0 < \delta_1 < b_1 < \theta_1 < a_1 < \ldots < \delta_n < b_n < \theta_n < a_n < \ldots,$$

where $B_n = u(b_n)$ is a strict unique negative local minimum of u in (δ_n, θ_n) and $A_n = u(a_n)$ is a strict unique positive local maximum of u in (θ_n, δ_{n+1}), $n \in \mathbb{N}$. $\{\delta_n\}_{n=1}^\infty$ and $\{\theta_n\}_{n=1}^\infty$ are unbounded sequences of zeros of u. By the same arguments as in Step 4 in the proof of Theorem 4.2, we can prove that amplitudes of u are decreasing.

If $u_0 \in (\bar{B}, 0)$, we argue similarly. □

Example 4.7 Consider problem (7), (8) with

$$f(x) = \begin{cases} -(x+2^\lambda+2) & \text{for } x \le -2, \\ |x|^\lambda \operatorname{sgn} x & \text{for } x \in (-2,1), \\ 2-x & \text{for } x \ge 1, \end{cases} \qquad p(t) = t^\alpha,\ q(t) = t^\beta,\ t \in [0,\infty),$$

where $\lambda \ge 1$, $\alpha \in (0,1]$, $\beta \ge \alpha$. Here $L_0 = -2-2^\lambda$, $L = 2$ and assumptions (9)–(15), (22) and (36) are satisfied for all such α, β and λ. Therefore the assertion of Theorem 2.6 holds. Since

$$\int_1^\infty \frac{1}{s^\alpha}\mathrm{d}s = \infty, \qquad \lim_{t\to\infty} \frac{1}{t^\alpha}\int_1^t s^\beta \mathrm{d}s = \frac{1}{\beta+1}\lim_{t\to\infty}\left(t^{\beta-\alpha+1} - \frac{1}{t^\alpha}\right) = \infty,$$

we have fulfilled all assumptions of Theorem 4.6 and thus it is applicable here.

In the following example the function p is bounded.

Example 4.8 Let us consider problem (7), (8) with

$$f(x) = |x|^\beta \operatorname{sgn}(x)(1-x)(x+2), \qquad p(t) = (\arctan t)^\gamma,\ q(t) = t^\alpha,\ t \in [0,\infty),$$

where $\alpha > \gamma - 1$, $\beta \ge 1$, $\gamma \ge 1$. We can verify that assumptions (9)–(15), (22) and (36) are satisfied for all such α, β, γ. In particular,

$$\lim_{t\to 0^+} \frac{1}{(\arctan t)^\gamma}\int_0^t s^\alpha \,\mathrm{d}s = \lim_{t\to 0^+}\left(\frac{t}{\arctan t}\right)^\gamma t^{\alpha+1-\gamma} = 0.$$

Consequently, Theorem 2.6 can be applied. Since

$$\lim_{t\to\infty}(\arctan t)^{-\gamma} = \left(\frac{2}{\pi}\right)^\gamma \ne 0 \Longrightarrow \int_1^\infty \frac{1}{(\arctan s)^\gamma}\,\mathrm{d}s = \infty,$$

$$\lim_{t\to\infty}\frac{1}{(\arctan t)^\gamma}\int_1^t s^\alpha \mathrm{d}s = \frac{1}{(\alpha+1)\left(\frac{\pi}{2}\right)^\gamma}\left(\lim_{t\to\infty} t^{\alpha+1} - 1\right) = \infty,$$

we have fulfilled all assumptions of Theorem 4.6.

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