

# Additional Experiment and Linear Statistical Models\*

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## Abstract

An accuracy of parameter estimates need not be sufficient for their unforeseen utilization. Therefore some additional measurement is necessary in order to attain the required precision. The problem is to express the correction to the original estimates in an explicit form.

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## 1 Introduction

A linear statistical model is considered in the form

$$\mathbf{Y}_1 \sim_n (\mathbf{X}_1\boldsymbol{\beta}, \boldsymbol{\Sigma}_1),$$

where  $\mathbf{Y}_1$  is an  $n$ -dimensional random vector (observation vector) with the mean value  $E(\mathbf{Y}_1) = \mathbf{X}_1\boldsymbol{\beta}$  and the covariance matrix  $\text{Var}(\mathbf{Y}_1) = \boldsymbol{\Sigma}_1$ . The  $n \times k$  matrix  $\mathbf{X}_1$  is given, the  $k$ -dimensional vector  $\boldsymbol{\beta}$  is unknown and the matrix  $\boldsymbol{\Sigma}_1$  is known.

In the following text it is assumed that the rank  $r(\mathbf{X}_1) = k \leq n$  and the matrix  $\boldsymbol{\Sigma}_1$  is positive definite.

The accuracy of the estimator  $\hat{\boldsymbol{\beta}}$  is characterized by its covariance matrix  $\text{Var}(\hat{\boldsymbol{\beta}})$ . If it is not satisfactory, then it is necessary to realize an additional experiment, e.g.

$$\mathbf{Y}_2 \sim_m (\mathbf{X}_2\boldsymbol{\beta}, \boldsymbol{\Sigma}_2).$$

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(Another forms of additional experiments are described in the following sections.)

Thus the joint model (original and additional) can be written as

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n+m} \left[ \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{pmatrix} \right].$$

The covariance matrix of the estimator  $\widehat{\boldsymbol{\beta}}$  in this model is obviously more satisfactory than the original covariance matrix.

In the following text the following notation will be used.

$\widehat{\boldsymbol{\beta}}$  ... the best linear unbiased estimator (BLUE) in a model without constraints;

$\widehat{\boldsymbol{\beta}}$  ... the BLUE in the model with constraints;

$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)$ ,  $\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)$  ... the BLUEs based on the observation vector  $\mathbf{Y}_1$  and  $\mathbf{Y}_1, \mathbf{Y}_2$ , respectively;

$\mathbf{A}^+$  ... the Moore–Penrose generalized inverse of the matrix  $\mathbf{A}$

(i.e.  $\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A}$ ,  $\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+$ ,  $\mathbf{A}\mathbf{A}^+ = (\mathbf{A}\mathbf{A}^+)'$ ,  $\mathbf{A}^+\mathbf{A} = (\mathbf{A}^+\mathbf{A})'$ );

$\mathbf{A}_{m(N)}^-$  ... the minimum  $\mathbf{N}$ -seminorm generalized inverse of the matrix  $\mathbf{A}$

(i.e.  $\mathbf{A}\mathbf{A}_{m(N)}^-\mathbf{A} = \mathbf{A}$ ,  $\mathbf{N}\mathbf{A}_{m(N)}^-\mathbf{A} = (\mathbf{N}\mathbf{A}_{m(N)}^-\mathbf{A})'$ ,  $\mathbf{N}$  is a positive semidefinite matrix; in more detail see in [3]);

$\mathbf{b}_{q,1} + \mathbf{B}_{q,k}\boldsymbol{\beta} = \mathbf{0}$  ... constraints in the original model;

$\mathbf{g}_{r,1} + \mathbf{G}_{r,k}\boldsymbol{\beta} = \mathbf{0}$  ... constraints in the additional model;

$\mathcal{M}(\mathbf{A})$  denotes the column space of the matrix  $\mathbf{A}$ , i.e.

$$\mathcal{M}(\mathbf{A}) = \{\mathbf{A}\mathbf{u} : \mathbf{u} \in R^n\}.$$

The original model can be either of the form (model without constraints)

$$\mathbf{Y}_1 \sim_n (\mathbf{X}_1\boldsymbol{\beta}, \boldsymbol{\Sigma}_1),$$

or of the form (model with constraints)

$$\mathbf{Y}_1 \sim_n (\mathbf{X}_1\boldsymbol{\beta}, \boldsymbol{\Sigma}_1), \quad \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0},$$

where the rank of the matrix  $\mathbf{B}$  is  $r(\mathbf{B}) = q < k$ .

The additional model can be either of the form (model without constraints)

$$\mathbf{Y}_2 \sim_m (\mathbf{X}_2\boldsymbol{\beta}, \boldsymbol{\Sigma}_2), \quad r(\mathbf{X}_2) = k < m, \quad \boldsymbol{\Sigma}_2 \text{ is positive definite,}$$

or of the forms

model with additional parameters  $\boldsymbol{\gamma} \in R^l$

$$\mathbf{Y}_2 \sim_m \left[ (\mathbf{D}_{m,k}, \mathbf{X}_{2,(m,l)}) \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \boldsymbol{\Sigma}_2 \right], \quad r(\mathbf{D}, \mathbf{X}_2) = k + l < m,$$

the matrix  $\boldsymbol{\Sigma}_2$  is positive definite,

model with additional constraints

$$\mathbf{Y}_2 \sim_m (\mathbf{X}_2\boldsymbol{\beta}, \boldsymbol{\Sigma}_2), \quad \mathbf{g}_{r,1} + \mathbf{G}_{r,k}\boldsymbol{\beta} = \mathbf{0}, \quad r(\mathbf{X}_2) = l < m, \quad r(\mathbf{G}) = r < k,$$

models with additional parameters and additional constraints

$$\mathbf{Y}_2 \sim_m \left[ (\mathbf{D}_{m,k}, \mathbf{X}_{2,(m,l)}) \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \boldsymbol{\Sigma}_2 \right], \quad \mathbf{g}_{r,1} + \mathbf{G}_{r,k}\boldsymbol{\beta} = \mathbf{0},$$

$$r(\mathbf{D}, \mathbf{X}_2) = k + l < m, \quad \boldsymbol{\Sigma}_2 \text{ is positive definite,}$$

$$\mathbf{Y}_2 \sim_m \left[ (\mathbf{D}_{m,k}, \mathbf{X}_{2,(m,l)}) \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \boldsymbol{\Sigma}_2 \right], \quad \mathbf{g}_{r,1} + \mathbf{G}_{1,(r,k)}\boldsymbol{\beta} + \mathbf{G}_{2,(r,l)}\boldsymbol{\gamma} = \mathbf{0},$$

$$r(\mathbf{D}, \mathbf{X}_2) = k + l < m, \quad \boldsymbol{\Sigma}_2 \text{ is positive definite,}$$

$$r(\mathbf{G}_1, \mathbf{G}_2) = r, \quad r(\mathbf{G}_2) = l < r.$$

The problem is to find the vector  $\mathbf{k}$  either in the equation

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{k}, \quad \text{or} \quad \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) + \mathbf{k},$$

or

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) + \mathbf{k}, \quad \text{or} \quad \widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1) + \mathbf{k}$$

( $\widehat{\widehat{\boldsymbol{\beta}}}$  denotes an estimator respecting two constraints) in dependence on the form of the joint model.

All models considered are assumed to be regular.

## 2 Original models without constraints

**Lemma 1** Let  $\mathbf{A}$  be an  $n \times n$  positive semidefinite matrix,  $\mathbf{C}$  be an  $m \times m$  positive semidefinite matrix and  $\mathbf{B}$  be an  $n \times m$  matrix with the properties  $\mathcal{M}(\mathbf{B}) \subset \mathcal{M}(\mathbf{A})$  and  $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C})$ . Then

$$(\mathbf{A} - \mathbf{B}\mathbf{C}^+\mathbf{B}')^+ = \mathbf{A}^+ + \mathbf{A}^+\mathbf{B}(\mathbf{C} - \mathbf{B}'\mathbf{A}^+\mathbf{B})^+\mathbf{B}'\mathbf{A}^+,$$

$$(\mathbf{A} - \mathbf{B}\mathbf{C}^+\mathbf{B}')^+\mathbf{B}\mathbf{C}^+ = \mathbf{A}^+\mathbf{B}(\mathbf{C} - \mathbf{B}'\mathbf{A}^+\mathbf{B})^+.$$

**Proof** It is sufficient to verify the properties of the Moore–Penrose generalized inverse of a matrix (in more detail see [3]).  $\square$

**Lemma 2** Let  $\mathbf{A}$  be any  $n \times k$  matrix,  $\mathbf{B}$  be any  $q \times k$  matrix and  $\boldsymbol{\Sigma}$  be any  $n \times n$  positive semidefinite matrix. Then

$$\left[ \begin{pmatrix} \mathbf{A}', \mathbf{B}' \\ \boldsymbol{\Sigma}, \mathbf{0} \\ \mathbf{0}, \mathbf{0} \end{pmatrix}^{-} \right]' = \left( \left[ (\mathbf{M}_{B'}\mathbf{A}')_{m(\boldsymbol{\Sigma})}^- \right]', \left\{ \mathbf{I} - \left[ (\mathbf{M}_{B'}\mathbf{A}')_{m(\boldsymbol{\Sigma})}^- \right]' \mathbf{A} \right\} \mathbf{B}^+ \right).$$

Here  $\mathbf{M}_{B'} = \mathbf{I} - \mathbf{P}_{B'}$ ,  $\mathbf{P}_{B'} = \mathbf{B}'(\mathbf{B}')^+$ .

**Proof** It is valid that

$$\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \left( \left[ (\mathbf{M}_{B'}\mathbf{A}')_{m(\boldsymbol{\Sigma})}^- \right]', \left\{ \mathbf{I} - \left[ (\mathbf{M}_{B'}\mathbf{A}')_{m(\boldsymbol{\Sigma})}^- \right]' \mathbf{A} \right\} \mathbf{B}^+ \right) \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{A}\mathbf{M}_{B'} \left[ (\mathbf{M}_{B'}\mathbf{A}')_{m(\boldsymbol{\Sigma})}^- \right]' \mathbf{A}\mathbf{M}_{B'} + \mathbf{A}\mathbf{P}_{B'} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix}$$

and

$$\begin{aligned} & \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \left( \left[ (\mathbf{M}_{B'} \mathbf{A}')_{m(\Sigma)}^- \right]', \left\{ \mathbf{I} - \left[ (\mathbf{M}_{B'} \mathbf{A}')_{m(\Sigma)}^- \right]' \mathbf{A} \right\} \mathbf{B}^+ \right) \begin{pmatrix} \boldsymbol{\Sigma}, \mathbf{0} \\ \mathbf{0}, \mathbf{0} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{A} \mathbf{M}_{B'} \left[ (\mathbf{M}_{B'} \mathbf{A}')_{m(\Sigma)}^- \right]' \boldsymbol{\Sigma}, \mathbf{0} \\ \mathbf{0}, \mathbf{0} \end{pmatrix} \end{aligned}$$

is a symmetric matrix.

The relationship

$$\left[ (\mathbf{M}_{B'} \mathbf{A}')_{m(\Sigma)}^- \right]' = \mathbf{M}_{B'} \left[ (\mathbf{M}_{B'} \mathbf{A}')_{m(\Sigma)}^- \right]'$$

was utilized.  $\square$

**Lemma 3** *Let the model*

$$\begin{pmatrix} \tilde{\boldsymbol{\beta}} \\ -\mathbf{g} \end{pmatrix} \sim_{k+q} \left[ \begin{pmatrix} \mathbf{I} \\ \mathbf{G} \end{pmatrix} \boldsymbol{\beta}, \begin{pmatrix} \mathbf{W}, \mathbf{0} \\ \mathbf{0}, \mathbf{0} \end{pmatrix} \right]$$

be considered ( $\mathbf{W}$  need not be regular and the rank  $r(\mathbf{G}_{q,k})$  need not be  $q < k$ ). The BLUE of  $\boldsymbol{\beta}$  is

$$\widehat{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}} - \mathbf{W} \mathbf{G}' (\mathbf{G} \mathbf{W} \mathbf{G}')^+ (\mathbf{G} \tilde{\boldsymbol{\beta}} + \mathbf{g})$$

if  $\mathbf{W}$  is p.d. and

$$\text{Var}(\widehat{\boldsymbol{\beta}}) = \mathbf{W} - \mathbf{W} \mathbf{G}' (\mathbf{G} \mathbf{W} \mathbf{G}')^+ \mathbf{G} \mathbf{W}$$

if  $\mathbf{W}$  is p.d.

In general

$$\widehat{\boldsymbol{\beta}} = \tilde{\boldsymbol{\beta}} - [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}' \mathbf{G}]^{-1} \mathbf{G}' \left\{ \mathbf{G} [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}' \mathbf{G}]^{-1} \mathbf{G}' \right\}^+ (\mathbf{G} \tilde{\boldsymbol{\beta}} + \mathbf{g})$$

and

$$\begin{aligned} \text{Var}(\widehat{\boldsymbol{\beta}}) &= [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}' \mathbf{G}]^{-1} - [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}' \mathbf{G}]^{-1} \mathbf{G}' \\ &\times \left\{ \mathbf{G} [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}' \mathbf{G}]^{-1} \mathbf{G}' \right\}^+ \mathbf{G} [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}' \mathbf{G}]^{-1} - \mathbf{M}_{G'}. \end{aligned}$$

**Proof** Since  $\boldsymbol{\beta}$  is unbiasedly estimable (in more detail see in [1], p. 337 and 346), the BLUE in the model with constraints is (see the preceding lemma)

$$\begin{aligned}
\widehat{\beta} &= \left[ (\mathbf{I}, \mathbf{G}')^{-1} \begin{pmatrix} \mathbf{W}, 0 \\ 0, 0 \end{pmatrix} \right]' \begin{pmatrix} \tilde{\beta} \\ -\mathbf{g} \end{pmatrix} \\
&= \left( [(\mathbf{M}_{G'})_{m(W)}^-]', \left\{ \mathbf{I} - [(\mathbf{M}_{G'})_{m(W)}^-] \mathbf{G}^+ \right\} \right) \begin{pmatrix} \tilde{\beta} \\ -\mathbf{g} \end{pmatrix} = [\mathbf{M}_{G'}(\mathbf{W} + \mathbf{M}_{G'})^+ \mathbf{M}_{G'}]^+ \\
&\quad \times \mathbf{M}_{G'}(\mathbf{W} + \mathbf{M}_{G'})^+ \tilde{\beta} + \left\{ \mathbf{I} - [\mathbf{M}_{G'}(\mathbf{W} + \mathbf{M}_{G'})^+ \mathbf{M}_{G'}]^+ (\mathbf{W} + \mathbf{M}_{G'})^+ \right\} \mathbf{G}^+ (-\mathbf{g}) \\
&= \left\{ [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} - [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \mathbf{G}' \right. \\
&\quad \left. \times \left\{ \mathbf{G} [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \mathbf{G}' \right\}^+ \mathbf{G} [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \right\} \\
&\quad \times [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}] \tilde{\beta} + \left( \mathbf{I} - \left\{ [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} - [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \right\} \right. \\
&\quad \left. \times \mathbf{G}' \left\{ \mathbf{G} [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \mathbf{G}' \right\}^+ \mathbf{G} [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \right) \\
&\quad \left. \times [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}] \right) \mathbf{G}^+ (-\mathbf{g}) \\
&= \tilde{\beta} - [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \mathbf{G}' \left\{ \mathbf{G} [(\mathbf{W} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \mathbf{G}' \right\}^+ (\mathbf{G} \tilde{\beta} + \mathbf{g}).
\end{aligned}$$

The expression for  $\text{Var}(\widehat{\beta})$  can be obtained easily.

If  $\mathbf{W}$  is p.d., then (see [1], p. 337)

$$[(\mathbf{M}_{G'})_{m(W)}^-]^{-1} = (\mathbf{M}_{G'} \mathbf{W}^{-1} \mathbf{M}_{G'})^+ \mathbf{M}_{G'} \mathbf{W}^{-1} = \mathbf{I} - \mathbf{W} \mathbf{G}' (\mathbf{G} \mathbf{W} \mathbf{G}')^+ \mathbf{G}$$

and the proof can be proceeded analogously.  $\square$

In the following text  $\mathbf{C}_1 = \mathbf{X}'_1 \boldsymbol{\Sigma}_1^{-1} \mathbf{X}_1$ ,  $\mathbf{C}_2 = \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} \mathbf{X}_2$ .

**Theorem 1** *If*

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n+m} \left[ \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \beta, \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{pmatrix} \right],$$

then

$$\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\beta}(\mathbf{Y}_1) + (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} [\mathbf{Y}_2 - \mathbf{X}_2 \widehat{\beta}(\mathbf{Y}_1)]$$

and

$$\text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)] = \text{Var} [\widehat{\beta}(\mathbf{Y}_1)] - \mathbf{C}_1^{-1} \mathbf{X}'_2 (\boldsymbol{\Sigma}_2 + \mathbf{X}_2 \mathbf{C}_1^{-1} \mathbf{X}'_2)^{-1} \mathbf{X}_2 \mathbf{C}_1^{-1}.$$

**Proof**

$$\begin{aligned}
\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) &= (\mathbf{C}_1 + \mathbf{C}_2)^{-1} (\mathbf{X}'_1 \boldsymbol{\Sigma}_1^{-1} \mathbf{Y}_1 + \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} \mathbf{Y}_2) \\
&= [\mathbf{C}_1^{-1} - \mathbf{C}_1^{-1} \mathbf{X}'_2 (\boldsymbol{\Sigma}_2 + \mathbf{X}_2 \mathbf{C}_1^{-1} \mathbf{X}'_2)^{-1} \mathbf{X}_2 \mathbf{C}_1^{-1}] (\mathbf{X}'_1 \boldsymbol{\Sigma}_1^{-1} \mathbf{Y}_1 + \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} \mathbf{Y}_2) \\
&= \widehat{\beta}(\mathbf{Y}_1) - \mathbf{C}_1^{-1} \mathbf{X}'_2 (\boldsymbol{\Sigma}_2 + \mathbf{X}_2 \mathbf{C}_1^{-1} \mathbf{X}'_2)^{-1} \mathbf{X}_2 \mathbf{C}_1^{-1} \mathbf{X}'_1 \boldsymbol{\Sigma}_1^{-1} \mathbf{Y}_1 + (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} \mathbf{Y}_2 \\
&= \widehat{\beta}(\mathbf{Y}_1) - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} \mathbf{X}_2 \mathbf{C}_1^{-1} \mathbf{X}'_1 \boldsymbol{\Sigma}_1^{-1} \mathbf{Y}_1 + (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} \mathbf{Y}_2 \\
&= \widehat{\beta}(\mathbf{Y}_1) + (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} [\mathbf{Y}_2 - \mathbf{X}_2 \widehat{\beta}(\mathbf{Y}_1)].
\end{aligned}$$

Since

$$\text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)] = (\mathbf{C}_1 + \mathbf{C}_2)^{-1}$$

and

$$(\mathbf{C}_1 + \mathbf{C}_2)^{-1} = \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1} \mathbf{X}'_2 (\boldsymbol{\Sigma}_2 + \mathbf{X}_2 \mathbf{C}_1^{-1} \mathbf{X}'_2)^{-1} \mathbf{X}_2 \mathbf{C}_1^{-1},$$

we have

$$\text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)] = \text{Var} [\widehat{\beta}(\mathbf{Y}_1)] - \mathbf{C}_1^{-1} \mathbf{X}'_2 (\boldsymbol{\Sigma}_2 + \mathbf{X}_2 \mathbf{C}_1^{-1} \mathbf{X}'_2)^{-1} \mathbf{X}_2 \mathbf{C}_1^{-1}.$$

□

**Theorem 2** *If*

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n+m} \left[ \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{D} & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{pmatrix} \right],$$

then

$$\begin{aligned}
\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\beta}(\mathbf{Y}_1) - \mathbf{C}_1^{-1} \mathbf{D}' [\mathbf{M}_{X_2} (\boldsymbol{\Sigma}_2 + \mathbf{D} \mathbf{C}_1^{-1} \mathbf{D}') \mathbf{M}_{X_2}]^+ \mathbf{D} \widehat{\beta}(\mathbf{Y}_1) \\
&\quad + [\mathbf{C}_1 + \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{D}]^{-1} \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{Y}_2
\end{aligned}$$

and

$$\text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)] = \text{Var} [\widehat{\beta}(\mathbf{Y}_1)] - \mathbf{C}_1^{-1} \mathbf{D}' [\mathbf{M}_{X_2} (\boldsymbol{\Sigma}_2 + \mathbf{D} \mathbf{C}_1^{-1} \mathbf{D}') \mathbf{M}_{X_2}]^+ \mathbf{D} \mathbf{C}_1^{-1}.$$

**Proof**

$$\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) = (\mathbf{I}, \mathbf{0}) \begin{pmatrix} \boxed{11} & \boxed{12} \\ \boxed{21} & \boxed{22} \end{pmatrix} \begin{pmatrix} \mathbf{X}'_1 \boldsymbol{\Sigma}_1^{-1} \mathbf{Y}_1 + \mathbf{D}' \boldsymbol{\Sigma}_2^{-1} \mathbf{Y}_2 \\ \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} \mathbf{Y}_2 \end{pmatrix},$$

where

$$\begin{aligned}
\begin{pmatrix} \boxed{11} & \boxed{12} \\ \boxed{21} & \boxed{22} \end{pmatrix} &= \left[ \begin{pmatrix} \mathbf{X}'_1 & \mathbf{D}' \\ \mathbf{0} & \mathbf{X}'_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\Sigma}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{D} & \mathbf{X}_2 \end{pmatrix} \right]^{-1} \\
&= \begin{pmatrix} \mathbf{C}_1 + \mathbf{D}' \boldsymbol{\Sigma}_2^{-1} \mathbf{D} & \mathbf{D}' \boldsymbol{\Sigma}_2^{-1} \mathbf{X}_2 \\ \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} \mathbf{D} & \mathbf{C}_2 \end{pmatrix}^{-1},
\end{aligned}$$

$$\begin{aligned}
\boxed{11} &= [\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1}, \\
\boxed{12} &= -[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}, \\
\boxed{21} &= -\mathbf{C}_2^{-1}\mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}\mathbf{D}[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1}, \\
\boxed{22} &= [\mathbf{C}_2 - \mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2]^{-1} \\
&= [\mathbf{X}'_2(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2]^{-1}.
\end{aligned}$$

Thus

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= [\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D} - \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}\mathbf{D}]^{-1}(\mathbf{X}'_1\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1 \\
&\quad + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2 - \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}\mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2) \\
&= \left\{ (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1} + (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2[\mathbf{C}_2 - \mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}\mathbf{D} \right. \\
&\quad \times (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2]^{-1}\mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1} \left. \right\} \mathbf{X}'_1\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1 \\
&\quad + [\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{Y}_2 \\
&= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2 \\
&\quad \times [\mathbf{X}'_2(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2]^{-1}\mathbf{X}'_2(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) \\
&\quad + [\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{Y}_2 \\
&= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{D}'[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}]^+\mathbf{D}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) \\
&\quad + [\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{Y}_2. \\
\text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] &= (\mathbf{I}, \mathbf{0}) \begin{pmatrix} \mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}, & \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2 \\ \mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}\mathbf{D}, & \mathbf{C}_2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{I} \\ \mathbf{0} \end{pmatrix} \\
&= (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1} + (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2[\mathbf{C}_2 - \mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}\mathbf{D} \\
&\quad \times (\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2]^{-1}\mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}\mathbf{D}(\mathbf{C}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{D})^{-1} \\
&= \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1} + \mathbf{C}_1^{-1}\mathbf{D}'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2 \\
&\quad \times [\mathbf{X}'_2(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2]^{-1}\mathbf{X}'_2(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1} \\
&= \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{D}'[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}]^+\mathbf{D}\mathbf{C}_1^{-1}.
\end{aligned}$$

□

**Theorem 3** *If*

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n+m} \left[ \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{pmatrix} \right], \quad \mathbf{g} + \mathbf{G}\boldsymbol{\beta} = \mathbf{0},$$

then

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{G}'[\mathbf{G}(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{G}']^{-1}[\mathbf{G}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{g}] \\
&\quad + [\mathbf{M}_{G'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{G'}]^+\mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)]
\end{aligned}$$

and

$$\begin{aligned} \text{Var} [\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2)] &= \text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)] - \mathbf{C}_1^{-1} \mathbf{X}_2' (\boldsymbol{\Sigma}_2 + \mathbf{X}_2 \mathbf{C}_1^{-1} \mathbf{X}_2')^{-1} \mathbf{X}_2 \mathbf{C}_1^{-1} \\ &\quad - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{G}' [\mathbf{G}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{G}']^{-1} \mathbf{G}(\mathbf{C}_1 + \mathbf{C}_2)^{-1}. \end{aligned}$$

**Proof** With respect to Lemma 3

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) - \mathbf{V} \mathbf{G}' (\mathbf{G} \mathbf{V} \mathbf{G}')^+ [\mathbf{G} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) + \mathbf{g}],$$

where  $\mathbf{V} = (\mathbf{C}_1 + \mathbf{C}_2)^{-1}$  and (see Theorem 1)

$$\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{X}_2' \boldsymbol{\Sigma}_2^{-1} [\mathbf{Y}_2 - \mathbf{X}_2 \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)]$$

Thus

$$\begin{aligned} \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{X}_2' \boldsymbol{\Sigma}_2^{-1} [\mathbf{Y}_2 - \mathbf{X}_2 \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)] \\ &\quad - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{G}' [\mathbf{G}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{G}']^{-1} [\mathbf{G} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) + \mathbf{g}] \\ &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{X}_2' \boldsymbol{\Sigma}_2^{-1} [\mathbf{Y}_2 - \mathbf{X}_2 \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)] \\ &\quad - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{G}' [\mathbf{G}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{G}']^{-1} \\ &\quad \times (\mathbf{G} \{ \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{X}_2' \boldsymbol{\Sigma}_2^{-1} [\mathbf{Y}_2 - \mathbf{X}_2 \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)] \} + \mathbf{g}) \\ &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + [\mathbf{M}_{\mathbf{G}'} (\mathbf{C}_1 + \mathbf{C}_2) \mathbf{M}_{\mathbf{G}'}]^+ \mathbf{X}_2' \boldsymbol{\Sigma}_2^{-1} [\mathbf{Y}_2 - \mathbf{X}_2 \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)] \\ &\quad - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{G}' [\mathbf{G}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{G}']^{-1} [\mathbf{G} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{g}]. \end{aligned}$$

With respect to Lemma 3

$$\begin{aligned} \text{Var} [\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2)] &= \text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] \\ &\quad - \text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] \mathbf{G}' \{ \mathbf{G} \text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] \mathbf{G}' \}^{-1} \mathbf{G} \text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] \\ &= \text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)] - \mathbf{C}_1^{-1} \mathbf{X}_2' (\boldsymbol{\Sigma}_2 + \mathbf{X}_2 \mathbf{C}_1^{-1} \mathbf{X}_2')^{-1} \mathbf{X}_2 \mathbf{C}_1^{-1} \\ &\quad - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{G}' [\mathbf{G}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{G}']^{-1} \mathbf{G}(\mathbf{C}_1 + \mathbf{C}_2)^{-1}. \end{aligned}$$

□

**Theorem 4** *If*

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n+m} \left[ \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{D} & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{pmatrix} \right], \quad \mathbf{g} + \mathbf{G} \boldsymbol{\beta} = \mathbf{0},$$

then

$$\begin{aligned} \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1} \mathbf{D}' [\mathbf{M}_{\mathbf{X}_2} (\boldsymbol{\Sigma}_2 + \mathbf{D} \mathbf{C}_1^{-1} \mathbf{D}') \mathbf{M}_{\mathbf{X}_2}]^+ \mathbf{D} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) \\ &\quad + [\mathbf{C}_1 + \mathbf{D}' (\mathbf{M}_{\mathbf{X}_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{\mathbf{X}_2})^+ \mathbf{D}]^{-1} \mathbf{D}' (\mathbf{M}_{\mathbf{X}_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{\mathbf{X}_2})^+ \mathbf{Y}_2 \\ &\quad - [\mathbf{C}_1 + \mathbf{D}' (\mathbf{M}_{\mathbf{X}_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{\mathbf{X}_2})^+ \mathbf{D}]^{-1} \mathbf{G}' \{ \mathbf{G} [\mathbf{C}_1 + \mathbf{D}' (\mathbf{M}_{\mathbf{X}_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{\mathbf{X}_2})^+ \mathbf{D}]^{-1} \mathbf{G}' \}^{-1} \\ &\quad \times [\mathbf{G} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) + \mathbf{g}], \end{aligned}$$



where

$$\begin{aligned}\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\beta}(\mathbf{Y}_1) - \mathbf{C}_1^{-1} \mathbf{D}' [\mathbf{M}_{X_2} (\boldsymbol{\Sigma}_2 + \mathbf{D} \mathbf{C}_1^{-1} \mathbf{D}') \mathbf{M}_{X_2}]^+ \mathbf{D} \widehat{\beta}(\mathbf{Y}_1) \\ &\quad + [\mathbf{C}_1 + \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{D}]^{-1} \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{Y}_2.\end{aligned}$$

$$\begin{aligned}\text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)] &= \text{Var} [\widehat{\beta}(\mathbf{Y}_1)] - \mathbf{C}_1^{-1} \mathbf{D}' [\mathbf{M}_{X_2} (\boldsymbol{\Sigma}_2 + \mathbf{D} \mathbf{C}_1^{-1} \mathbf{D}') \mathbf{M}_{X_2}]^+ \mathbf{D} \mathbf{C}_1^{-1} \\ &\quad - \text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)] \mathbf{G}' \left\{ \mathbf{G} \text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)] \mathbf{G}' \right\}^{-1} \mathbf{G} \text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)],\end{aligned}$$

and

$$\text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)] = \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1} \mathbf{D}' [\mathbf{M}_{X_2} (\boldsymbol{\Sigma}_2 + \mathbf{D} \mathbf{C}_1^{-1} \mathbf{D}') \mathbf{M}_{X_2}]^+ \mathbf{D} \mathbf{C}_1^{-1}.$$

**Proof**

$$\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) - \mathbf{V} \mathbf{G}' (\mathbf{G} \mathbf{V} \mathbf{G}')^{-1} [\mathbf{G} \widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) + \mathbf{g}],$$

where

$$\begin{aligned}\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\beta}(\mathbf{Y}_1) - \mathbf{C}_1^{-1} \mathbf{D}' [\mathbf{M}_{X_2} (\boldsymbol{\Sigma}_2 + \mathbf{D} \mathbf{C}_1^{-1} \mathbf{D}') \mathbf{M}_{X_2}]^+ \mathbf{D} \widehat{\beta}(\mathbf{Y}_1) \\ &\quad + [\mathbf{C}_1 + \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{D}]^{-1} \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{Y}_2,\end{aligned}$$

(see Theorem 2)

$$\mathbf{V} = \text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)] = [\mathbf{C}_1 + \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{D}]^{-1}.$$

Now the proof can be easily finished.  $\square$

**Theorem 5** *If*

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n+m} \left[ \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{D} & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{pmatrix} \right], \quad \mathbf{g} + \mathbf{G}_1 \boldsymbol{\beta} + \mathbf{G}_2 \boldsymbol{\gamma} = \mathbf{0},$$

then

$$\begin{aligned}\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\beta}(\mathbf{Y}_1) - \mathbf{C}_1' \mathbf{D} [\mathbf{M}_{X_2} (\boldsymbol{\Sigma}_2 + \mathbf{D} \mathbf{C}_1^{-1} \mathbf{D}') \mathbf{M}_{X_2}]^+ \mathbf{D} \widehat{\beta}(\mathbf{Y}_1) \\ &\quad + [\mathbf{C}_1 + \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{D}]^{-1} \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{Y}_2 - (\mathbf{V}_{1,1}, \mathbf{V}_{1,2}) \begin{pmatrix} \mathbf{G}'_1 \\ \mathbf{G}'_2 \end{pmatrix} \\ &\quad \times \left[ (\mathbf{G}_1, \mathbf{G}_2) \begin{pmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1} & \mathbf{V}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{G}'_1 \\ \mathbf{G}'_2 \end{pmatrix} \right]^{-1} \left[ (\mathbf{G}_1, \mathbf{G}_2) \begin{pmatrix} \widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\gamma}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix} + \mathbf{g} \right],\end{aligned}$$

where

$$\begin{aligned}\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\beta}(\mathbf{Y}_1) - \mathbf{C}_1 \mathbf{D} [\mathbf{M}_{X_2} (\boldsymbol{\Sigma}_2 + \mathbf{D} \mathbf{C}_1^{-1} \mathbf{D}') \mathbf{M}_{X_2}]^+ \mathbf{D} \widehat{\beta}(\mathbf{Y}_1) \\ &\quad + [\mathbf{C}_1 + \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{D}]^{-1} \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{Y}_2, \\ \widehat{\gamma}(\mathbf{Y}_1, \mathbf{Y}_2) &= \mathbf{C}_2^{-1} \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} \mathbf{Y}_2 - \mathbf{C}_2^{-1} \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} \mathbf{D} [\mathbf{C}_1 + \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{D}]^{-1} \\ &\quad \times [\mathbf{X}'_1 \boldsymbol{\Sigma}_1^{-1} \mathbf{Y}_1 + \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{Y}_2], \\ &\quad \begin{pmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1} & \mathbf{V}_{2,2} \end{pmatrix} = \text{Var} \begin{pmatrix} \widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\gamma}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\mathbf{V}_{1,1} &= [\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1}, \\ \mathbf{V}_{1,2} &= -[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1}\mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{X}_2\mathbf{C}_2^{-1}, \\ \mathbf{V}_{2,2} &= [\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2]^{-1}\end{aligned}$$

and

$$\begin{aligned}-\mathbf{C}_2^{-1}\mathbf{X}_2'\boldsymbol{\Sigma}_2^{-1}\mathbf{D}[\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1} = \\ -[\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2]^{-1}\mathbf{X}_2'(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1}.\end{aligned}$$

**Proof** It is valid that

$$\begin{aligned}\begin{pmatrix} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix} = \left[ \begin{pmatrix} \mathbf{X}'_1 & \mathbf{D}' \\ \mathbf{0} & \mathbf{X}'_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\Sigma}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{D} & \mathbf{X}_2 \end{pmatrix} \right]^{-1} \\ \times \begin{pmatrix} \mathbf{X}'_1\boldsymbol{\Sigma}_1^{-1}\mathbf{Y}_1 + \mathbf{D}'\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2 \\ \mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}\mathbf{Y}_2 \end{pmatrix}.\end{aligned}$$

Now the expressions for  $\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)$  and  $\widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1, \mathbf{Y}_2)$  and also for

$$\begin{pmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1} & \mathbf{V}_{2,2} \end{pmatrix}$$

can be obtained easily.

With respect to Lemma 3

$$\begin{aligned}\begin{pmatrix} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix} = \begin{pmatrix} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix} - \begin{pmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1} & \mathbf{V}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{G}'_1 \\ \mathbf{G}'_2 \end{pmatrix} \\ \times \left[ (\mathbf{G}_1, \mathbf{G}_2) \begin{pmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1} & \mathbf{V}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{G}'_1 \\ \mathbf{G}'_2 \end{pmatrix} \right]^{-1} \left[ (\mathbf{G}_1, \mathbf{G}_2) \begin{pmatrix} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix} + \mathbf{g} \right].\end{aligned}$$

□

### 3 Original models with constraints

**Theorem 6** *If*

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n+m} \left[ \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_2 \end{pmatrix} \right], \quad \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0},$$

then

$$\begin{aligned}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + [\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{B'}]^+\mathbf{X}'_2\boldsymbol{\Sigma}_2^{-1}[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)] \\ + (\mathbf{B}_{m(C_1)}^- - \mathbf{B}_{m(C_1+C_2)}^-)[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{b}]\end{aligned}$$

and

$$\text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] = \text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)] - \left\{ (\mathbf{M}_{B'}\mathbf{C}_1\mathbf{M}_{B'})^+ - [\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{B'}]^+ \right\}.$$

**Proof** Regarding Theorem 1 and Lemma 3 we have

$$\begin{aligned}\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) &= (\mathbf{C}_1 + \mathbf{C}_2)^{-1}(\mathbf{X}'_1 \boldsymbol{\Sigma}_1^{-1} \mathbf{Y}_1 + \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} \mathbf{Y}_2), \\ \widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}' [\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}']^{-1} [\mathbf{B} \widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) + \mathbf{b}]\end{aligned}$$

(see also in [2], p. 152).

Thus

$$\begin{aligned}\widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\beta}(\mathbf{Y}_1) + (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} [\mathbf{Y}_2 - \mathbf{X}_2 \widehat{\beta}(\mathbf{Y}_1)] \\ &\quad - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}' [\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}']^{-1} \\ &\quad \times \left\{ \mathbf{B} \widehat{\beta}(\mathbf{Y}_1) + \mathbf{b} + \mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} [\mathbf{Y}_2 - \mathbf{X}_2 \widehat{\beta}(\mathbf{Y}_1)] \right\} \\ &= \widehat{\beta}(\mathbf{Y}_1) + \left\{ (\mathbf{C}_1 + \mathbf{C}_2)^{-1} - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}' [\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}']^{-1} \mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \right\} \\ &\quad \times \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} [\mathbf{Y}_2 - \mathbf{X}_2 \widehat{\beta}(\mathbf{Y}_1)] - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}' [\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}']^{-1} [\mathbf{B} \widehat{\beta}(\mathbf{Y}_1) + \mathbf{b}] \\ &= \widehat{\beta}(\mathbf{Y}_1) + [\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2) \mathbf{M}_{B'}]^+ \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} [\mathbf{Y}_2 - \mathbf{X}_2 \widehat{\beta}(\mathbf{Y}_1)] \\ &\quad - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}' [\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}']^{-1} [\mathbf{B} \widehat{\beta}(\mathbf{Y}_1) + \mathbf{b}].\end{aligned}$$

Now it is necessary to re-establish the expression

$$\widehat{\beta}(\mathbf{Y}_1) - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}' [\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}']^{-1} [\mathbf{B} \widehat{\beta}(\mathbf{Y}_1) + \mathbf{b}];$$

it is valid that

$$\begin{aligned}&\widehat{\beta}(\mathbf{Y}_1) - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}' [\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}']^{-1} [\mathbf{B} \widehat{\beta}(\mathbf{Y}_1) + \mathbf{b}] \\ &= \widehat{\widehat{\beta}}(\mathbf{Y}_1) + \left\{ \mathbf{C}_1^{-1} \mathbf{B}' (\mathbf{B} \mathbf{C}_1^{-1} \mathbf{B}')^{-1} - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}' [\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}']^{-1} \right\} \\ &\quad \times [\mathbf{B} \widehat{\beta}(\mathbf{Y}_1) + \mathbf{b}] = \widehat{\widehat{\beta}}(\mathbf{Y}_1) + \left\{ \mathbf{B}_{m(C_1)}^- - \mathbf{B}_{m(C_1+C_2)}^- \right\} [\mathbf{B} \widehat{\beta}(\mathbf{Y}_1) + \mathbf{b}].\end{aligned}$$

As far as  $\text{Var} [\widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)]$  is concerned, it is valid that

$$\begin{aligned}\text{Var} [\widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] &= (\mathbf{C}_1 + \mathbf{C}_2)^{-1} - (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}' [\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{B}']^{-1} \mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \\ &= [\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2) \mathbf{M}_{B'}]^+ = (\mathbf{M}_{B'} \mathbf{C}_1 \mathbf{M}_{B'})^+ \\ &\quad - \left\{ (\mathbf{M}_{B'} \mathbf{C}_1 \mathbf{M}_{B'})^+ - [\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2) \mathbf{M}_{B'}]^+ \right\} \\ &= \text{Var} [\widehat{\beta}(\mathbf{Y}_1)] - \left\{ (\mathbf{M}_{B'} \mathbf{C}_1 \mathbf{M}_{B'})^+ - [\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2) \mathbf{M}_{B'}]^+ \right\}.\end{aligned}$$

□

**Theorem 7** *If*

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n+m} \left[ \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \boldsymbol{\beta}, \begin{pmatrix} \boldsymbol{\Sigma}_1, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_2 \end{pmatrix} \right], \quad \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0}, \quad \mathbf{g} + \mathbf{G}\boldsymbol{\beta} = \mathbf{0},$$

then ( $\widehat{\widehat{\beta}}$  denotes the estimator satisfying both constraints)

$$\begin{aligned} \widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\beta}(\mathbf{Y}_1) + \mathbf{V}\mathbf{X}'_2\Sigma_2^{-1}[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\beta}(\mathbf{Y}_1)] \\ &\quad + [\mathbf{B}_{m(C_1)}^- - \mathbf{B}_{m(C_1+C_2)}^-][\mathbf{B}\widehat{\beta}(\mathbf{Y}_1) + \mathbf{b}] \\ &- [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1}\mathbf{G}'\left\{\mathbf{G}[(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1}\mathbf{G}'\right\}^+ [\mathbf{G}\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) + \mathbf{g}], \end{aligned}$$

where

$$\begin{aligned} \mathbf{V} &= \text{Var} [\widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] \\ &= (\mathbf{C}_1 + \mathbf{C}_2)^{-1} - (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \\ &= [\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{B'}]^+ \end{aligned}$$

and

$$\begin{aligned} \text{Var} [\widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] &= [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} - [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1}\mathbf{G}' \\ &\quad \times \left\{\mathbf{G}[(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1}\mathbf{G}'\right\}^+ [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} - \mathbf{M}_{G'}. \end{aligned}$$

**Proof** If

$$\begin{aligned} \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} &\sim_{n+m} \left[ \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{pmatrix} \beta, \begin{pmatrix} \Sigma_1, & \mathbf{0} \\ \mathbf{0}, & \Sigma_2 \end{pmatrix} \right], \\ \mathbf{b} + \mathbf{B}\beta &= \mathbf{0}, \quad \mathbf{g} + \mathbf{G}\beta = \mathbf{0}, \end{aligned}$$

then with respect to Theorem 6 and Lemma 3

$$\begin{aligned} \widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\beta}(\mathbf{Y}_1) + [\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{B'}]^+\mathbf{X}'_2\Sigma_2^{-1}[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\beta}(\mathbf{Y}_1)] \\ &\quad + (\mathbf{B}_{m(C_1)}^- - \mathbf{B}_{m(C_1+C_2)}^-)[\mathbf{B}\widehat{\beta}(\mathbf{Y}_1) + \mathbf{b}] \\ &- [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1}\mathbf{G}'\left\{\mathbf{G}[(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1}\mathbf{G}'\right\}^+ [\mathbf{G}\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) + \mathbf{g}], \end{aligned}$$

where

$$\begin{aligned} \mathbf{V} &= \text{Var} [\widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] \\ &= (\mathbf{C}_1 + \mathbf{C}_2)^{-1} - (\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}'[\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1}\mathbf{B}']^{-1}\mathbf{B}(\mathbf{C}_1 + \mathbf{C}_2)^{-1} \end{aligned}$$

and

$$\begin{aligned} \widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\beta}(\mathbf{Y}_1) + [\mathbf{M}_{B'}(\mathbf{C}_1 + \mathbf{C}_2)\mathbf{M}_{B'}]^+\mathbf{X}'_2\Sigma_2^{-1}[\mathbf{Y}_2 - \mathbf{X}_2\widehat{\beta}(\mathbf{Y}_1)] \\ &\quad + \left\{\mathbf{B}_{m(C_1)}^- - \mathbf{B}_{m(C_1+C_2)}^-\right\}[\mathbf{B}\widehat{\beta}(\mathbf{Y}_1) + \mathbf{b}]. \end{aligned}$$

The expression for  $\text{Var} [\widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)]$  can be easily obtained.  $\square$

**Theorem 8** *If*

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n+m} \left[ \begin{pmatrix} \mathbf{X}_1, & \mathbf{0} \\ \mathbf{D}, & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_1, & \mathbf{0} \\ \mathbf{0}, & \boldsymbol{\Sigma}_2 \end{pmatrix} \right], \quad \mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0},$$

then

$$\begin{aligned} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - [\mathbf{V}_1 + \mathbf{I} - \mathbf{V}_1(\mathbf{V}_1 + \mathbf{V}_2)^+\mathbf{V}_1][\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] \\ &\quad \times [\mathbf{V}_2 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}]^+ [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \widehat{\boldsymbol{\beta}}(\mathbf{Y}_2)], \end{aligned}$$

where

$$\begin{aligned} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}_1^{-1}\mathbf{B}')^{-1}[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) + \mathbf{b}], \\ \widehat{\boldsymbol{\beta}}(\mathbf{Y}_2) &= [\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{Y}_2, \\ \widehat{\boldsymbol{\beta}}(\mathbf{Y}_2) &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_2) - \mathbf{W}_2\mathbf{B}'(\mathbf{B}\mathbf{W}_2\mathbf{B}')^{-1}[\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_2) + \mathbf{b}], \\ \mathbf{V}_1 &= \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}_1^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}_1^{-1}, \\ \mathbf{V}_2 &= \mathbf{W}_2 - \mathbf{W}_2\mathbf{B}'(\mathbf{B}\mathbf{W}_2\mathbf{B}')^{-1}\mathbf{B}\mathbf{W}_2, \\ \mathbf{W}_2 &= [\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1} \end{aligned}$$

and

$$\mathbf{V} = \text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] = \text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)] - \mathbf{V}_1(\mathbf{V}_1 + \mathbf{V}_2)^+\mathbf{V}_1.$$

**Proof** In the model

$$\begin{aligned} \begin{pmatrix} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) \\ \widehat{\boldsymbol{\beta}}(\mathbf{Y}_2) \end{pmatrix} &\sim_{2k} \left[ \begin{pmatrix} \mathbf{I} \\ \mathbf{I} \end{pmatrix} \boldsymbol{\beta}, \begin{pmatrix} \mathbf{V}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{V}_2 \end{pmatrix} \right], \\ \mathbf{V}_1 &= \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}_1^{-1}\mathbf{B}')^{-1}\mathbf{B}\mathbf{C}_1^{-1}, \\ \mathbf{V}_2 &= \mathbf{W}_2 - \mathbf{W}_2\mathbf{B}'(\mathbf{B}\mathbf{W}_2\mathbf{B}')^{-1}\mathbf{B}\mathbf{W}_2, \\ \mathbf{W}_2 &= [\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1}, \end{aligned}$$

the BLUE of  $\boldsymbol{\beta}$  is

$$\begin{aligned} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \left[ (\mathbf{I}, \mathbf{I})^{-1} \begin{pmatrix} \mathbf{V}_1, & \mathbf{0} \\ \mathbf{0}, & \mathbf{V}_2 \end{pmatrix} \right]' \begin{pmatrix} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) \\ \widehat{\boldsymbol{\beta}}(\mathbf{Y}_2) \end{pmatrix} \\ &= \left[ (\mathbf{I}, \mathbf{I}) \begin{pmatrix} \mathbf{V}_1 + \mathbf{I}, & \mathbf{I} \\ \mathbf{I}, & \mathbf{V}_2 + \mathbf{I} \end{pmatrix}^+ \begin{pmatrix} \mathbf{I} \\ \mathbf{I} \end{pmatrix} \right]^{-1} (\mathbf{I}, \mathbf{I}) \begin{pmatrix} \mathbf{V}_1 + \mathbf{I}, & \mathbf{I} \\ \mathbf{I}, & \mathbf{V}_2 + \mathbf{I} \end{pmatrix}^+ \begin{pmatrix} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) \\ \widehat{\boldsymbol{\beta}}(\mathbf{Y}_2) \end{pmatrix}. \end{aligned}$$

Here

$$\begin{aligned}
& \left[ (\mathbf{I}, \mathbf{I}) \begin{pmatrix} \mathbf{V}_1 + \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{V}_2 + \mathbf{I} \end{pmatrix}^+ \begin{pmatrix} \mathbf{I} \\ \mathbf{I} \end{pmatrix} \right]^{-1} = \\
& = \left\{ (\mathbf{V}_1 + \mathbf{I})^{-1} + [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] [\mathbf{V}_2 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}]^+ [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] \right\}^{-1} \\
& \quad = \mathbf{V}_1 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I}) [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] \\
& \quad \times \left\{ \mathbf{V}_2 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1} + [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] (\mathbf{V}_1 + \mathbf{I}) [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] \right\}^+ \\
& \quad \quad \times [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] (\mathbf{V}_1 + \mathbf{I}) \\
& = \mathbf{V}_1 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I}) [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] (\mathbf{V}_1 + \mathbf{V}_2)^+ [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] (\mathbf{V}_1 + \mathbf{I}) \\
& \quad = \mathbf{V}_1 + \mathbf{I} - \mathbf{V}_1 (\mathbf{V}_1 + \mathbf{V}_2)^+ \mathbf{V}_1
\end{aligned}$$

and

$$\begin{aligned}
& (\mathbf{I}, \mathbf{I}) \begin{pmatrix} \mathbf{V}_1 + \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{V}_2 + \mathbf{I} \end{pmatrix}^+ \begin{pmatrix} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) \\ \widehat{\boldsymbol{\beta}}(\mathbf{Y}_2) \end{pmatrix} = \\
& = \left\{ (\mathbf{V}_1 + \mathbf{I})^{-1} - [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] [\mathbf{V}_2 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}]^+ (\mathbf{V}_1 + \mathbf{I})^{-1} \right\} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) \\
& \quad + [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] [\mathbf{V}_2 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}]^+ \widehat{\boldsymbol{\beta}}(\mathbf{Y}_2).
\end{aligned}$$

Thus

$$\begin{aligned}
& \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - (\mathbf{V}_1 + \mathbf{I}) [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] \left\{ \mathbf{V}_2 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1} \right. \\
& \quad \left. + [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] (\mathbf{V}_1 + \mathbf{I}) [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] \right\}^+ [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) \\
& - \left( \mathbf{V}_1 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I}) [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] (\mathbf{V}_1 + \mathbf{V}_2)^+ [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] (\mathbf{V}_1 + \mathbf{I}) \right) \\
& \quad \times [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] [\mathbf{V}_2 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}]^+ [(\mathbf{V}_1 + \mathbf{I})^{-1} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \widehat{\boldsymbol{\beta}}(\mathbf{Y}_2)].
\end{aligned}$$

Since

$$\begin{aligned}
& (\mathbf{V}_1 + \mathbf{I}) [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] (\mathbf{V}_1 + \mathbf{V}_2)^+ [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) \\
& = \left( (\mathbf{V}_1 + \mathbf{I})^{-1} + [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] [\mathbf{V}_2 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}]^+ [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] \right)^+ \\
& \quad \times [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] [\mathbf{V}_2 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}]^+ [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1),
\end{aligned}$$

the statement concerning the estimator is obvious.

As far as  $\text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)]$  is concerned it is valid that

$$\begin{aligned} \text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)] &= \left[ (\mathbf{I}, \mathbf{I}) \begin{pmatrix} \mathbf{V}_1 + \mathbf{I} & \mathbf{I} \\ \mathbf{I} & \mathbf{V}_2 + \mathbf{I} \end{pmatrix}^+ \begin{pmatrix} \mathbf{I} \\ \mathbf{I} \end{pmatrix} \right]^{-1} - \mathbf{I} \\ &= \mathbf{V}_1 - \mathbf{V}_1(\mathbf{V}_1 + \mathbf{V}_2)^+ \mathbf{V}_1, \end{aligned}$$

what was already shown.  $\square$

**Theorem 9** *If*

$$\begin{aligned} \begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} &\sim_{n+m} \left[ \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} \\ \mathbf{D} & \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix} \right], \\ \mathbf{b} + \mathbf{B}\beta &= \mathbf{0}, \quad \mathbf{g} + \mathbf{G}\beta = \mathbf{0}, \end{aligned}$$

then

$$\begin{aligned} \widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\beta}(\mathbf{Y}_1) - \mathbf{A} [\widehat{\beta}(\mathbf{Y}_1) - \widehat{\beta}(\mathbf{Y}_2)] - [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \mathbf{G}' \\ &\quad \times \left\{ \mathbf{G} [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \mathbf{G}' \right\}^+ [\mathbf{G}\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) + \mathbf{g}] \end{aligned}$$

where  $\mathbf{V}$  is given in Theorem 8,

$$\mathbf{A} = \left( \mathbf{V}_1 + \mathbf{I} - \mathbf{V}_1(\mathbf{V}_1 + \mathbf{V}_2)^+ \mathbf{V}_1 \right) [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] [\mathbf{V}_2 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}]^+$$

and

$$\begin{aligned} \text{Var} [\widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] &= [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} - [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \mathbf{G}' \\ &\quad \times \left\{ \mathbf{G} [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \mathbf{G}' \right\}^+ \mathbf{G} [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} - \mathbf{M}_{G'}. \end{aligned}$$

Here

$$\begin{aligned} \mathbf{V} &= \text{Var} [\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2)] = \mathbf{V}_1 - \mathbf{V}_1(\mathbf{V}_1 + \mathbf{V}_2)^+ \mathbf{V}_1, \\ \mathbf{V}_1 &= \text{Var} [\widehat{\beta}(\mathbf{Y}_1)] = \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1} \mathbf{B}' (\mathbf{B} \mathbf{C}_1^{-1} \mathbf{B}')^{-1} \mathbf{B} \mathbf{C}_1^{-1}, \\ \mathbf{V}_2 &= \text{Var} [\widehat{\beta}(\mathbf{Y}_2)] = \mathbf{W} - \mathbf{W} \mathbf{B}' (\mathbf{B} \mathbf{W} \mathbf{B}')^{-1} \mathbf{B} \mathbf{W}, \\ \mathbf{W} &= [\mathbf{D}' (\mathbf{M}_{X_2} \Sigma_2 \mathbf{M}_{X_2})^+ \mathbf{D}]^{-1}. \end{aligned}$$

**Proof** With respect to Lemma 3 it is valid that

$$\begin{aligned} \widehat{\widehat{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) - [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \mathbf{G}' \\ &\quad \times \left\{ \mathbf{G} [(\mathbf{V} + \mathbf{M}_{G'})^+ + \mathbf{G}'\mathbf{G}]^{-1} \mathbf{G}' \right\}^+ [\mathbf{G}\widehat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) + \mathbf{g}], \end{aligned}$$

where (see Theorem 8)

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) - \mathbf{A}[\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) - \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_2)],$$

$$\mathbf{A} = \left( \mathbf{V}_1 + \mathbf{I} - \mathbf{V}_1(\mathbf{V}_1 + \mathbf{V}_2)^+\mathbf{V}_1 \right) [\mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}] [\mathbf{V}_2 + \mathbf{I} - (\mathbf{V}_1 + \mathbf{I})^{-1}]^+.$$

□

If

$$\begin{pmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{pmatrix} \sim_{n+m} \left[ \begin{pmatrix} \mathbf{X}_1, \mathbf{0} \\ \mathbf{D}, \mathbf{X}_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\gamma} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Sigma}_1, \mathbf{0} \\ \mathbf{0}, \boldsymbol{\Sigma}_2 \end{pmatrix} \right],$$

$$\mathbf{b} + \mathbf{B}\boldsymbol{\beta} = \mathbf{0}, \quad \mathbf{g} + \mathbf{G}_1\boldsymbol{\beta} + \mathbf{G}_2\boldsymbol{\gamma} = \mathbf{0},$$

then the explicit expression for  $\mathbf{k}$  in the relationship

$$\widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) + \mathbf{k}$$

is rather complicated. Therefore a sequence of relationships is given which enables us to obtain the vector  $\mathbf{k}$  in the actual situation at least. It is assumed that  $r(\mathbf{X}_1) = k < m$ ,  $r(\mathbf{D}, \mathbf{X}_2) = k + l < m$ ,  $\boldsymbol{\Sigma}_1$  and  $\boldsymbol{\Sigma}_2$  are positive definite and  $r(\mathbf{B}) = q < k$ ,  $r(\mathbf{G}_1, \mathbf{G}_2) = r < k + l$ ,  $r(\mathbf{G}_2) = l < r$ . The notation

$$\begin{pmatrix} \mathbf{T}_{1,1}, \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1}, \mathbf{T}_{2,2} \end{pmatrix} = \left\{ \left[ \begin{pmatrix} \mathbf{W}_{1,1}, \mathbf{W}_{1,2} \\ \mathbf{W}_{2,1}, \mathbf{W}_{2,2} \end{pmatrix} + \mathbf{M} \begin{pmatrix} \mathbf{G}'_1 \\ \mathbf{G}'_2 \end{pmatrix} \right]^+ + \begin{pmatrix} \mathbf{G}'_1 \\ \mathbf{G}'_2 \end{pmatrix} (\mathbf{G}_1, \mathbf{G}_2) \right\}^{-1}.$$

will be used.

It is valid that

$$\begin{aligned} \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{D}'[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}]^+\mathbf{D}\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) \\ &\quad + [\mathbf{C}_1 + \mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{D}]^{-1}\mathbf{D}'(\mathbf{M}_{X_2}\boldsymbol{\Sigma}_2\mathbf{M}_{X_2})^+\mathbf{Y}_2, \end{aligned}$$

$$\widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1, \mathbf{Y}_2) = [\mathbf{X}'_2(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2]^{-1}\mathbf{X}'_2(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}[\mathbf{Y}_2 - \mathbf{D}\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1)],$$

$$\mathbf{V}_{1,1} = \text{Var}[\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2)] = \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1}\mathbf{D}'[\mathbf{M}_{X_2}(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')\mathbf{M}_{X_2}]^+\mathbf{D}\mathbf{C}_1^{-1},$$

$$\mathbf{V}_{2,1} = -[\mathbf{X}'_2(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2]^{-1}\mathbf{X}'_2(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{D}\mathbf{C}_1^{-1},$$

$$\mathbf{V}_{2,2} = [\mathbf{X}'_2(\boldsymbol{\Sigma}_2 + \mathbf{D}\mathbf{C}_1^{-1}\mathbf{D}')^{-1}\mathbf{X}_2]^{-1},$$

$$\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}_1^{-1}\mathbf{B}')^{-1}[\mathbf{B}\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1) + \mathbf{b}],$$

$$\widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1, \mathbf{Y}_2) = \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) - \mathbf{V}_{1,1}\mathbf{B}'\{\mathbf{B}\mathbf{V}_{1,1}\mathbf{B}'\}^{-1}[\mathbf{B}\widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) + \mathbf{b}],$$



$$\begin{aligned}
\begin{pmatrix} \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\widehat{\boldsymbol{\gamma}}}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix} &= \begin{pmatrix} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix} - \begin{pmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1} & \mathbf{V}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{B}' \\ \mathbf{0} \end{pmatrix} \\
&\times \left[ (\mathbf{B}, \mathbf{0}) \begin{pmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1} & \mathbf{V}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{B}' \\ \mathbf{0} \end{pmatrix} \right]^{-1} [\mathbf{B}\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) + \mathbf{b}], \\
\text{Var} \begin{pmatrix} \widehat{\widehat{\boldsymbol{\beta}}}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\widehat{\boldsymbol{\gamma}}}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix} &= \begin{pmatrix} \mathbf{W}_{1,1} & \mathbf{W}_{1,2} \\ \mathbf{W}_{2,1} & \mathbf{W}_{2,2} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1} & \mathbf{V}_{2,2} \end{pmatrix} \\
&- \begin{pmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1} & \mathbf{V}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{B}' \\ \mathbf{0} \end{pmatrix} (\mathbf{B}\mathbf{V}_{1,1}\mathbf{B}')^{-1} (\mathbf{B}, \mathbf{0}) \begin{pmatrix} \mathbf{V}_{1,1} & \mathbf{V}_{1,2} \\ \mathbf{V}_{2,1} & \mathbf{V}_{2,2} \end{pmatrix}, \\
\begin{pmatrix} \widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\widehat{\widehat{\boldsymbol{\gamma}}}}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix} &= \begin{pmatrix} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix} - \begin{pmatrix} \mathbf{T}_{1,1} & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1} & \mathbf{T}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{G}'_1 \\ \mathbf{G}'_2 \end{pmatrix} \\
&\times \left[ (\mathbf{G}_1, \mathbf{G}_2) \begin{pmatrix} \mathbf{T}_{1,1} & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1} & \mathbf{T}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{G}'_1 \\ \mathbf{G}'_2 \end{pmatrix} \right]^{-1} \left[ (\mathbf{G}_1, \mathbf{G}_2) \begin{pmatrix} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\boldsymbol{\gamma}}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix} + \mathbf{g} \right], \\
\text{Var} \begin{pmatrix} \widehat{\widehat{\widehat{\boldsymbol{\beta}}}}(\mathbf{Y}_1, \mathbf{Y}_2) \\ \widehat{\widehat{\widehat{\boldsymbol{\gamma}}}}(\mathbf{Y}_1, \mathbf{Y}_2) \end{pmatrix} &= \begin{pmatrix} \mathbf{T}_{1,1} & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1} & \mathbf{T}_{2,2} \end{pmatrix} - \begin{pmatrix} \mathbf{T}_{1,1} & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1} & \mathbf{T}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{G}'_1 \\ \mathbf{G}'_2 \end{pmatrix} \\
&\times \left[ (\mathbf{G}_1, \mathbf{G}_2) \begin{pmatrix} \mathbf{T}_{1,1} & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1} & \mathbf{T}_{2,2} \end{pmatrix} \begin{pmatrix} \mathbf{G}'_1 \\ \mathbf{G}'_2 \end{pmatrix} \right]^{-1} \times (\mathbf{G}_1, \mathbf{G}_2) \begin{pmatrix} \mathbf{T}_{1,1} & \mathbf{T}_{1,2} \\ \mathbf{T}_{2,1} & \mathbf{T}_{2,2} \end{pmatrix} - \mathbf{M} \begin{pmatrix} \mathbf{G}'_1 \\ \mathbf{G}'_2 \end{pmatrix}.
\end{aligned}$$

#### 4 Numerical examples

Many examples can be found when levelling networks in geodesy are designed.

##### Example 1

$$\begin{aligned}
\mathbf{Y}_1 &\sim \left[ \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \sigma_1^2 \mathbf{I}_4 \right], \\
\mathbf{Y}_2 &\sim \left[ \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}, \sigma_2^2 \mathbf{I}_4 \right], \\
\sigma_1^2 &= (0.01 \text{ m})^2, \quad \sigma_2^2 = (0.001 \text{ m})^2.
\end{aligned}$$

Here  $\beta_1, \beta_2, \beta_3$  are heights of points  $P_1, P_2, P_3$  and  $\{\mathbf{Y}_1\}_1$  means a measurement of a difference of the heights between the point  $P_1$  and a reference point  $A$  with the height equal to zero. Analogously  $\{\mathbf{Y}_1\}_2$  means a measurement of the difference between the heights of the point  $P_1$  and  $P_2$ , etc.

$$\mathbf{C}_1 = (0.01)^{-2} \begin{pmatrix} 2, & -1, & 0 \\ -1, & 2, & -1 \\ 0, & -1, & 2 \end{pmatrix}, \quad \mathbf{C}_2 = (0.001)^{-2} \begin{pmatrix} 2, & -1, & -1 \\ -1, & 3, & -1 \\ -1, & -1, & 2 \end{pmatrix},$$

$$\mathbf{C}_1^{-1} = (0.01)^2 \frac{1}{4} \begin{pmatrix} 3, & 2, & 1 \\ 2, & 4, & 2 \\ 1, & 2, & 3 \end{pmatrix}, \quad \mathbf{C}_2^{-1} = (0.001)^2 \frac{1}{3} \begin{pmatrix} 5, & 3, & 4 \\ 3, & 3, & 3 \\ 4, & 3, & 5 \end{pmatrix},$$

$$\mathbf{Y}_1 = (5.18 \text{ m}, -0.35 \text{ m}, 2.48 \text{ m}, -7.29 \text{ m})',$$

$$\mathbf{Y}_2 = (-2.134 \text{ m}, -0.351 \text{ m}, 2.485 \text{ m}, 4.825 \text{ m})',$$

$$\begin{pmatrix} \hat{\beta}_1(\mathbf{Y}_1) \\ \hat{\beta}_2(\mathbf{Y}_1) \\ \hat{\beta}_3(\mathbf{Y}_1) \end{pmatrix} = \mathbf{C}_1^{-1} \mathbf{X}'_1 \boldsymbol{\Sigma}_1^{-1} \mathbf{Y}_1 = \begin{pmatrix} 5.175 \text{ m} \\ 4.820 \text{ m} \\ 7.295 \text{ m} \end{pmatrix}$$

$$(\mathbf{C}_1 + \mathbf{C}_2)^{-1} = 10^{-6} \begin{pmatrix} 1.617, & 0.971, & 1.286 \\ 0.971, & 0.981, & 0.971 \\ 1.286, & 0.971, & 1.617 \end{pmatrix}, \quad \mathbf{C}_1^{-1} = 10^{-6} \begin{pmatrix} 75, & 50, & 25 \\ 50, & 100, & 50 \\ 25, & 50, & 75 \end{pmatrix}$$

$$\hat{\beta}(\mathbf{Y}_1, \mathbf{Y}_2) = \hat{\beta}(\mathbf{Y}_1) + (\mathbf{C}_1 + \mathbf{C}_2)^{-1} \mathbf{X}'_2 \boldsymbol{\Sigma}_2^{-1} [\mathbf{Y}_2 - \mathbf{X}_2 \hat{\beta}(\mathbf{Y}_1)] = \begin{pmatrix} 5.176 \text{ m} \\ 4.825 \text{ m} \\ 7.310 \text{ m} \end{pmatrix},$$

$$\mathbf{Y}_2 - \mathbf{X}_2 \hat{\beta}(\mathbf{Y}_1) = \begin{pmatrix} -0.014 \text{ m} \\ 0.004 \text{ m} \\ 0.010 \text{ m} \\ 0.005 \text{ m} \end{pmatrix} \quad (\text{a discrepancy between the original and the additional experiment})$$

The improvement of the estimator is obvious.

**Example 2** The original model describes the measurement of heights of three points  $P_1, P_2, P_3$ . The additional model involves new point  $P_4$  of the height  $\gamma$  and it describes the measurement of height differences between the points  $P_4P_3, P_1P_4$ , and  $P_2P_1$ , respectively.

$$\mathbf{Y}_1 \sim_3 (\mathbf{I}_3 \boldsymbol{\beta}, \sigma_1^2 \mathbf{I}_3), \quad \mathbf{Y}_2 \sim_3 \left[ (\mathbf{D}, \mathbf{X}_2) \begin{pmatrix} \boldsymbol{\beta} \\ \gamma \end{pmatrix}, \sigma_2^2 \mathbf{I}_3 \right],$$

$$\sigma_1^2 = (0.01 \text{ m})^2, \quad \sigma_2^2 = (0.001 \text{ m})^2,$$

$$\mathbf{D} = \begin{pmatrix} 0, & 0, & -1 \\ 1, & 0, & 0 \\ -1, & 1, & 0 \end{pmatrix}, \quad \mathbf{X}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

$$\mathbf{Y}_1 = (5.18 \text{ m}, 4.82 \text{ m}, 7.30 \text{ m})', \quad \mathbf{Y}_2 = (-4.360 \text{ m}, 2.226 \text{ m}, -0.351 \text{ m})',$$

$$\begin{aligned}
\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) &= \mathbf{Y}_1, \\
\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - \mathbf{C}_1^{-1} \mathbf{D}' [\mathbf{M}_{X_2} (\boldsymbol{\Sigma}_2 + \mathbf{D} \mathbf{C}_1^{-1} \mathbf{D}') \mathbf{M}_{X_2}]^+ \mathbf{D} \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) \\
&\quad + [\mathbf{C}_1 + \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{D}]^{-1} \mathbf{D}' (\mathbf{M}_{X_2} \boldsymbol{\Sigma}_2 \mathbf{M}_{X_2})^+ \mathbf{Y}_2, \\
&= \begin{pmatrix} 5.18 \\ 4.82 \\ 7.30 \end{pmatrix} - \begin{pmatrix} -0.5797 \\ -0.9304 \\ 1.5101 \end{pmatrix} + \begin{pmatrix} -0.5873 \\ -0.9290 \\ 1.5164 \end{pmatrix} = \begin{pmatrix} 5.1724 \text{ m} \\ 4.8214 \text{ m} \\ 7.3063 \text{ m} \end{pmatrix}, \\
\text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2)] &= \text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1)] - \mathbf{C}_1^{-1} \mathbf{D}' [\mathbf{M}_{X_2} (\boldsymbol{\Sigma}_2 + \mathbf{D} \mathbf{C}_1^{-1} \mathbf{D}') \mathbf{M}_{X_2}]^+ \mathbf{D} \mathbf{C}_1^{-1} \\
&= 10^{-4} \begin{pmatrix} 0.337, 0.333, 0.330 \\ 0.333, 0.340, 0.327 \\ 0.330, 0.327, 0.343 \end{pmatrix}.
\end{aligned}$$

**Example 3** A free levelling traverse consists of points  $P_1, P_2, P_3, P_4$ . The original model describes the measurement of the height differences  $\beta_1 \sim P_2P_1$ ,  $\beta_2 \sim P_3P_2$ ,  $\beta_3 \sim P_4P_3$ . In the additional experiment the height difference  $\gamma \sim P_1P_4$  is measured. After the measurement of the additional experiment the fact that the levelling traverse  $P_1, P_2, P_3, P_4, P_1$ , is closed must be taken into account.

$$\begin{aligned}
\mathbf{Y}_1 &\sim_3 (\mathbf{I}_3 \boldsymbol{\beta}, \sigma_1^2 \mathbf{I}_3), \quad Y_2 \sim_1 \left[ (\mathbf{0}', 1) \begin{pmatrix} \boldsymbol{\beta} \\ \gamma \end{pmatrix}, \sigma_2^2 \right], \\
g + \mathbf{G}_1 \boldsymbol{\beta} + G_2 \gamma &= 0, \quad g = 0, \quad \mathbf{G}_1 = (1, 1, 1), \quad G_2 = 1, \\
\sigma_1^2 &= (0.01 \text{ m})^2, \quad \sigma_2^2 = (0.001 \text{ m})^2, \\
\mathbf{Y}_1 &= (3.51 \text{ m}, 2.70 \text{ m}, 1.32 \text{ m})', \quad Y_2 = -7.516 \text{ m}, \\
\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) &= \mathbf{Y}_1, \quad \widehat{\gamma}(Y_2) = Y_2, \\
\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, \mathbf{Y}_2) &= \widehat{\boldsymbol{\beta}}(\mathbf{Y}_1) - (\mathbf{C}_1^{-1}, \mathbf{0}) \begin{pmatrix} \mathbf{1} \\ 1 \end{pmatrix} \left[ (\mathbf{1}', 1) \begin{pmatrix} \mathbf{C}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ 1 \end{pmatrix} \right]^{-1} \\
&\quad \times (\mathbf{1}', 1) \begin{pmatrix} \mathbf{Y}_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 3.512 \\ 2.70 \\ 1.32 \end{pmatrix} - \frac{1}{301} \begin{pmatrix} 0.014 \\ 0.014 \\ 0.014 \end{pmatrix} = \begin{pmatrix} 3.5054 \text{ m} \\ 2.6954 \text{ m} \\ 1.3154 \text{ m} \end{pmatrix}, \\
\text{Var} [\widehat{\boldsymbol{\beta}}(\mathbf{Y}_1, Y_2)] &= \mathbf{C}_1^{-1} - \mathbf{C}_1^{-1} \mathbf{1} \left[ (\mathbf{1}', 1) \begin{pmatrix} \mathbf{C}_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1} \\ 1 \end{pmatrix} \right]^{-1} \mathbf{1}' \mathbf{C}_1^{-1} \\
&= 10^{-4} (\mathbf{I}_3 - 0.332 \times \mathbf{1}\mathbf{1}') = 10^{-4} \begin{pmatrix} 0.668, -0.332, -0.332 \\ -0.332, 0.668, -0.332 \\ -0.332, -0.332, 0.668 \end{pmatrix}.
\end{aligned}$$

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