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Adjoint Semilattice and Minimal Brouwerian Extensions of a Hilbert Algebra*

Jānis CĪRULIS

Faculty of Computing, University of Latvia,
Raiņa b. 19, Riga LV-1586, Latvia
e-mail: jc@lanet.lv

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Abstract

Let $A := (A, \to, 1)$ be a Hilbert algebra. The monoid of all unary operations on A generated by operations $\alpha_p \colon x \mapsto (p \to x)$, which is actually an upper semilattice w.r.t. the pointwise ordering, is called the adjoint semilattice of A. This semilattice is isomorphic to the semilattice of finitely generated filters of A, it is subtractive (i.e., dually implicative), and its ideal lattice is isomorphic to the filter lattice of A. Moreover, the order dual of the adjoint semilattice is a minimal Brouwerian extension of A, and the embedding of A into this extension preserves all existing joins and certain "compatible" meets.

Key words: adjoint semilattice, Brouwerian extension, closure endomorphism, compatible meet, filter, Hilbert algebra, implicative semilattice, subtraction

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1 Introduction

Let $A := (A, \to, 1)$ be a Hilbert algebra. A mapping $\varphi \colon A \to A$ is called a *closure endomorphism* if it is simultaneously a closure operator and an endomorphism.

This notion goes back to [15], where Glivenko operators on implicative, or Brouwerian, semilattices were discussed. In [16], it was shown that the Glivenko operators are precisely the closure endomorphisms and that all such endomorphisms form a distributive lattice. Connections of this lattice with the filter

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lattice and with a certain sublattice of subalgebras of an implicative semilattice were discovered in [17].

Closure endomorphisms on Hilbert algebras were introduced by the present author in [1] and further studied in [2]. The identity mapping ε, the unit mapping $\iota\colon x \mapsto 1$ and, for any $p \in A$, the mappings α_p and β_p defined by

$$\alpha_p x := p \to x, \quad \beta_p x := (x \to p) \to x,$$

respectively are examples of closure endomorphisms. The set *CE* of all closure endomorphisms on A is closed under composition $\circ$ and pointwise defined meets. The algebra $(CE, \circ, \wedge, \varepsilon, \iota)$ also is a bounded distributive lattice [2], in which $\circ$ acts as join and the natural ordering may be defined pointwise. Furthermore, an endomorphism φ is a closure operator if and only if

$$\varphi(x \to y) = x \to \varphi y. \tag{1}$$

In this paper, we pay attention to closure endomorphisms α_p and their compositions. For every finite subset $P := \{p_1, p_2, \ldots, p_n\}$ of A (in symbols, $P \subseteq_{\text{fin}} A$), we set $\alpha_P := \alpha_{p_n} \circ \cdots \circ \alpha_{p_2} \circ \alpha_{p_1}$ (we shall usually drop the symbol '$\circ$' in notation). In the case when P is empty, this means that $\alpha_P = \varepsilon$. Of course, each mapping α_P also is a closure endomorphism; we shall call them *finitely generated* (cf. Proposition 2 below). The set CE^f of all such mappings is closed under composition, and the algebra $(CE^f, \circ, \varepsilon)$ is a lower bounded join semilattice. In the dual context of BCI/BCK-algebras, the counterpart of CE^f is usually called the adjoint semigroup (or monoid) of an algebra under consideration (see, for example, [8, 9, 13]). We adopt this term and call CE^f the *adjoint semilattice* of the initial Hilbert algebra A. It is shown in Section 3 to be isomorphic to the semilattice of finitely generated filters of A and subtractive, i.e., dually implicative, while its generating set turns out to be closed under subtraction and is an order dual of A (Section 4). The lattice of ideals of CE^f is isomorphic to the lattice of filters of A (Section 3). A minimal Brouwerian extension of A is a minimal implicative semilattice of which A is a subreduct; in Section 4 such an extension is shown to be dually isomorphic to the adjoint semilattice of A. Embedding of A into its minimal Brouwerian extension preserves all existing joins; we characterize also the preserved meets (Section 5).

2 Preliminaries

We assume that the reader is acquainted with the notion of Hilbert algebra and with elementary arithmetics in such algebras. This information can be found, e.g., in [2, 3, 5, 6, 14]. Recall that Hilbert algebras were introduced in [5] as the order duals of L. Henkin's *implicational models* [6]. In [2, 3] also the notion of compatible meet (suggested by [14]) in a Hilbert algebra was introduced and discussed. We now list some basic facts concerning it.

Let $A := (A, \to, 1)$ be a Hilbert algebra. Elements $a, b \in A$ are said to be *compatible* (in symbols, $a\ C\ b$) if there is a lower bound c of $\{a, b\}$ such that

$a \leq b \to c$. If this is the case, then c is the g.l.b. of a and b; we call this element the *compatible meet* of a and b and denote it by $a \curlywedge b$. In this way, we come to a partial operation $\curlywedge$ on A. It is total if and only if A is actually an implicative semilattice. For example, if $\varphi, \psi \in CE$, then always $\varphi x \; C \; \psi x$. A *relative subsemilattice* of A is any subset of A closed under existing compatible meets. Thus, the subset $CE(a) := \{\varphi a \colon \varphi \in CE\}$ is a relative subsemilattice for every $a \in A$: all meets in it exist and are compatible. (In [2, 3], we used the notation xy for the meet of x and y, and wrote $x \wedge y$ for it if it was compatible.)

Examples of relative subsemilattices are provided also by filters. A *filter* (an implicative filter, a deductive system) of A is a subset J containing 1 and such that $y \in J$ whenever $x, x \to y \in J$. According to [3, Lemma 3.2], J is an implicative filter if and only if it is a semilattice filter, i.e., an upwards closed relative subsemilattice of A.

The above definition of compatibility is equivalent to the original one presented in [14]. It was also observed there that elements x and y are compatible if and only if the filter generated by $\{x, y\}$ is principal, and then $x \curlywedge y$ is the least element of the filter. We now consider an arbitrary finite subset $P \subseteq A$ as *compatible* if P has a lower bound c (necessary unique) in $[P)$, and say that then c is the *compatible meet of* P (denoted by $\bigcurlywedge P$). This is the case if and only if $[P) = [c)$. Equivalently, a subset P is compatible if $\alpha_P = \alpha_p$ for some $p \in A$. In particular, the empty set is compatible; of course, $\bigcurlywedge \varnothing = 1$.

Where $X \subseteq A$, we denote by $\alpha_P(X)$ the set $\{\alpha_P x \colon x \in X\}$. A one-element subset of A is identified with its single element. The following notation will be convenient (cf. Definition 6.4 in [7]): given two finite subsets, P and Q, of A, we shall write $Q \to p$ for the element $\alpha_Q(p)$, and $Q \to P$, for the set $\alpha_Q(P)$, i.e., $\{Q \to p \colon p \in P\}$. If Q is empty, then $Q \to p = p$, and if P is empty, then $Q \to P$ also is the empty set. At last, $\{q\} \to \{p\} = \{q \to p\} = q \to p$. Observe that, if $P = \{p_1, p_2, \ldots, p_m\}$, then

$$\alpha_{(Q \to P)} = \alpha_{(Q \to p_m)} \cdots \alpha_{(Q \to p_2)} \alpha_{(Q \to p_1)}. \tag{2}$$

For example, $P \to P = 1$, and if $P = \{p_1, p_2\}$ and $Q := \{q_1, q_2\}$, then

$$\begin{aligned} \alpha_{(Q \to P)} x = \alpha_{(Q \to p_1)} \alpha_{(Q \to p_2)} x &= (Q \to p_1) \to ((Q \to p_2) \to x) \\ &= (q_1 \to (q_2 \to p_1)) \to ((q_1 \to (q_2 \to p_2)) \to x). \end{aligned}$$

For further reference, we list some properties of the operations α_P, where $\leq$ is the natural ordering of the semilattice CE^f (recall that it is defined pointwise).

Lemma 1 *In A,*

(a) $\alpha_P \alpha_Q = \alpha_Q \alpha_P$,

(b) $\alpha_Q \leq \alpha_P \alpha_Q$,

(c) $\alpha_P \leq \alpha_Q$ *if and only if* $\alpha_P \alpha_Q = \alpha_Q$.

(d) $\alpha_P \alpha_Q = \alpha_{(P \cup Q)}$,

(e) *if* $P \subseteq Q$, *then* $\alpha_P \leq \alpha_Q$,

(f) $\alpha_Q\alpha_{(Q\to P)} = \alpha_P\alpha_Q$,

(g) $\alpha_P \leq \alpha_Q$ *iff* $Q \to P = 1$.

Proof Items (a), (b) and (c) are obvious, and (e) follows from (c) and (d). In virtue of (a), items (d) and (f) generalize the Hilbert algebra identities $x \to (y \to z) = y \to (x \to z)$ and $x \to (x \to y) = x \to y$, respectively. For the "only if" part of (g), observe that $\alpha_P(p) = 1$ whenever $p \in P$. At last, if $Q \to P = 1$, then (f) and (c) lead us to left-side inequality of (g). □

3 The adjoint semilattice of A

We first extend to Hilbert algebras a result stated for implicative semilattices in [16, Proposition 3.6].

Proposition 2 *The subset CE^f is join-dense in the poset CE, i.e., every closure endomorphism is a join of a subset of CE^f. More exactly, if $\varphi \in CE$, then $\varphi = \bigvee(\alpha_p \colon p \in K_\varphi)$, where K_φ is the kernel of φ.*

Proof At first, $\alpha_p \leq \varphi$ for every $p \in K_\varphi$: if $\varphi p = 1$, then, for every x, $(p \to x) \to \varphi x = \varphi((p \to x) \to x) = (\varphi p \to \varphi x) \to \varphi x = 1$ (see (1)). At second, if ψ is another upper bound of $\{\alpha_p \colon p \in K_\varphi\}$ and $p = \varphi a \to a$ for some $a \in A$, then $\psi a \geq \alpha_p a = p \to a \geq \varphi a$. Thus, φ is the least upper bound of the set. □

A well-known description of the filter $[X)$ generated by some subset X of A, which goes back to [5, 14], may be formulated in terms of closure endomorphisms as follows: $a \in [X)$ if and only if a belongs to the kernel of some α_P with $P \subseteq_{\text{fin}} X$. If X is finite, one may put $P = X$. Therefore, the kernel of α_P is the filter $[P)$ generated by P; this correspondence between endomorphisms from CE^f and finitely generated filters is bijective. By Lemma 1(g), it is even order-preserving: $Q \to P = 1$ iff $P \subseteq K_{\alpha_Q}$ iff $[P) \subseteq [Q)$. Moreover, the kernel of $\alpha_P\alpha_Q$ is the standard join of filters $[P)$ and $[Q)$, i.e, the least filter including both $[P)$ and $[Q)$. Indeed, $K_{\alpha_P}, K_{\alpha_Q} \subseteq K_{\alpha_P\alpha_Q}$ (Lemma 1(b)). Suppose that, on the other hand, $K_{\alpha_P}, K_{\alpha_Q} \subseteq K_{\alpha_R}$ for some $R \subseteq_{\text{fin}} A$. If now $x \in K_{\alpha_P\alpha_Q}$, then $\alpha_Q x \in K_{\alpha_P}$ and, further, $\alpha_Q x \in K_{\alpha_R}$. Then $\alpha_R x \in K_{\alpha_Q}$ by Lemma 1(a), and, further $\alpha_R x \in K_{\alpha_R}$, i.e., $x \in K_{\alpha_R}$ (Lemma 1(d)). Therefore, $K_{\alpha_P\alpha_Q}$ is the least upper bound of K_{α_P} and K_{α_Q}.

These considerations are summed up in the next proposition.

Proposition 3 *The transformation $\alpha_P \mapsto [P)$ is an isomorphism of CE^f onto the semilattice of finitely generated filters.*

A *subtractive semilattice* [4] is the order dual of an implicative semilattice. We are going to show that the adjoint semilattice of a Hilbert algebra A is subtractive, i.e., that there is a binary operation $-$ (*subtraction*) on CE^f such that, for all $P, Q, R \subseteq_{\text{fin}} A$,

$$\alpha_P - \alpha_Q \leq \alpha_R \text{ if and only if } \alpha_P \leq \alpha_Q \circ \alpha_R. \tag{3}$$

Taking into account Proposition 3, this fact could be derived from Theorem 2.3 of [3], where the set of finitely generated filters of a Hilbert algebra was shown to be a subtractive semilattice. The theorem itself was proved referring to several constructions from Section 6 in [7]. We shall give for the adjoint semilattice a concise direct proof.

Theorem 4 *The operation $-$ defined on CE^f by*

$$\alpha_P - \alpha_Q := \alpha_{(Q\to P)}$$

is a subtraction.

Proof We have to prove that

$$\alpha_{(Q\to P)}x \le \alpha_R x \text{ for all } x \text{ if and only if } \alpha_P x \le \alpha_Q\alpha_R x \text{ for all } x.$$

The "only if" part holds by virtue of Lemma 1(f):

$$\alpha_P \le \alpha_Q\alpha_P = \alpha_Q\alpha_{(Q\to P)} \le \alpha_Q\alpha_R.$$

Conversely, from the right-side inequality, $1 = \alpha_P p \le \alpha_Q\alpha_R p$ for any $p \in P$. Now observe that $\alpha_Q p \le (\alpha_Q p \to x) \to x = \alpha_{(Q\to p)}x \to x$. Hence, for every x,

$$1 = \alpha_R\alpha_Q p \le \alpha_R(\alpha_{(Q\to p)}x \to x) = \alpha_{(Q\to p)}x \to \alpha_R x$$

(see (1)) and, further, $\alpha_{(Q\to p)}x \le \alpha_R x$. By (2), then $\alpha_{(Q\to P)}x \le \alpha_R x$ for all x. □

We next show that the transfer from Hilbert algebras to their adjoint (subtractive) semilattices is functorial. Suppose that A and A' are Hilbert algebras and that CE^f and CE'^f are the respective adjoint semilattices. Given a homomorphism $f\colon A \to A'$, let $f^*\colon CE^f \to CE'^f$ be the mapping defined by $f^*(\alpha_P) := \alpha_{f(P)}$.

Theorem 5 *Suppose that A, A' and A'' are Hilbert algebras, ε is the identity endomorphism of A, and $f\colon A \to A'$, $g\colon A' \to A''$ are homomorphisms. Then*

(a) *f^* and g^* are subtractive homomorphisms.*

(b) *ε^* is the identity morphism of CE^f,*

(c) *$(gf)^* = g^*f^*$.*

Proof (a) f^* is a semilattice homomorphism:

$$f^*(\alpha_P\alpha_Q) = f^*(\alpha_{P\cup Q}) = \alpha_{f(P\cup Q)} = \alpha_{f(P)\cup f(Q)} = \alpha_{f(P)}\alpha_{f(Q)} = f^*(\alpha_P)f^*(\alpha_Q),$$

and preserves subtraction: for $P = \{p_1, p_2, \ldots, p_n\}$,

$$f^*(\alpha_P - \alpha_Q) = f^*(\alpha_{(Q\to P)}) = \alpha_{f(Q\to P)}$$
$$= \alpha_{(f(Q)\to f(P))} = \alpha_{f(P)} - \alpha_{f(Q)} = f^*(\alpha_P) - f^*(\alpha_Q).$$

(b) is evident, as $\varepsilon = \alpha_1$.

(c) $(gf)^*(\alpha_P) = \alpha_{g(f(P))} = g^*(\alpha_{f(P)}) = g^*(f^*(\alpha_P))$. □

Finally, we characterise the lattice of ideals of CE^f. The subsequent theorem is partly suggested by various general results in [13] on ideals of a BCI-algebra.

Theorem 6 *The filter lattice of a Hilbert algebra is isomorphic to the ideal lattice of its adjoint semilattice.*

Proof It consists of several steps. Suppose that A is a Hilbert algebra, and CE^f, its adjoint semilattice. Let I stand for the ideal lattice of the semilattice CE^f, and F, for the filter lattice of A.

(a) For every filter J of A, the subset $\mathsf{i}(J) := \{\alpha_P\colon P \subseteq_{\text{fin}} J\}$ is an ideal of CE^f:

(a1) the identity closure endomorphism belongs to $\mathsf{i}(J)$, for $\varepsilon = \alpha_1$;

(a2) if $\alpha_P, \alpha_Q \in \mathsf{i}(J)$ with $P, Q \subseteq_{\text{fin}} J$, then also $P \cup Q \subseteq_{\text{fin}} J$ and, further, $\alpha_{P\cup Q} \in \mathsf{i}(J)$, i.e., $\alpha_P\alpha_Q \in \mathsf{i}(J)$;

(a3) if $\alpha_Q \in \mathsf{i}(J)$ with $Q \subseteq_{\text{fin}} J$, and if $\alpha_P \le \alpha_Q$ for some finite P, then $\alpha_Q x \ge \alpha_P x = 1$ and $x \in J$ for every $x \in P$. Therefore, $P \subseteq_{\text{fin}} J$ and $\alpha_P \in \mathsf{i}(J)$.

(b) The transformation $\mathsf{i}\colon F \to I$ is order-preserving: if $J \subseteq J'$, then every finite subset of J is also a subset of J', and then $\mathsf{i}(J) \subseteq \mathsf{i}(J')$.

(c) For every ideal $N \in I$, the subset $\mathsf{j}(N) := \{p\colon \alpha_p \in N\}$ is a filter of A:

(c1) $1 \in \mathsf{j}(N)$, for $\alpha_1 = \varepsilon \in N$;

(c2) if p and q are compatible elements of $\mathsf{j}(N)$ and $r := p \curlywedge q$, then $\alpha_p, \alpha_q \in N$, $\alpha_r = \alpha_p \circ \alpha_q \in N$ and, further, $r \in \mathsf{j}(N)$;

(c3) if $p \in \mathsf{j}(N)$ and $q \ge p$, then $\alpha_p \in N$, $\alpha_q \le \alpha_p$ and, furthermore, $\alpha_q \in N$, i.e., $q \in J$.

(d) The transformation $\mathsf{j}\colon I \to F$ is order-preserving: if $N \subseteq N'$, i.e., $\alpha_p \in N'$ whenever $\alpha_p \in N$, then $p \in \mathsf{j}(N')$ for all $p \in \mathsf{j}(N)$, and $\mathsf{j}(N) \subseteq \mathsf{j}(N')$.

(e) The transformations i and j are mutually inverse:

(e1) $\mathsf{j}(\mathsf{i}(J)) = J$: if $q \in J$, then $\alpha_q \in \mathsf{i}(J)$ and $q \in \mathsf{j}(\mathsf{i}(J))$, and if $q \in \mathsf{j}(\mathsf{i}(J))$, then $\alpha_q \in \mathsf{i}(J)$, i.e., $\alpha_q = \alpha_P$ for some $P \subseteq_{\text{fin}} J$. Hence, $q \in P$ and $q \in J$;

(e2) $\mathsf{i}(\mathsf{j}(N)) = N$: if $\alpha_P \in \mathsf{i}(\mathsf{j}(N))$ with $P \subseteq_{\text{fin}} \mathsf{j}(N)$, then $\alpha_p \in N$ for all $p \in P$, and α_P, being the join of all these α_p, also belongs to N. Conversely, if $\alpha_P \in N$, then $\alpha_p \le \alpha_P$, $\alpha_p \in N$ and $p \in \mathsf{j}(N)$ for all $p \in P$, i.e., $P \subseteq \mathsf{j}(N)$ and, further, $\alpha_P \in \mathsf{i}(\mathsf{j}(N))$.

Eventually, i and j are order isomorphisms from F to I and from I to F, respectively. Therefore, the lattices F and I are isomorphic. □

4 Principal closure endomorphisms

Each principal filter $[p)$ is the kernel of α_p and conversely; for this reason we call closure endomorphisms α_p *principal*. Let CE^α stand for the set of all such endomorphisms. We now can say more about the transformation $p \mapsto \alpha_p$.

Theorem 7 *In A,*

(a) $p \le q$ *if and only if* $\alpha_q \le \alpha_p$,

(b) $\alpha_{p\to q} = \alpha_q - \alpha_p$,

(c) $\alpha_{p\vee q} = \alpha_p \wedge \alpha_q$ *whenever* $p \vee q$ *exists,*

(d) *$p\ C\ q$ if and only $\alpha_p \circ \alpha_q$ is a principal closure endomorphism, and then $\alpha_{p \wedge\!\!\!\wedge q} = \alpha_p \circ \alpha_q$.*

(e) *a finite subset P of A is compatible if and only if the closure endomorphism α_P is principal, and then $\alpha_{(\bigwedge\!\!\!\bigwedge P)} = \alpha_P$.*

Proof (a) Clearly, if $p \leq q$, then $\alpha_q \leq \alpha_p$. If, conversely, $q \to x \leq p \to x$ for all x, then substitution of q for x shows that $p \leq q$.

(b) By the definition of subtraction.

(c) is an easy consequence of (a). Suppose that $p \vee q$ exists in A, then $\alpha_{p \vee q} \leq \alpha_p$, α_q. On the other hand, if $\alpha_r \leq \alpha_p$, α_q for some r, then $p, q \leq r$ and $p \vee q \leq r$. Hence, $\alpha_r \leq \alpha_{p \vee q}$, i.e., $\alpha_{p \vee q}$ is indeed the least upper bound of α_p and α_q.

(d) If $p \wedge\!\!\!\wedge q$ exists in A, then similarly, $\alpha_{p \wedge\!\!\!\wedge q} = \alpha_p \alpha_q$. Conversely, if $\alpha_p \alpha_q = \alpha_r$ for some r, then $r \to x = p \to (q \to x)$ for all x, whence $r \leq p, q$ and $p \leq q \to r$ (put $x := p, q, r$). Thus, $r = p \wedge\!\!\!\wedge q$, and $p\ C\ q$.

(e) If $\bigwedge\!\!\!\bigwedge P$ exists in A, then $[P) = [\bigwedge\!\!\!\bigwedge P)$ and, further $\alpha_P = \alpha_{(\bigwedge\!\!\!\bigwedge P)}$. If $\alpha_P = \alpha_r$ for some r, then $[P) = [r)$ and r is the compatible meet of P. □

The item (d) of the theorem is, in fact, contained in Theorem 3 of [14]. In virtue of items (b) and (a), the set of principal closure endomorphisms of A turns out to be closed under subtraction and is actually an order-dual copy of A; we shall call it the *dual algebra* of A. Therefore, CE^α is an implicative model or, as we prefer to say, a Henkin algebra. (It is now known well that the class of Henkin algebras coincides with that of positive implicative BCK-algebras described in [10].)

Corollary 8 *The set of principal closure endomorphisms of a Hilbert algebra A is a Henkin algebra dual to A.*

If every pair of elements of A is compatible, then, according to item (e) of the above theorem, all finitely generated closure endomorphisms are principal. We thus come to the following conclusion.

Corollary 9 *The adjoint semilattice of an implicative semilattice A is dually isomorphic to A.*

It follows that every subtractive semilattice is isomorphic to the adjoint semilattice of a Hilbert algebra. Also, non-isomorphic Hilbert algebras may have isomorphic adjoint semilattices. We obtain one more conclusion by help of Theorem 7(c).

Corollary 10 *If a Hilbert algebra A is an upper semilattice, then its adjoint semilattice is a sublattice of CE.*

Proof It suffices to prove that CE^f is closed under meets whenever all joins $p \vee q$ exist in A. As the lattice *CE* is distributive (see [2, Corollary 3.5]), for all finite P and Q,

$$\alpha_P \wedge \alpha_Q = \bigvee(\alpha_p \colon p \in P) \wedge \bigvee(\alpha_q \colon q \in Q)$$
$$= \bigvee(\alpha_p \wedge \alpha_q \colon p \in P, q \in Q) = \bigvee(\alpha_{p\vee q} \colon p \in P, q \in Q) = \alpha_{(P\vee Q)},$$

where $P \vee Q := \{p \vee q \colon p \in P, q \in Q\}$. □

The items (c) and (e) of Theorem 7 can be extended to joins and certain meets of arbitrary subsets of A. We call a subset $Y \subseteq A$ *K-compatible* if it has a lower bound which belongs to $[P)$ for every finite P such that $Y \subseteq [P)$. Let z be such a lower bound. If u is any other lower bound of Y, then $Y \subseteq [u)$ and, further, $z \in [u)$, i.e., $u \leq z$. Thus, z is the greatest lower bound; we say that it is a *K-compatible meet* of Y and denote it by $\bigwedge\!\!\!\!\bigwedge Y$. Evidently, if Y is finite, then this version of compatibility agrees with that discussed in Section 2: Y is K-compatible if and only if the filter generated by Y is principal, and $p = \bigwedge\!\!\!\!\bigwedge Y$ if and only if $[p) = [Y)$. Observe that even infinite subset Y is K-compatible, if it generates a principal filter; the converse need not hold.

Theorem 11 *Let Y be an arbitrary subset of A. Then*

(a) $\alpha_{(\bigvee Y)} = \bigwedge\{\alpha_y \colon y \in Y\}$ *whenever* $\bigvee Y$ *exists,*

(b) *Y is K-compatible if and only if the set $\{\alpha_y \colon y \in Y\}$ has a join in CE^f that is a principal closure endomorphism, say, α_r, and then $\alpha_{(\bigwedge\!\!\!\!\bigwedge Y)} = \alpha_r$.*

Proof (a) Similarly to item (c) of the previous theorem.

(b) By Lemma 1(g), $\alpha_q \leq \alpha_P$ iff $P \to q = 1$ iff $q \in K_{\alpha_P} = [P)$. Now suppose that r is a K-compatible meet of Y. Then $r \leq y$ for all $y \in Y$ and, for every finite P with $Y \subseteq [P)$, also $r \in [P)$. Consequently, $\alpha_y \leq \alpha_r$ for these y, i.e., α_r is an upper bound of $\{\alpha_y \colon y \in Y\}$. It is actually a least upper bound: if $\alpha_y \leq \alpha_P$ for all $y \in Y$ and some finite P, then $y \in [P)$; so $Y \subseteq [P)$ and, by the choice of r, $r \in [P)$, i.e., $[r) \subseteq [P)$. Now Proposition 3 implies that $\alpha_r \leq \alpha_P$.

Conversely, suppose that α_r is the join of $\{\alpha_y \colon y \in Y\}$. Then, in particular, $\alpha_y \leq \alpha_r$ and, by Theorem 7(a), $r \leq y$ for all $y \in Y$. Thus r is a lower bound of Y. On the other hand, if $Y \subset [P)$ for some finite P, then, for every $y \in Y$, $[y) \subseteq [P)$ and $\alpha_y \leq \alpha_P$ (Proposition 3). By choice of r, also $\alpha_r \leq \alpha_P$ and further $r \in P$. Hence, Y is K-compatible. □

A significant consequence of this theorem will be obtained in Section 5 (Corollary 15).

5 Minimal Brouwerian extensions of A

Corollary 8 and the observations preceding it motivate a transfer from the adjoint semilattice of a Hilbert algebra to certain extensions of the latter.

We say that an implicative semilattice B is a *Brouwerian extension* of a Hilbert algebra A if A is a subreduct of B. If this is the case, then, for all $x, y, z \in A$, $z = x \wedge\!\!\!\wedge y$ if and only if $z = x \wedge y$ in B [3, Lemma 3.3]. Such an extension of A is said to be *minimal*, if it is generated by A. Equivalently, B is a minimal Brouwerian extension if every its element can be presented as a join of a finite number of elements of A. Indeed, the set of those elements of B which can be so presented is closed under $\to$, as the following identity (with $P = \{p_1, p_2, \ldots, p_m\}$) shows:

$$\bigwedge Q \to \bigwedge P = (Q \to p_1) \wedge (Q \to p_2) \wedge \cdots \wedge (Q \to p_m).$$

Theorem 12 *A Brouwerian extension of a Hilbert algebra A is minimal if and only it is dually isomorphic to CE^f.*

Proof The condition is sufficient by the definition of CE^f (and Corollary 8). Now suppose that B is a minimal extension of A. Then every closure endomorphism in the adjoint semilattice of B can be presented in the form α_P with $P \subseteq_{\text{fin}} A$. Indeed, all closure endomorphisms of an implicative semilattice are principal. As any element $p \in B$ is a meet of a finite subset P of A, it follows that $\alpha_p = \alpha_P$. Now, the restriction $\alpha_P|A$ coincides with the closure endomorphism α_P of A; thus, there is a bijection between adjoint semilattices of B and A. Furthermore, restriction to A preserves composition and subtraction and respects the identity endomorphism of B. Therefore, the adjoint semilattices are isomorphic. Hence, B is dually isomorphic to the adjoint semilattice of A. □

Corollary 13 *Every Hilbert algebra has a unique (up to isomorphism) minimal Brouwerian extension.*

Remark 14 A construction of a minimal Brouwerian extension is implicit in A. Horn's paper [7]; see Theorem 8.5 therein (his C-algebras are just Hilbert algebras). Starting from a Hilbert algebra A, the author builds up an algebra B of non-empty subsets of A with operations $\cup$ and $\to$ and a constant 1 (cf. Section 2 above). The relation eq on B defined by

$$P \mathrel{\mathsf{eq}} Q \quad \text{iff} \quad P \to Q = Q \to P = 1.$$

is shown to be a congruence, and the quotient algebra B/eq is an implicative semilattice. Moreover, $\{p\} \mathrel{\mathsf{eq}} \{q\}$ iff $p = q$. In this way, A is embedded into an implicative semilattice. The author does not prove that the obtained extension of A is minimal; this follows from our Lemma 1(g).

Observe that $P \mathrel{\mathsf{eq}} Q$ iff $\alpha_P(Q) = \alpha_Q(P) = 1$ iff $[P) = [Q)$ iff $\alpha_P = \alpha_Q$.

Due to the above theorem, we may infer several properties of minimal Brouwerian extensions from the results of previous sections. Thus, any minimal Brouwerian extension of a Hilbert algebra is an example of the implicative semilattice mentioned in the next corollary.

Corollary 15 *Every Hilbert algebra can be embedded into an implicative semilattice with preservation of arbitrary existing joins and exactly K-compatible meets.*

Remark 16 Up to order duality, this corollary is a concise version of the extension theorem for L. Henkin's implicative models which was announced by Carol R. Karp in 1954 [11] (see also the first chapter in her Ph.D. thesis [12]). The above condition of K-compatibility is just a conjunction of her conditions (i) (borrowed from [6]) and (ii), while the subcondition (2) of (i) is actually a particular case of (ii). She also observed that every implicative model is isomorphic to a subspace of closed sets of a topological space, thus anticipating the topological representation theorem of Hilbert algebras stated by A. Diego in [5].

The subsequent counterpart of [7, Theorem 8.4] is the dual of Corollary 8.

Corollary 17 *A minimal Brouwerian extension of an upper Hilbert semilattice is an implicative lattice.*

It follows from Corollary 2.4 in [3] that the filter lattice of a Hilbert algebra is isomorphic to the filter lattice of some implicative semilattice. Theorem 6 above allows us to improve this observation.

Corollary 18 *The filter lattices of a Hilbert algebra and its minimal Brouwerian extension are isomorphic.*

Namely, if A is a Hilbert algebra and B is its minimal Brouwerian extension, then, as analysis of the transformations i and j in the proof of Theorem 6 shows, the filter J^* of B which corresponds to a filter J of A is given by $J^* := \{\bigwedge P \colon P \subseteq_{\mathrm{fin}} J\}$, and then $J = J^* \cap A$.

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