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On weakly ϕ-symmetric Kenmotsu Manifolds

Shyamal Kumar HUI

Nikhil Banga Sikshan Mahavidyalaya
Bishnupur – 722122, Bankura, West Bengal, India
e-mail: shyamal_hui@yahoo.co.in

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Abstract

The object of the present paper is to study weakly ϕ-symmetric and weakly ϕ-Ricci symmetric Kenmotsu manifolds. It is shown that weakly ϕ-symmetric and weakly ϕ-Ricci symmetric Kenmotsu manifolds are η-Einstein.

Key words: weakly ϕ-symmetric, weakly ϕ-Ricci symmetric, Kenmotsu manifold, Einstein manifold, η-Einstein manifold

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1 Introduction

In [29] Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing ξ is a constant, say c. He proved that they could be divided into three classes: (i) homogeneous normal contact Riemannian manifolds with $c > 0$, (ii) global Riemannian products of a line or a circle with a Kähler manifold of constant holomorphic sectional curvature if $c = 0$ and (iii) a warped product space $\mathbb{R} \times_f \mathbb{C}^n$ if $c < 0$. It is known that the manifolds of class (i) are characterized by admitting a Sasakian structure. The manifolds of class (ii) are characterized by a tensorial relation admitting a cosymplectic structure. Kenmotsu [9] characterized the differential geometric properties of the manifolds of class (iii) which are nowadays called Kenmotsu manifolds and later studied by several authors.

As a generalization of both Sasakian and Kenmotsu manifolds, Oubiña [14] introduced the notion of trans-Sasakian manifolds, which are closely related to the locally conformal Kähler manifolds. A trans-Sasakian manifold of type (0,0), $(\alpha, 0)$ and $(0, \beta)$ are called the cosympletic, α-Sasakian and β-Kenmotsu manifolds respectively, α, β being scalar functions. In particular, if $\alpha = 0$, $\beta = 1$, and $\alpha = 1$, $\beta = 0$ then a trans-Sasakian manifold will be a Kenmotsu and Sasakian manifold respectively.

The study of Riemann symmetric manifolds began with the work of Cartan [2]. A Riemannian manifold (M^n, g) is said to be locally symmetric due to Cartan [2] if its curvature tensor R satisfies the relation $\nabla R = 0$, where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g.

During the last five decades the notion of locally symmetric manifolds has been weakened by many authors in several ways to a different extent such as recurrent manifold by Walker [33], semisymmetric manifold by Sinyukov [27] and Szabó [28], pseudosymmetric manifold in the sense of Mikeš ([11],[12]) and Deszcz [7], pseudosymmetric manifold in the sense of Chaki [3], generalized pseudosymmetric manifold by Chaki [4], weakly symmetric manifold by Selberg [18] and weakly symmetric manifold by Támassy and Binh [31]. It may be noted that the notion of weakly symmetric Riemannian manifolds by Selberg [18] is different and are not equivalent to that of Támassy and Binh [31]. In this connection it is mentioned that Mikeš [10] studied projective-symmetric and projective-recurrent affinely connected spaces. Also in [13] Mikeš and Tolobaev studied symmetric and projectively symmetric affinely connected spaces and it is shown that [13] there exist projectively m-symmetric spaces, the differ from k-symmetric spaces and projectively k-symmetric spaces $(k < m)$.

A non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called a weakly symmetric manifold [31] if its curvature tensor R of type (0,4) satisfies the condition

$$\begin{aligned}(\nabla_W R)(X,Y,Z,U) &= A(W)R(X,Y,Z,U) + B(X)R(W,Y,Z,U)\\ &+ H(Y)R(X,W,Z,U) + D(Z)R(X,Y,W,U)\\ &+ E(U)R(X,Y,Z,W) \qquad (1)\end{aligned}$$

for all vector fields W, X, Y, Z, $U \in \chi(M^n)$, where A, B, H, D and E are 1-forms (not simultaneously zero) and ∇ denotes the operator of covariant differentiation with respect to the Riemannian metric g. The 1-forms are called the associated 1-forms of the manifold and an n-dimensional manifold of this kind is denoted by $(WS)_n$. The existence of a $(WS)_n$ is proved by Prvanović [16]. Then De and Bandyopadhyay [6] gave an example of a $(WS)_n$ by a metric of Roter [17] and proved that in a $(WS)_n$, $B = H$ and $D = E$ [6]. Hence the defining condition of a $(WS)_n$ reduces to the following form:

$$\begin{aligned}(\nabla_W R)(X,Y,Z,U) &= A(W)R(X,Y,Z,U) + B(X)R(W,Y,Z,U)\\ &+ B(Y)R(X,W,Z,U) + D(Z)R(X,Y,W,U)\\ &+ D(U)R(X,Y,Z,W), \qquad (2)\end{aligned}$$

i.e.,

$$\begin{aligned}(\nabla_W R)(X,Y)Z &= A(W)R(X,Y)Z + B(X)R(W,Y)Z \\ &+ B(Y)R(X,W)Z + D(Z)R(X,Y)W \\ &+ g(R(X,Y)Z,W)\rho, \end{aligned} \qquad (3)$$

where A, B and D are 1-forms (not simultaneously zero) and ρ is the vector field associated to the 1-form D such that $D(Z) = g(Z, \rho)$.

The $(WS)_n$ is also studied by Shaikh and Hui ([8], [19], [20], [21], [22], [23]), Shaikh and Jana ([24], [25]) and many others.

In 1993, Tamássy and Binh [32] introduced the notion of weakly Ricci symmetric manifolds. A Riemannian manifold (M^n, g) $(n > 2)$ is called weakly Ricci symmetric if its Ricci tensor S of type (0,2) is not identically zero and satisfies the condition

$$(\nabla_X S)(Y,Z) = A(X)S(Y,Z) + B(Y)S(Z,X) + D(Z)S(Y,X), \qquad (4)$$

where A, B and D are 1-forms (not simultaneously zero). Such an n-dimensional manifold is denoted by $(WRS)_n$.

The relation (4) can be written as

$$(\nabla_X Q)(Y) = A(X)Q(Y) + B(Y)Q(X) + S(Y,X)\rho, \qquad (5)$$

where ρ is the vector field associated to the 1-form D such that $D(Z) = g(Z, \rho)$ and Q is the Ricci operator, i.e., $g(QX, Y) = S(X, Y)$ for all X, Y.

The notion of locally ϕ-symmetric Sasakian manifolds was introduced by Takahashi [30]. In this connection De [5] introduced and studied ϕ-symmetric Kenmotsu manifolds. Recently Shukla and Shukla [26] introduced and studied ϕ-Ricci symmetric Kenmotsu manifolds. Again Özgür [15] studied weakly symmetric and weakly Ricci symmetric Kenmotsu manifolds and proved that in such a manifold the sum of the associated 1-forms is zero everywhere and hence such a manifold does not exist unless the sum of the associated 1-forms is everywhere zero.

The object of the present paper is to study weakly ϕ-symmetric and weakly ϕ-Ricci symmetric Kenmotsu manifolds. The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of weakly ϕ-symmetric Kenmotsu manifolds and it is shown that a weakly ϕ-symmetric Kenmotsu manifold is η-Einstein and hence such a structure is always exist. In section 4, we have studied weakly ϕ-Ricci symmetric Kenmotsu manifolds. It is proved that a weakly ϕ-Ricci symmetric Kenmotsu manifold is η-Einstein and consequently such a structure is always exist.

2 Preliminaries

A smooth manifold (M^n, g) $(n = 2m+1 > 1)$ is said to be an almost contact metric manifold [1] if it admits a (1, 1) tensor field ϕ, a vector field ξ, an 1-form

η and a Riemannian metric g which satisfy

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \quad \phi^2 X = -X + \eta(X)\xi, \tag{6}$$

$$g(\phi X, Y) = -g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \eta(\xi) = 1, \tag{7}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{8}$$

for all vector fields X, Y on M.

An almost contact metric manifold $M^n(\phi, \xi, \eta, g)$ is said to be Kenmotsu manifold if the following condition holds [9]:

$$\nabla_X \xi = X - \eta(X)\xi, \tag{9}$$

$$(\nabla_X \phi)(Y) = g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{10}$$

where ∇ denotes the Riemannian connection of g.

In a Kenmotsu manifold, the following relations hold [9]:

$$(\nabla_X \eta)(Y) = g(X, Y) - \eta(X)\eta(Y), \tag{11}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{12}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{13}$$

$$\eta(R(X, Y)Z) = \eta(Y)g(X, Z) - \eta(X)g(Y, Z), \tag{14}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{15}$$

$$S(\xi, \xi) = -(n-1), \text{ i.e., } Q\xi = -(n-1)\xi, \tag{16}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y), \tag{17}$$

$$(\nabla_W R)(X, Y)\xi = g(X, W)Y - g(Y, W)X - R(X, Y)W \tag{18}$$

for any vector field X, Y, Z on M and R is the Riemannian curvature tensor and S is the Ricci tensor of type (0,2) such that $g(QX, Y) = S(X, Y)$.

Definition 2.1 A Kenmotsu manifold M is said to be η-Einstein if its Ricci tensor S of type (0,2) is of the form

$$S = ag + b\eta \otimes \eta, \tag{19}$$

where a, b are smooth functions on M.

3 Weakly ϕ-symmetric Kenmotsu manifolds

Definition 3.1 A Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ $(n = 2m + 1 > 1)$ is said to be weakly ϕ-symmetric if the curvature tensor R satisfies

$$\begin{aligned} \phi^2((\nabla_W R)(X, Y)Z) = {} & A(W)\phi^2(R(X, Y)Z) + B(X)\phi^2(R(W, Y)Z) \\ & + B(Y)\phi^2(R(X, W)Z) + D(Z)\phi^2(R(X, Y)W) \\ & + g(R(X, Y)Z, W)\phi^2(\rho), \end{aligned} \tag{20}$$

where A, B and D are 1-forms (not simultaneously zero). If, in particular, $A = B = D = 0$ then the manifold is said to be ϕ-symmetric [5].

We now consider a Kenmotsu manifold $M^n(\phi, \xi, \eta, g)$ $(n = 2m + 1 > 1)$, which is weakly ϕ-symmetric. Then by virtue of (6), it follows from (20) that

$$\begin{aligned} -(\nabla_W R)(X,Y)Z + \eta((\nabla_W R)(X,Y)Z)\xi \\ = A(W)[-R(X,Y)Z + \eta(R(X,Y)Z)\xi] \\ + B(X)[-R(W,Y)Z + \eta(R(W,Y)Z)\xi] \\ + B(Y)[-R(X,W)Z + \eta(R(X,W)Z)\xi] \\ + D(Z)[-R(X,Y)W + \eta(R(X,Y)W)\xi] \\ + g(R(X,Y)Z,W)[-\rho + \eta(\rho)\xi]. \end{aligned} \tag{21}$$

Setting $Z = \xi$ in (21) and using (12), (14) and (18), we get

$$\begin{aligned} \{1 + D(\xi)\}R(X,Y)W = g(X,W)Y - g(Y,W)X \\ + A(W)[\eta(Y)X - \eta(X)Y] + B(X)[\eta(Y)W - \eta(W)Y] \\ + B(Y)[\eta(W)X - \eta(X)W] \\ + [\eta(Y)g(X,W) - \eta(X)g(Y,W)]\rho. \end{aligned} \tag{22}$$

This leads to the following:

Theorem 3.2 *In a weakly ϕ-symmetric Kenmotsu manifold, the curvature tensor is of the form* (22)*.*

From (22), we get

$$\begin{aligned} \{1 + D(\xi)\}S(Y,W) = -[(n-1) + D(\xi)]g(Y,W) + (n-1)A(W)\eta(Y) \\ + (n-2)B(Y)\eta(W) + \{B(W) + D(W)\}\eta(Y). \end{aligned} \tag{23}$$

Replacing Y by ϕY and W by ϕW in (23), we get

$$\{1 + D(\xi)\}S(\phi Y, \phi W) = -[(n-1) + D(\xi)]g(\phi Y, \phi W). \tag{24}$$

By virtue of (8) and (17), we have from (24) that

$$S(Y,W) = \alpha g(Y,W) + \beta\eta(Y)\eta(W), \tag{25}$$

where

$$\alpha = -\frac{n - 1 + D(\xi)}{1 + D(\xi)} \quad \text{and} \quad \beta = -\frac{(n-2)D(\xi)}{1 + D(\xi)},$$

provided $1 + D(\xi) \neq 0$.

This leads to the following:

Theorem 3.3 *A weakly ϕ-symmetric Kenmotsu manifold is an η-Einstein manifold.*

Corollary 3.4 [5] *A ϕ-symmetric Kenmotsu manifold is an Einstein manifold.*

4 Weakly ϕ-Ricci symmetric Kenmotsu manifolds

Definition 4.1 A Kenmotsu manifold $M^n(\phi,\xi,\eta,g)$ $(n = 2m+1 > 1)$ is said to be weakly ϕ-Ricci symmetric if the Ricci operator satisfies

$$\phi^2((\nabla_X Q)(Y)) = A(X)\phi^2(Q(Y)) + B(Y)\phi^2(Q(X)) + S(Y,X)\phi^2(\rho). \tag{26}$$

Especially, if the 1-forms $A = B = D = 0$, then (26) turns into the notion of ϕ-Ricci symmetric introduced by Shukla and Shukla [26].

Let us take a Kenmotsu manifold $M^n(\phi,\xi,\eta,g)$ $(n = 2m+1 > 1)$, which is weakly ϕ-Ricci symmetric. Then by virtue of (6) it follows from (26) that

$$\begin{aligned}-(\nabla_X Q)(Y) + \eta((\nabla_X Q)(Y))\xi &= A(X)[-QY + \eta(QY)\xi]\\ &+ B(Y)[-QX + \eta(QX)\xi] + S(Y,X)[-\rho + \eta(\rho)\xi]\end{aligned}$$

from which it follows that

$$\begin{aligned}&-g(\nabla_X Q(Y), Z) + S(\nabla_X Y, Z) + \eta((\nabla_X Q)(Y))\eta(Z)\\ &\quad = A(X)[-S(Y,Z) + \eta(QY)\eta(Z)] + B(Y)[-S(X,Z) + \eta(QX)\eta(Z)]\\ &\quad + S(Y,X)[-D(Z) + \eta(\rho)\eta(Z)].\end{aligned} \tag{27}$$

Putting $Y = \xi$ in (27) and using (9), (15) and (16), we get

$$\begin{aligned}[1 + B(\xi)]S(X,Z) = -(n-1)[&g(X,Z) - \eta(X)D(Z)\\ &+ \{B(\xi) + \eta(\rho)\}\eta(X)\eta(Z)].\end{aligned} \tag{28}$$

Replacing X by ϕX and Z by ϕZ in (28), we have

$$[1 + B(\xi)]S(\phi X, \phi Z) = -(n-1)g(\phi X, \phi Z). \tag{29}$$

By virtue of (8) and (17), we have from (29) that

$$S(X,Z) = \gamma g(X,Z) + \delta\eta(X)\eta(Z), \tag{30}$$

where

$$\gamma = -\frac{n-1}{1 + B(\xi)} \quad \text{and} \quad \delta = -\frac{(n-1)B(\xi)}{1 + B(\xi)},$$

provided $1 + B(\xi) \neq 0$.

Thus we can state the following:

Theorem 4.2 *A weakly ϕ-Ricci symmetric Kenmotsu manifold is an η-Einstein manifold.*

Corollary 4.3 *[26] A ϕ-Ricci symmetric Kenmotsu manifold is an Einstein manifold.*

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