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Optimal Convective Heat-Transport*

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Dedicated to Lubomír Kubáček on the occasion of his 80th birthday

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Abstract

The one-dimensional steady-state convection-diffusion problem for the unknown temperature $y(x)$ of a medium entering the interval (a, b) with the temperature $y_{\min}$ and flowing with a positive velocity $v(x)$ is studied. The medium is being heated with an intensity corresponding to $y_{\max} - y(x)$ for a constant $y_{\max} > y_{\min}$. We are looking for a velocity $v(x)$ with a given average such that the outflow temperature $y(b)$ is maximal and discuss the influence of the boundary condition at the point b on the "maximizing" function $v(x)$.

Key words: convective heat-transport, two-point convection-diffusion boundary-value problem, optimization of the amount of heat

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1 Introduction

There is an enormous interest in the development of effective tools able to control the processes of production, transport and dissipation of energy in technical systems. Classical models of these processes as well as sophisticated tools for their approximate solutions can be found in many monographs; see [2] and [1] for example. In spite of this, the relation between the distribution of speed of a medium flowing and exchanging heat with the neighbourhood and its outflow temperature has not been analysed although this involves the energy transport through long-distant pipes, solar collectors etc. The amount of heat inside of a flowing medium is influenced by the following essential processes: The transfer of heat between the flowing medium and its environment and the diffusive transport of heat inside of the flowing medium. The aim is to construct the transportation system, so that the efficiency of the energy transport is optimal.

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In order to get some insight into this problem, one of the typical processes of this kind is studied by means of a simple one-dimensional differential model.

PROBLEM: *For an inflow temperature $y_{\min}$ and $A > 0$ given, find such a velocity $v(x)$ of an average A that the temperature of the flowing medium at the outflow is maximal.*

According to [4], the following one-dimensional, steady-state *convection-diffusion* problem

$$\begin{gathered} -\varepsilon y'' + v(x)y' + \alpha y = \alpha y_{\max} \text{ in } (a,b), \\ y(a) = y_{\min}, \quad v(b)y'(b) + \alpha y(b) = \alpha y_{\max} \end{gathered} \tag{1}$$

is a simple model of both the transfer and the diffusive flow of heat and the following *convection* problem

$$v(x)y' + \alpha y = \alpha y_{\max} \text{ in } (a,b), \quad y(a) = y_{\min} \tag{2}$$

is a model of the heat-transfer only. In these problems, $\varepsilon > 0$, $\alpha > 0$, $v \in C[a,b]$, $v(x) > 0$ for $x \in [a,b]$ and $y_{\min} < y_{\max}$. We interpret $y = y(x)$ as the temperature of the flowing medium and the constants ε and α as the heat conductivity and heat-transfer coefficients. Due to [3], p. 185, problem (1) has a unique solution $y \in C^2[a,b]$ if and only if the associated homogeneous problem

$$-\varepsilon \overline{y}'' + v(x)\overline{y}' + \alpha \overline{y} = 0 \text{ in } (a,b), \quad \overline{y}(a) = 0 = v(b)\overline{y}'(b) + \alpha \overline{y}(b) \tag{3}$$

has the trivial solution only. We can see that any solution of (3) satisfies $\overline{y}(b) = 0 = \overline{y}'(b)$, so that $\overline{y}$ is a solution of the initial-value problem

$$-\varepsilon \overline{y}'' + v(x)\overline{y}' + \alpha \overline{y} = 0, \qquad \overline{y}(b) = \overline{y}'(b) = 0$$

which has the trivial solution only due to [3], p. 92. The initial-value problem (2) has a unique solution $y \in C^1[a,b]$ according to [3], p. 20.

2 Optimization, the case of pure convection

An essential information concerning the role of the convective heat-flow gives us the following statement.

Theorem 1 *For a positive velocity v with average A on $\langle a,b \rangle$, the solution y of the problem (2) satisfies*

$$y(b) = y_{\min} + (y_{\max} - y_{\min})e^{-\alpha(b-a)/A}. \tag{4}$$

Proof The statement becomes very easy to prove and also to understand from the physical point of view whenever problem (2) will be transformed into the following Lagrangian coordinates: Denoting $x(t)$ the position at time t of the material point situated in the inflow boundary point a at time 0, we obtain the following initial-value problem for the trajectory $x(t)$:

$$\frac{dx(t)}{dt} = v(x(t)), \text{ in } (0,(b-a)/A), \quad x(0) = a. \tag{5}$$

Integrating the identity (5), we obtain the following condition guaranteeing that the average of velocity v on (a, b) is A:

$$x\left(\frac{b-a}{A}\right) - x(0) = \int_0^{(b-a)/A} v(x(t))\,dt = b - a. \tag{6}$$

The substitution of x by $x(t)$ in (2) gives us

$$v(x(t))\frac{dy(x(t))}{dx} + \alpha y(x(t)) = \alpha y_{\max} \quad \text{for } t \in (0, (b-a)/A).$$

Denoting $y(x(t))$ by $Y(t)$ in this equation and using (5), we obtain the initial-value problem

$$\frac{dY}{dt} + \alpha Y = \alpha y_{\max} \quad \text{in } (0, (b-a)/A), \quad Y(0) = y_{\min} \tag{7}$$

with the solution

$$Y(t) = y_{\min} + (y_{\max} - y_{\min})e^{-\alpha t}. \tag{8}$$

Putting $t = (b-a)/A$ into (8) and using (6), we get

$$y(b) = y\left(x\left(\frac{b-a}{A}\right)\right) = Y\left(\frac{b-a}{A}\right) = y_{\min} + (y_{\max} - y_{\min})e^{-\alpha(b-a)/A}.$$

□

Definition 1 We say that the temperature (4) is *critical*.

3 Optimization, the case of convection-diffusion

Our Theorem 1 says that, in the case of pure convection, the distribution of speed of the velocity function v with a given average has no impact on the outflow temperature $y(b)$. As the diffusive flow is present in every real convective heat-transport, we experimentally illustrate the way in which the value $y(b)$ of the solution of the convection-diffusion problem depends on the size of ε, the choice of velocity and on the boundary condition in the outflow boundary b.

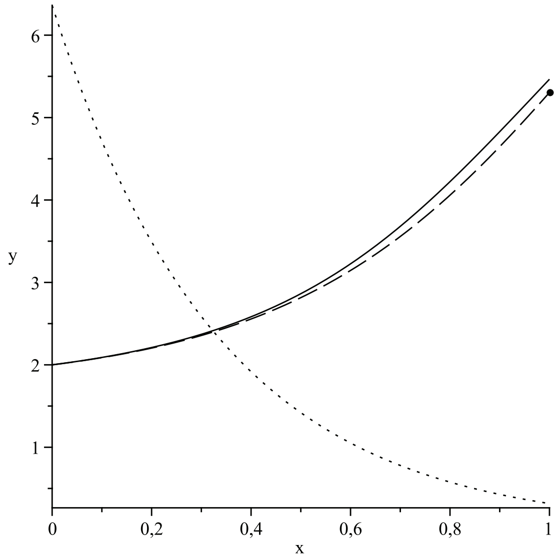
Let us set $(a, b) = (0, 1)$, $y_{\min} = 2$, $y_{\max} = 8$, $\alpha = 0.8$ and

$$v(x) = v_\mu(x) = \begin{cases} \frac{1-e^{-\mu}}{\mu} e^{\mu x} & \text{for } \mu \neq 0 \\ 1 & \text{for } \mu = 0 \end{cases}$$

for $\mu \in \langle -3, 3 \rangle$ in the problems (1), (2). The average of all these functions v_μ on $(0, 1)$ is $A = 1$ and, of course, they change continuously from the strongly decreasing v_{-3} to the strongly increasing v_3. In the following Fig. 1, 2 (a), (b), the velocities v_μ used in Examples 1, 2 are illustrated by the dotted graphs, the exact solutions of the problem (1) with $v = v_\mu$ by the solid graphs and those of the problem (2) with $v = v_\mu$ by the dashed graphs.

Example 1 If we insert the boundary condition of the problem (1) into the equation for $x = b$, we can see that the solution of (1) satisfies $y''(b) = 0$. Among the functions v_μ, the choice $v = v_{-3}$ leads to exact solutions $y(x)$ of our problem (1) with the values $y(1)$ maximal both for $\varepsilon = 0.05$ and $\varepsilon = 1$; see Fig. 1 (a) and (b). Observe that the values y(1) are bigger than the critical value in both cases.

a) $\varepsilon = 0.05$:



b) $\varepsilon = 1$:

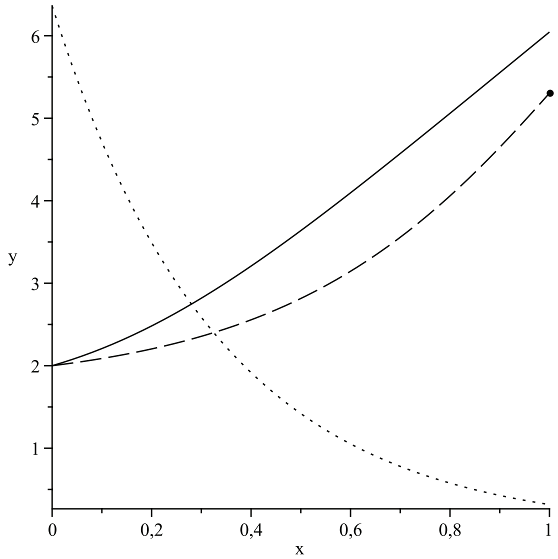
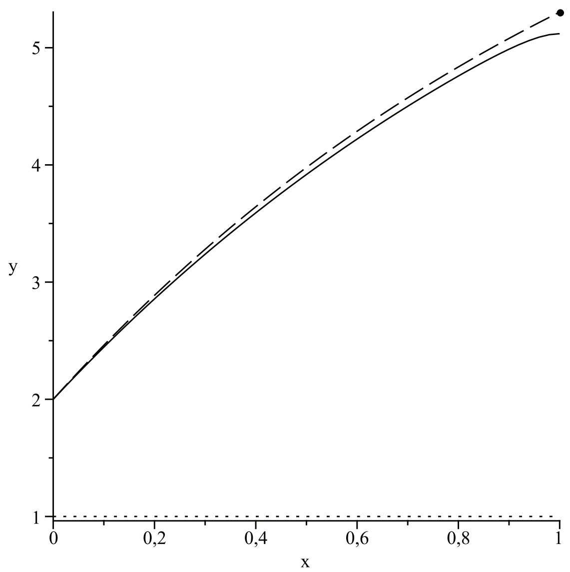


Fig. 1: Solution of problem (1) satisfying $y''(b) = 0$

Example 2 Let us modify the boundary condition of problem (1) in $b = 1$ to $y'(1) = 0$. The choice $v = v_0$ leads to exact solutions $y(x)$ with the values $y(1)$ maximal both for $\varepsilon = 0.05$ and $\varepsilon = 1$; see Fig. 2 (a) and (b). The values y(1) are less than the critical value in both cases.

a) $\varepsilon = 0.05$:



b) $\varepsilon = 1$:

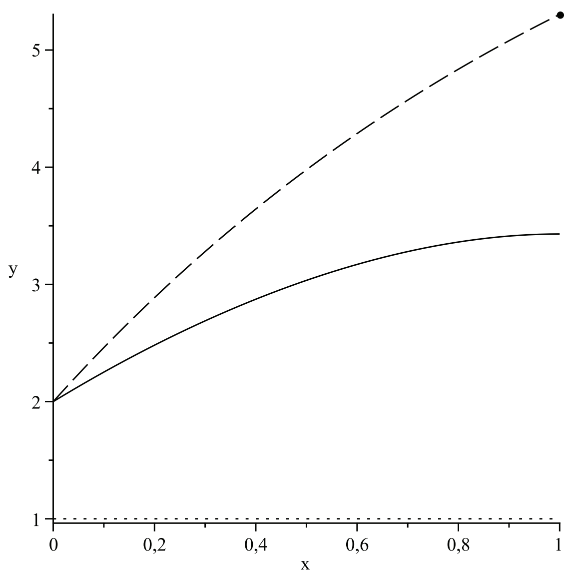


Fig. 2: Solution of problem (1) satisfying $y'(b) = 0$

4 Conclusions

The boundary condition in the point b of problem (1) depends on the concrete implementation of the real transportation system. Examples 1 and 2 illustrate that this condition changes the properties of the solutions of problem (1) essentially. As the typical purpose of the technical systems transporting heat is to save as much of the energy inside of the system as possible, the boundary condition in the point b should require neither any remarkable gain nor any remarkable loss of heat by any process different from the transfer and diffusive flow included in the model (1). From this point of view the boundary condition $y''(b) = 0$, saying that the solution y behaves in the point b similarly as in the points from (a, b) near to b, can be characterized as natural. On the contrary, as the flowing medium is being heated with intensity $\alpha(y_{\max} - y(x)) > 0$, the temperature should be increasing in (a, b). From this point of view, the boundary condition $y'(b) = 0$ can be called non-natural.

This article provides a basic information about the relationship between the distribution of the speed of a medium flowing through a technical transportation system and its outflow temperature as a result of an analysis of the simple one-dimensional linear convection-diffusion boundary-value problem. The results indicate that a suitable choice of the speed distribution leads to an increase of the output temperature. This one depends on the temperature conditions at the outflow boundary of the transportation system, too.

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