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On the Stability of Jungck–Mann, Jungck–Krasnoselskij and Jungck Iteration Processes in Arbitrary Banach Spaces

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Abstract

In this paper, we establish some stability results for the Jungck–Mann, Jungck–Krasnoselskij and Jungck iteration processes in arbitrary Banach spaces. These results are proved for a pair of nonselfmappings using the Jungck–Mann, Jungck–Krasnoselskij and Jungck iterations. Our results are generalizations and extensions to a multitude of stability results in literature including those of Imoru and Olatinwo [8], Jungck [10], Berinde [1] and many others.

Key words: stability, nonselfmappings, Jungck–Mann, Jungck–Krasnoselskij and Jungck iteration processes

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1 Introduction

Let (E, d) be a complete metric space, $T\colon E \longrightarrow E$ a selfmap of E. Suppose that $F_T = \{p \in E\colon Tp = p\}$ is the set of fixed points of T in E.

Let $\{x_n\}_{n=0}^{\infty} \subset E$ be the sequence generated by an iteration procedure involving the operator T, that is,

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, 2, \ldots, \tag{1}$$

where $x_0 \in E$ is the initial approximation and f is some function. If in (1),

$$f(T, x_n) = Tx_n, \quad n = 0, 1, 2, \ldots, \tag{2}$$

then, we have the Picard iteration process, which has been employed to approximate the fixed points of mappings satisfying

$$d(Tx,Ty) \leq ad(x,y), \quad \forall x,y \in E,\ a \in [0,1), \tag{3}$$

called the *Banach's contraction condition* and is of great importance in the celebrated Banach's fixed point Theorem, see [2]. An operator satisfying (3) is called a *strict contraction*.

Let $x_0 \in E$ be the initial approximation, where E is a Banach space. If in (1),

$$f(T,x_n) = (1-\alpha_n)x_n + \alpha_n T(x_n), \quad n = 0,1,2,\ldots, \tag{4}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of real numbers in $[0,1]$, then we have the Mann iteration process, see [12].

Singh et al [14] introduced the following iteration to obtain some common fixed points and stability results: Let S and T be operators on an arbitrary set Y with values in E such that $T(Y) \subseteq S(Y)$, $S(Y)$ is a complete subspace of E. For arbitrary $x_0 \in Y$, the sequence $\{S(x_n)\}_{n=0}^{\infty}$ defined by

$$S(x_{n+1}) = (1-\alpha_n)S(x_n) + \alpha_n T(x_n), \quad n = 0,1,2,\ldots, \tag{5}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence of real numbers in [0,1], is called the *Jungck–Mann iteration process*.

If in (5), $\alpha_n = 1$ and $Y = E$, then we have

$$S(x_{n+1}) = T(x_n), \quad n = 0,1,2,\ldots, \tag{6}$$

which is the *Jungck iteration*. (For example, see [10].)

Jungck [10] proved that the maps S and T satisfying

$$d(Tx,Ty) \leq ad(Sx,Sy), \quad \forall x,y \in E,\ a \in [0,1), \tag{7}$$

have a unique common fixed point in a complete metric space E, provided that S and T commute, $T(Y) \subseteq S(Y)$ and S is continuous.

Singh et al [14] also introduced the following general iteration process: Let $S,T\colon Y \longrightarrow E$ and $T(Y) \subseteq S(Y)$. For any $x_0 \in Y$, let

$$S(x_{n+1}) = f(T,x_n), \quad n = 0,1,2,\ldots, \tag{8}$$

where f is some function.

In 2006, Imoru and Olatinwo [8] proved some stability results for Picard and Mann iteration processes using the following contractive conditions: there exist a constant $a \in [0,1)$, and a monotone increasing function $\varphi\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, with $\varphi(0) = 1$ such that $\forall x,y \in E$,

$$d(Tx,Ty) \leq ad(x,y)\varphi(d(x,Tx)), \tag{9}$$

where φ may be a comparison function or just a monotone increasing function.

In 2008, Olatinwo [13] used the following Jungck–Ishikawa iteration process to establish some stability and strong convergence results: Let $(E, \|.\|)$ be a Banach space and Y an arbitrary set. Let $S, T\colon Y \longrightarrow E$ be two nonselfmappings such that $T(Y) \subseteq S(Y)$, $S(Y)$ is a complete subspace of E and S is injective. For $x_0 \in Y$, the Jungck–Ishikawa iteration is a sequence $\{S(x_n)\}_{n=0}^{\infty}$ defined iteratively by

$$\begin{aligned} S(x_{n+1}) &= (1-\alpha_n)S(x_n) + \alpha_n T(z_n) \\ S(z_n) &= (1-\beta_n)S(x_n) + \beta_n T(x_n), \end{aligned} \tag{10}$$

where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are sequences of real numbers in [0,1].

Olatinwo [13] also used the following contractive definitions to prove his stability and strong convergence results on the Jungck–Ishikawa iteration process:

(a) there exist a real number $a \in [0, 1)$ and a monotone increasing function $\varphi\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\varphi(0) = 0$ and $\forall x, y \in Y$, we have

$$\|T(x) - T(y)\| \leq \varphi(\|S(x) - T(x)\|) + a\,\|S(x) - S(y)\|\,; \tag{11}$$

(b) there exist real numbers $M \geq 0$, $a \in [0, 1)$ and a monotone increasing function $\varphi\colon \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that $\varphi(0) = 0$ and $\forall x, y \in Y$, we have

$$\|T(x) - T(y)\| \leq \frac{\varphi(\|S(x) - T(x)\|) + a\,\|S(x) - S(y)\|}{1 + M\,\|S(x) - T(x)\|}. \tag{12}$$

Recently, Bosede and Rhoades [6] proved the stability of Picard and Mann iterations for a general class of functions.

In 2010, Bosede [5] also proved some strong convergence results for the Jungck–Ishikawa and Jungck–Mann iteration processes.

Our aim in this paper is to prove some stability results for Jungck–Mann, Jungck–Krasnoselskij and Jungck iteration processes considered in Banach spaces using a contractive condition independent of those of Olatinwo [13]. Also, the assumption that S is injective in [13] is no longer necessary in our contractive condition and this is the novelty of our results in this paper.

Consequently, we shall employ the following contractive definition:

Let $(E, \|.\|)$ be a Banach space and Y an arbitrary set. Suppose that $S, T\colon Y \longrightarrow E$ are two nonselfmappings such that $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of E. Suppose also that $z \in Y$ is a coincidence point of S and T, with $p := S(z) = T(z)$ and that there exist a constant $a \in [0, 1)$ and a monotone increasing function $\varphi\colon (0, \infty) \longrightarrow (0, \infty)$, with $\varphi(0) = 1$ such that

$$\|T(x) - T(y)\| \leq a\,\|S(x) - S(y)\|\,\varphi(\|S(x) - T(x)\|), \quad x, y \in Y. \tag{13}$$

In a complete metric space setting, (13) becomes

$$d(Tx, Ty) \leq ad(Sx, Sy)\varphi(d(Sx, Tx)). \tag{14}$$

Remark 1 The contractive condition (14) is more general than those considered by Imoru and Olatinwo [8], Jungck [10], Berinde [1], and several others in the following sense: If $S = I$, (the identity operator), in the contractive condition (14), then we obtain the contraction condition (9) used by Imoru and Olatinwo which is itself a generalization of those of Berinde and many others in literature.

The following definition of the stability of iteration process due to Singh et al [14] shall be required in the sequel:

Definition 1 Let $S, T\colon Y \longrightarrow E$, $T(Y) \subseteq S(Y)$ and z a coincidence point of S and T, that is, $Sz = Tz = p$ (say). For any $x_0 \in Y$, let the sequence $\{S(x_n)\}_{n=0}^{\infty}$ generated by the iteration procedure (8) converge to p. Let $\{S(y_n)\}_{n=0}^{\infty} \subset E$ be an arbitrary sequence, and set $\epsilon_n = d(Sy_{n+1}, f(T, y_n)),\quad n = 0, 1, \ldots$. Then, the iteration procedure (8) will be called *(S, T)-stable* if and only if $\lim_{n\longrightarrow\infty} \epsilon_n = 0$ implies that $\lim_{n\longrightarrow\infty} S(y_n) = p$.

We shall also use the following Lemma contained in Berinde, see [1], in the sequel.

Lemma 1 *Let δ be a real number satisfying $0 \leq \delta < 1$, and $\{\epsilon_n\}$ a positive sequence satisfying $\lim_{n\longrightarrow\infty} \epsilon_n = 0$. Then, for any positive sequence $\{u_n\}$ satisfying $u_{n+1} \leq \delta u_n + \epsilon_n$, $\;n = 0, 1, 2, \ldots$, it follows that $\lim_{n\longrightarrow\infty} u_n = 0$.*

2 Main results

Theorem 1 *Let $(E, \|.\|)$ be a Banach space, Y be a set, and $x_0 \in Y$. Suppose that $S, T\colon Y \longrightarrow E$ are such that $T(Y) \subseteq S(Y)$, that $S(Y)$ is a complete subspace of E, and that there is a constant $a \in [0, 1)$ and a monotone increasing function $\varphi\colon (0, \infty) \longrightarrow (0, \infty)$, with $\varphi(0) = 1$ such that*

$$\|T(x) - T(y)\| \leq a\,\|S(x) - S(y)\|\,\varphi(\|S(x) - T(x)\|) \quad \textit{whenever}\; x, y \in Y. \quad (15)$$

Assume that $z \in Y$ is a coincidence point of S and T, with $p := S(z) = T(z)$, that $\{S(x_n)\}_{n=0}^{\infty}$ is a sequence in E converging to p, $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in [0,1], and $\alpha > 0$ such that

$$S(x_{n+1}) = (1 - \alpha_n)S(x_n) + \alpha_n T(x_n), \quad \alpha \leq \alpha_n,\; n \in \{0, 1, 2, \ldots\}. \qquad (16)$$

Then the above Jungck–Mann iteration process is (S, T)-stable.

Proof Let $\{S(y_n)\}_{n=0}^{\infty}$ be an arbitrary sequence in E, and define

$$\epsilon_n = \|S(y_{n+1}) - (1 - \alpha_n)S(y_n) - \alpha_n T(y_n)\|\,, \quad n \in \{0, 1, 2, \ldots\}.$$

Assume that $\lim_{n\longrightarrow\infty} \epsilon_n = 0$. Then, we shall prove that $\lim_{n\longrightarrow\infty} S(y_n) = p$.

Using the contractive condition (15) and the triangle inequality, we have

$$\begin{aligned}
\|S(y_{n+1})-p\| &\leq \|S(y_{n+1})-(1-\alpha_n)S(y_n)-\alpha_n T(y_n)\| \\
&\quad + \|(1-\alpha_n)S(y_n)+\alpha_n T(y_n)-p\| \\
&= \epsilon_n + \|(1-\alpha_n)S(y_n)+\alpha_n T(y_n)-((1-\alpha_n)+\alpha_n)p\| \\
&= \epsilon_n + \|(1-\alpha_n)(S(y_n)-p)+\alpha_n(T(y_n)-p)\| \\
&\leq \epsilon_n + (1-\alpha_n)\|S(y_n)-p\| + \alpha_n\|T(y_n)-p\| \\
&= \epsilon_n + (1-\alpha_n)\|S(y_n)-p\| + \alpha_n\|T(z)-T(y_n)\| \\
&\leq \epsilon_n + (1-\alpha_n)\|S(y_n)-p\| \\
&\quad + \alpha_n a\|S(z)-S(y_n)\|\varphi(\|S(z)-T(z)\|) \\
&= \epsilon_n + (1-\alpha_n+\alpha_n a)\|S(y_n)-p\| \\
&\leq \epsilon_n + (1-\alpha(1-a))\|S(y_n)-p\|. \qquad (17)
\end{aligned}$$

Since $0 \leq (1-\alpha(1-a)) < 1$, then applying Lemma 1 in (17) yields

$$\lim_{n\longrightarrow\infty}\|S(y_n)-p\| = 0,$$

that is, $\lim_{n\longrightarrow\infty} S(y_n) = p$. This completes the proof. □

Remark 2 Theorem 1 of this paper is a generalization of the results obtained by Imoru and Olatinwo [8], Jungck [10], Berinde [1], and many others; and this is also a further improvement to many known stability results in literature.

A special case of Jungck–Mann iteration process is that of Jungck–Krasnoselskij iteration process which is Jungck–Mann iteration, with each $\alpha_n = \lambda$, for some $0 < \lambda < 1$.

For arbitrary $x_0 \in Y$, the sequence $\{S(x_n)\}_{n=0}^{\infty}$ defined by

$$S(x_{n+1}) = (1-\lambda)S(x_n) + \lambda T(x_n), \quad n = 0, 1, 2, \ldots, \qquad (18)$$

for some $0 < \lambda < 1$, is called the *Jungck–Krasnoselskij iteration process.*

Corollary 1 *Let E, Y, S, T, z and p be as in Theorem 1. For arbitrary $x_0 \in Y$, let $\{S(x_n)\}_{n=0}^{\infty}$ be the Jungck–Krasnoselskij iteration process defined by (18), for some $0 < \lambda < 1$.*

Then the Jungck–Krasnoselskij iteration process is (S,T)-stable.

Proof In Theorem 1, set each $\alpha_n = \lambda$. □

Theorem 2 *Let E, Y, S, T, z and p be as in Theorem 1. For arbitrary $x_0 \in Y$, let $\{S(x_n)\}_{n=0}^{\infty}$ be the Jungck iteration process defined by (6). Then the Jungck iteration process is (S,T)-stable.*

Proof Follows directly from Theorem 1 taking $\alpha_n = 1$, for each n. □

Remark 3 Theorem 2 is a generalization of many existing results in literature especially those of Imoru and Olatinwo [8], Jungck [10] and Berinde [1].

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