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A Common Fixed Point Theorem for Expansive Mappings under Strict Implicit Conditions on b-Metric Spaces

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Abstract

In the setting of a b-metric space (see [2] and [3]), we establish two general common fixed point theorems for two mappings satisfying the (E.A) condition (see [1]) under strict expansive conditions using two classes of implicit relations. These two theorems may be considered as extensions of the main result of [14] to b-metric spaces. Also, the main result of [1] is obtained as a consequence of our results.

Key words: common fixed point, metric space, b-metric space, property (E.A), implicit relation, expansive mapping

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1 Introduction

Let (X, d) be a metric space and S and T two self-mappings of X. In [7], Jungck defines S and T to be compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. This concept was frequently used to prove existence theorems in common fixed point theory. The study of common fixed points of noncompatible mappings is also very interesting. Work along these lines has been recently initiated by Pant [9], [10], [11]. Aamri and Moutawakil [1] introduced a generalization of the concept of noncompatible mappings.

Definition 1.1 [1] Let S and T be two self-mappings of a metric space (X, d). We say that T and S satisfy property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

Remark 1.1 It is clear that two self-mappings of a metric space (X, d) will be noncompatible if there exists at least a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, for some $t \in X$, but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or does not exist. Therefore, two noncompatible self-mappings of a metric space (X, d) satisfy property (E.A).

Definition 1.2 [8] Two self-mappings S and T of a metric space (X, d) are said to be weakly compatible if $Tu = Su$, for $u \in X$, then $STu = TSu$.

Remark 1.2 Two compatible mappings are weakly compatible.

Recently, V. Popa ([14]) has proved a general fixed point theorem for expansive weakly compatible mappings satisfying a strict implicit condition.

For other papers of V. Popa making use of implicit relations in metric fixed point theory, see [12] and [13].

The class $\mathbf{F}$ of implicit functions used in [14] is given as follows.

Let $\mathbf{F}$ be the set of all real continuous functions $F(t_1, \ldots, t_6) \colon \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:

(F1): $F(t, 0, 0, t, t, 0) < 0$, for all $t > 0$,

(F2): $F(t, t, 0, 0, t, t) \leq 0$, for all $t > 0$.

Using these functions, the following theorem was proved in [14].

Theorem 1.1 ([14]) *Let S and T be two weakly compatible self-mappings of a metric space (X, d) such that:*

(1) S and T satisfy property (E.A),

(2) $F(d(Tx, Ty), d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)) > 0$ for each $(x, y) \in X^2$, where $F \in \mathbf{F}$.

(3) $T(X) \subset S(X)$.

If $S(X)$ or $T(X)$ is a complete subspace of X, then T and S have a unique common fixed point.

For other recent papers studying fixed point theorems for expansive or non-expansive weakly compatible mappings satisfying implicit relations, the reader is invited to consult the papers [4], [5] and [6].

The aim of this paper is to investigate possible extensions of Theorem 1.1 due to V. Popa [14] to the general case of b-metric spaces introduced by S. Czerwik [2] and [3].

The main results of this paper are Theorem 4.1 and Theorem 5.1.

In our results, we assume only closedness of the ranges of the mappings not their completeness. Moreover, the contractive conditions used in these results are supposed to hold only for distinct elements.

In Theorem 4.1, we use the class $\mathcal{G}_s$ (defined below) to establish the existence of a unique common fixed point for a weakly compatible pair of self-mappings of a b-metric space (X, d).

In Theorem 5.1, if the b-metric d satisfies the property S_C (see Definition 5.1), we provide an extension of Theorem 1.1 by using the class $\mathbf{F}$ of implicit functions considered in [14]. In particular, the main result of [1] is obtained as a consequence of Theorem 5.1.

2 Implicit relations

We denote $\mathcal{F}$ the set of all real continuous functions $F(t_1, \ldots, t_6)\colon \mathbb{R}_+^6 \to \mathbb{R}$.

Let $s \geq 1$ be fixed and let $\mathcal{G}_s$ be the set of all $G \in \mathcal{F}$ satisfying the following conditions:

(P_m): G is nondecreasing in the variable t_1 and nonincreasing in the variable t_5,

(P_1): $G(st, 0, 0, t, \frac{1}{s}t, 0) < 0$, for all $t > 0$,

(P_2): $G(t, t, 0, 0, t, t) \leq 0$, for all $t > 0$.

We denote $\mathcal{F}_s$ the set of all $F \in \mathcal{F}$ satisfying (only) the conditions (P_1) and (P_2).

We observe that $\mathcal{F}_1 = \mathbf{F}$ and that $\mathcal{G}_1 \subset \mathbf{F}$. (This inclusion is strict).

Examples

Let s be a given number in the set $[1, \infty)$.

Example 2.1 $G(t_1, \ldots, t_6) := t_1 - q \max\{t_2, \ldots, t_6\}$, where $q > s$.

(P_m): is clear

(P_1): $G(st, 0, 0, t, \frac{1}{s}t, 0) = t(s - q) < 0$, for all $t > 0$.

(P_2): $G(t, t, 0, 0, t, t) = t(1 - q) < 0$, for all $t > 0$

Example 2.2 $G(t_1, \ldots, t_6) := t_1 - qs \max\{t_2, \ldots, t_6\}$, where $q > 1$.

(P_m): is clear

(P_1): $G(st, 0, 0, t, \frac{1}{s}t, 0) = st(1 - q) < 0$, for all $t > 0$.

(P_2): $G(t, t, 0, 0, t, t) = t(1 - qs) < 0$, for all $t > 0$

Example 2.3 $G(t_1, \ldots, t_6) := t_1^2 - at_2t_3 - bs^2t_4t_5 - ct_5t_6$, where $a \geq 0$, $b > s$ and $1 \leq c$.

(P_m): is clear

(P_1): $G(st, 0, 0, t, \frac{1}{s}t, 0) = st^2(s - b) < 0$, for all $t > 0$.

(P_2): $G(t, t, 0, 0, t, t) = t^2(1 - c) \leq 0$, for all $t > 0$.

Example 2.4 Let $s = 1$ and consider $F(t_1, \ldots, t_6) := t_1^3 - at_1^2t_2 - bt_1t_4t_5$, where $a \geq 1$ and $b > 1$.

(P_m): is not satisfied.

(P_1): $F(t, 0, 0, t, t, 0) = t^3(1 - b) < 0$, for all $t > 0$.

(P_2): $F(t, t, 0, 0, t, t) = t^3(1 - a) \leq 0$, for all $t > 0$.

We conclude that $F \in \mathbf{F} \setminus \mathcal{G}_1$. So, this example shows that the class $\mathbf{F}$ contains strictly the class $\mathcal{G}_1$.

Example 2.5 Let $s = 1$ and consider $F(t_1, \dots, t_6) := t_2 - qt_1$, where $q \geq 1$.
(P_m): is not satisfied.
(P_1): $F(t, 0, 0, t, t, 0) = -qt < 0$, for all $t > 0$.
(P_2): $F(t, t, 0, 0, t, t) = t(1 - q) \leq 0$, for all $t > 0$.
We conclude that $F \in \mathbf{F} \setminus \mathcal{G}_1$. This example shows again that the class $\mathbf{F}$ contains strictly the class $\mathcal{G}_1$.

3 Preliminaries

The concept of a b-metric space was introduced by S. Czerwik (see [2] and [3]). We recall from [3] the following definition.

Definition 3.1 ([2]) Let X be a (nonempty) set and $s \geq 1$ a given real number. A function $d\colon X \times X \to \mathbb{R}_+$ (nonnegative real numbers) is called a b-metric provided that, for all $x, y, z \in X$,
(bm_1) $d(x, y) = 0$ iff $x = y$,
(bm_2) $d(x, y) = d(y, x)$,
(bm_3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.
The pair (X, d) is called a b-metric space with parameter s.

We remark that a metric space is evidently a b-metric space. However, S. Czerwik (see [2],[3]) has shown that a b-metric on X need not be a metric on X.

Let d be a b-metric with parameter s on a set X. As in the metric case, the b-metric d induces a topology. For every $r > 0$ and any arbitrary $x \in X$, we set $B(x, r) = \{y \in X : d(x, y) < r\}$. The topology $\mathcal{T}(d)$ on X associated with d is given by setting $U \in \mathcal{T}(d)$ if, and only if, for each $x \in U$, there exists some $r > 0$ such that $B(x, r) \subset U$. The space X will be equipped with the topology $\mathcal{T}(d)$. In particular a sequence $\{x_n\}$ converges to a point $x \in X$ if $\lim_{n\to\infty} d(x_n, x) = 0$. Almost all the concepts and results obtained for metric spaces can be extended to the case of b-metric spaces. For a large amount of results concerning b-metric spaces, the reader is invited to consult the papers [2] and [3].

As in the metric case, we introduce the definition of the property (E.A) for b-metrics.

Definition 3.2 Let S and T be two selfmappings of a b-metric space (X, d). We say that T and S satisfy the property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} d(Tx_n, t) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(Sx_n, t) = 0$$

for some $t \in X$.

Example 3.1 Let $X = [0, +\infty)$ and set $d(x, y) = |x - y|^2$, for all $x, y \in X$. It is easy to see that d is a b-metric with parameter $s = 2$. Also, it is easy to see that d is not a metric. Define $T, S\colon X \to X$ as follows

$$Tx = 3x + 1 \quad \text{and} \quad Sx = x + 3, \quad \text{for all } x \in X.$$

Consider the sequence $x_n = \frac{n+2}{n+1}$ for every nonnegative integer n. Clearly, we have

$$\lim_{n\to\infty} d(Tx_n, 4) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(Sx_n, 4) = 0.$$

Therefore, the self-mappings T and S satisfy the property $(E.A)$.

As in the metric case, we introduce the definition of compatible self-mappings in b-metric spaces.

Definition 3.3 Let (X, d) be a b-metric space and S and T be two self-mappings of X. We say that S and T are compatible if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} d(Sx_n, t) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(Tx_n, t) = 0,$$

for some $t \in X$.

Remark 3.1 Clearly, two self mappings of a b-metric space (X, d) are noncompatible if, and only if there exists at least a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} d(Sx_n, t) = 0$ and $\lim_{n\to\infty} d(Tx_n, t) = 0$, for some $t \in X$, but $\lim_{n\to\infty} d(STx_n, TSx_n)$ is either nonzero or does not exist. Therefore, two noncompatible self-mappings of a b-metric space (X, d) satisfy property (E.A).

We introduce the definition of weakly compatible self-mappings in b-metric spaces.

Definition 3.4 Two self-mappings S and T of a b-metric space (X, d) are said to be weakly compatible if $Tu = Su$, for $u \in X$, then $STu = TSu$.

4 A common fixed point theorem for expansive mappings

The first main result of this paper reads as follows.

Theorem 4.1 *Let (X, d) be a b-metric space with parameter s. Let S and T be two weakly compatible self-mappings of X such that:*

(1) S and T satisfy property (E.A),

(2) $G(d(Tx, Ty), d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)) > 0$ for each $(x, y) \in X^2$ such that $x \neq y$, where $G \in \mathcal{G}_s$.

(3) $T(X) \subset S(X)$.

If $S(X)$ or $T(X)$ is a closed subspace of X, then T and S have a unique common fixed point.

Proof Since T and S satisfy the property (E.A), there exists in X a sequence $\{x_n\}$ satisfying

$$\lim_{n\to\infty} d(Tx_n, t) = \lim_{n\to\infty} d(Sx_n, t) = 0,$$

for some $t \in X$. Therefore $t \in \overline{S(X)} \cap \overline{T(X)}$, where $\overline{A}$ designates the closure of A in (X, d) for any subset $A \subset X$.

Suppose $S(X)$ is closed in (X, d). Then $\lim_{n\to\infty} Sx_n = t = Sa$ for some $a \in X$. Also, $t = \lim_{n\to\infty} Tx_n = Sa$. To get a contradiction, suppose that $Sa \neq Ta$.

If $x_n = a$ for all nonnegative integer $n \geq N_0$ for some nonnegative integer N_0, then we have $Sa = Ta$.

Otherwise, there exists a subsequence $\{x_{\phi(n)}\}$ such that $x_{\phi(n)} \neq a$ for all nonnegative integer n. So, without loss of generality, we may suppose that $x_n \neq a$ for all nonnegative integer n. In this case, by using (2) for $x = x_n$ and $y = a$, we obtain that

$$G(d(Tx_n, Ta), d(Sx_n, Sa), d(Sx_n, Tx_n), d(Sa, Ta), d(Sx_n, Ta), d(Sa, Tx_n)) > 0.$$

Since $d(Ta, Tx_n) \leq s[d(Ta, Sa) + d(Sa, Tx_n)]$ and G is nondecreasing in the first variable, we get

$$\begin{aligned} G(s[d(Ta, Sa) + d(Sa, Tx_n)], d(Sx_n, Sa), d(Sx_n, Tx_n), \\ d(Sa, Ta), d(Sx_n, Ta), d(Sa, Tx_n)) > 0. \end{aligned}$$

Since d is a b-metric with parameter s, we have

$$d(Sa, Ta) - sd(Sa, Sx_n) \leq sd(Sx_n, Ta).$$

Since G is nonincreasing in the fifth variable, we get

$$\begin{aligned} G(s[d(Ta, Sa) + d(Sa, Tx_n)], d(Sx_n, Sa), d(Sx_n, Tx_n), \\ d(Sa, Ta), \tfrac{1}{s}d(Sa, Ta) - d(Sa, Sx_n), d(Sa, Tx_n)) > 0. \end{aligned}$$

For every nonnegative integer n, we have $d(Sx_n, Tx_n) \leq s(d(Sx_n, t)+d(t, Tx_n))$. Therefore, we get $\lim_{n\to\infty} d(Sx_n, Tx_n) = 0$.

So, by letting n tend to infinity and using the continuity of G, we obtain:

$$G(sd(Sa, Ta), 0, 0, d(Sa, Ta), \tfrac{1}{s}d(Sa, Ta), 0) \geq 0,$$

a contradiction of (P_1). Hence, $Sa = Ta$. That is a is point of coincidence.

We set $z := Ta$. We show that z is a common fixed point of T and S. Since S and T are weakly compatible, $STa = TSa$ and therefore

$$Tz = TSa = Sz = SSa.$$

By (2) for $x = a$ and $y = z$ we have successively:

$$G(d(Ta, Tz), d(Sa, Sz), d(Sa, Ta), d(Tz, Sz), d(Sa, Tz), d(z, Sz)) > 0.$$
$$G(d(z, Tz), d(z, Tz), 0, 0, d(z, Tz), d(z, Tz)) > 0,$$

a contradiction of (P_2) if $d(z, Tz) \neq 0$. Hence, $Tz = z$ and $Sz = Tz = z$. Therefore z is a common fixed point of S and T.

The proof is similar when we suppose that $T(X)$ is closed, since $T(X) \subset S(X)$.

Suppose that $Su = Tu = u$ and $Sv = Tv = v$ for $u \neq v$. Then, by (2) we have successively:

$$G(d(Tu,Tv), d(Su,Sv), d(Su,Tu), d(Sv,Tv), d(Su,Tv), d(Sv,Tu)) > 0,$$
$$G(d(u,v), d(u,v), 0, 0, d(u,v), d(u,v)) > 0,$$

a contradiction of (P_2) if $d(u,v) \neq 0$. Hence, $u = v$. □

Corollary 4.1 *Let $s \geq 1$ and d be a b-metric on a set X with parameter s. Let S and T two noncompatible and weakly compatible self-mappings of X such that:*

(1) $G(d(Tx,Ty), d(Sx,Sy), d(Tx,Sx), d(Sy,Ty), d(Sx,Ty), d(Sy,Tx)) > 0$, for each $(x,y) \in X^2$ such that $x \neq y$, where $G \in \mathcal{G}_s$.

(2) $T(X) \subset S(X)$.

If $S(X)$ or $T(X)$ is a closed subspace of X, then S and T have a unique common fixed point.

The proof of this corollary follows by Remark 3.1 and Theorem 4.1.

5 A related result

Let (X,d) be a b-metric space with parameter $s \geq 1$. The aim of this section is to show that if d is continuous on the topological space $X \times X$, or merely, satisfies a continuity of weak type, then we can use the class $\mathbf{F}$ of V. Popa [14] to establish a unique common fixed point theorem.

We need to introduce the following definition.

Definition 5.1 Let (X,d) be a b-metric space. We say that (X,d) satisfies the property (S_C) if for every sequence $\{x_n\}$ in X and all x, y in X, we have

$$\lim_{n\to\infty} d(x_n, x) = 0 \Longrightarrow \lim_{n\to\infty} d(x_n, y) = d(x,y).$$

Remark 5.1 Let $X = [0, +\infty)$ be endowed with the b-metric $d(x,y) = |x-y|^2$, for all $x, y \in X$. Then it is easy to see that the b-metric space (X,d) satisfies the property (S_C).

Every metric space (X,d) satisfies the property (S_C).

The second main result of this paper reads as follows.

Theorem 5.1 *Let (X,d) be a b-metric space with parameter s. We suppose that (X,d) satisfies the property (S_C). Let S and T be two weakly compatible self-mappings of X such that:*

(1) S and T satisfy property (E.A),

(2) $F(d(Tx,Ty),d(Sx,Sy),d(Sx,Tx),d(Sy,Ty),d(Sx,Ty),d(Sy,Tx)) > 0$ for each $(x,y) \in X^2$ such that $x \neq y$, where $F \in \mathbf{F}$.

(3) $T(X) \subset S(X)$.

If $S(X)$ or $T(X)$ is a closed subspace of X, then T and S have a unique common fixed point.

Proof Since T and S satisfy the property (E.A), there exists in X a sequence $\{x_n\}$ satisfying

$$\lim_{n\to\infty} d(Tx_n,t) = \lim_{n\to\infty} d(Sx_n,t) = 0,$$

for some $t \in X$. Therefore $t \in \overline{S(X)} \cap \overline{T(X)}$. We recall that $\overline{A}$ designates the closure of A in (X,d) for any subset $A \subset X$.

Suppose $S(X)$ is closed in (X,d). Then $\lim_{n\to\infty} Sx_n = t = Sa$ for some $a \in X$. Also, $t = \lim_{n\to\infty} Tx_n = Sa$. To get a contradiction, suppose that $Sa \neq Ta$.

If $x_n = a$ for all nonnegative integer $n \geq N_0$ for some nonnegative integer N_0, then we have $Sa = Ta$.

Otherwise, we can find a subsequence $\{x_{\phi(n)}\}$ such that $x_{\phi(n)} \neq a$ for all nonnegative integer n. So, without loss of generality, we may suppose that $x_n \neq a$ for all nonnegative integer n. In this case, by using (2) for $x = x_n$ and $y = a$, we obtain that

$$F(d(Tx_n,Ta),d(Sx_n,Sa),d(Sx_n,Tx_n),d(Sa,Ta),d(Sx_n,Ta),d(Sa,Tx_n)) > 0.$$

Since d is a b-metric with parameter s, we have

$$d(Sx_n,Tx_n) \leq s(d(Sx_n,t) + d(t,Tx_n)).$$

Therefore, we have $\lim_{n\to\infty} d(Sx_n,Tx_n) = 0$.

By letting n tend to infinity and using the property (S_C) and the continuity of F, we get:

$$F(d(Sa,Ta),0,0,d(Sa,Ta),d(Sa,Ta),0) \geq 0,$$

a contradiction of (F1). Hence, $Sa = Ta$. That is a is point of coincidence.

We set $z := Ta$. We show that z is a common fixed point of T and S. Since S and T are weakly compatible, $STa = TSa$ and therefore

$$Tz = TSa = Sz = SSa.$$

By (2) for $x = a$ and $y = z$ we have successively:

$$F(d(Ta,Tz),d(Sa,Sz),d(Sa,Ta),d(Tz,Sz),d(Sa,Tz),d(z,Sz)) > 0.$$
$$F(d(z,Tz),d(z,Tz),0,0,d(z,Tz),d(z,Tz)) > 0,$$

a contradiction of (F2) if $d(z,Tz) \neq 0$. Hence, $Tz = z$ and $Sz = Tz = z$. Therefore z is a common fixed point of S and T.

The proof is similar when we suppose that $T(X)$ is closed, since $T(X) \subset S(X)$.

Suppose that $Su = Tu = u$ and $Sv = Tv = v$ for $u \neq v$. Then, by (2) we have successively:

$$F(d(Tu, Tv), d(Su, Sv), d(Su, Tu), d(Sv, Tv), d(Su, Tv), d(Sv, Tu)) > 0,$$
$$F(d(u, v), d(u, v), 0, 0, d(u, v), d(u, v)) > 0,$$

a contradiction of (F2) if $d(u, v) \neq 0$. Hence, $u = v$. □

As an application to the metric case, we provide the following.

Corollary 5.1 *Let (X, d) be a metric space. Let S and T be two weakly compatible mappings of X. Suppose that there exists a mapping $\phi : X \to \mathbb{R}_+$ such that*

(1) $d(Sx, Tx) < \phi(Sx) - \phi(Tx)$, *for each* $x \in X$.

(2) $F(d(Tx, Ty), d(Sx, Sy), d(Tx, Sx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)) > 0$, *for each* $(x, y) \in X^2$ *such that* $x \neq y$, *where* $F \in \mathbf{F}$.

(3) $T(X) \subset S(X)$.

If $S(X)$ or $T(X)$ is a closed subspace of X, then T and S have a unique common fixed point.

Proof As in the proof of Corollary 2 from [1] it follows from the condition (1) above that S and T satisfy property (E.A). Therefore, all conditions of the Theorem 5.1 are satisfied. So the desired conclusions follow from this theorem immediately. □

Another consequence of Theorem 5.1 is the following.

Corollary 5.2 *Let (X, d) be a metric space. Let S and T be two weakly compatible self-mappings of X such that*

(1) T and S satisfy the property (E.A),

(2) $d(Tx, Ty) < \max\{d(Sx, Sy), [d(Tx, Sx) + d(Sy, Ty)]/2, [d(Sx, Ty) + d(Sy, Tx)]/2\}$, *for each* $(x, y) \in X^2$ *such that* $x \neq y$,

(3) $T(X) \subset S(X)$.

If $S(X)$ or $T(X)$ is a closed subspace of X, then T and S have a unique common fixed point.

Proof We set

$$F(t_1, \ldots, t_6) := \max\{t_2, [t_3 + t_4]/2, [t_5 + t_6]/2\} - t_1.$$

We have

(P_1): $F(t, 0, 0, t, t, 0) = -\frac{t}{2} < 0$, for all $t > 0$.

(P_2): $F(t, t, 0, 0, t, t) = 0$, for all $t > 0$.

We conclude that $F \in \mathbf{F}$. Therefore, all conditions of the Theorem 5.1 are satisfied. So the desired conclusions follow from this theorem immediately. $\square$

Theorem 1 of [1] follows immediately from Corollary 5.2. We provide an example to support Theorem 5.1.

Example 5.1 Let $X = [1, +\infty)$ be endowed with the b-metric d defined for all $x, y \in X$ by $d(x, y) = |x - y|^2$. d is a b-metric with parameter $s = 2$. It is easy to see that the b-metric space (X, d) satisfies the property (S_C). Also, it is easy to see that d is not a metric. Define $T, S\colon X \to X$ as follows

$$Tx = x \quad \text{and} \quad Sx = x^2, \quad \text{for all } x \in X.$$

Consider the sequence $x_n = \frac{n+2}{n+1}$ for every nonnegative integer n. Clearly, we have

$$\lim_{n\to\infty} d(Tx_n, 1) = 0 \quad \text{and} \quad \lim_{n\to\infty} d(Sx_n, 1) = 0.$$

Therefore, the self-mappings T and S satisfy the property $(E.A)$.

We have $S(X) = T(X) = X$ is closed. Furthermore, for all $x, y \geq 1$ with $x \neq y$, we have the following inequalities

$$d(Sx, Sy) = \mid x^2 - y^2 \mid^2 = (x + y)^2 \mid x - y \mid^2 \geq 4d(Tx, Ty) > 3d(Tx, Ty).$$

We set

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_2 - 3t_1.$$

We know (see Example 2.5) that $F \in \mathbf{F}$.

From the inequalities above, we have

$$F(d(Tx, Ty), d(Sx, Sy), d(Sx, Tx), d(Sy, Ty), d(Sx, Ty), d(Sy, Tx)) > 0,$$

for each $(x, y) \in X^2$ such that $x \neq y$.

So, all the conditions of Theorem 5.1 are satisfied. By applying Theorem 5.1, we see that S and T have a unique common fixed point (which is the point $x = 1$).

Theorem 1.1 cannot be applied, since for all $y = x \geq 1$, we have

$$F(d(Tx, Tx), d(Sx, Sx), d(Sx, Tx), d(Sx, Tx), d(Sx, Tx), d(Sx, Tx)) = 0.$$

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