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A Bound Sets Technique for Dirichlet Problem with an Upper-Carathéodory Right-hand Side*

Martina PAVLAČKOVÁ

Department of Mathematical Analysis and Applications of Mathematics
Faculty of Science, Palacký University
17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: martina.pavlackova@upol.cz

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Abstract

In this paper, the existence and the localization result will be proven for vector Dirichlet problem with an upper-Carathéodory right-hand side. The result will be obtained by combining the continuation principle with bound sets technique.

Key words: Dirichlet problem, upper-Carathéodory differential inclusions, bounding functions.

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1 Introduction

Given an upper-Carathéodory multivalued mapping $F\colon [0,T]\times\mathbb{R}^n\times\mathbb{R}^n \multimap \mathbb{R}^n$, the existence and the localization result for multivalued vector Dirichlet problem

$$\ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0,T], \tag{1}$$

$$x(T) = x(0) = 0 \tag{2}$$

will be proven in this paper.

By a *solution* of problem (1), (2) we shall mean a function $x\colon [0,T] \to \mathbb{R}^n$ with absolutely continuous first derivative satisfying (1), (2).

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Vector Dirichlet problems for differential equations or inclusions were studied by many authors (see, e.g., [4, 6, 8, 9, 10, 11, 12, 13, 14]). In mentioned papers, various methods (like an upper and lower solutions technique, method of shift along trajectories or tube solution method) were applied for obtaining the existence results. In this paper, not only the existence but also the localization results are obtained by means of bound sets technique. This method was introduced in the single-valued case by Gaines and Mawhin in [7] for obtaining the existence of solutions of the first and the second-order differential equations. Bound sets technique was recently, among others, applied for multivalued Dirichlet problem with globally upper semi-continuous right-hand side (shortly, r.h.s.) in [10].

In this paper, we employ the bound sets technique for the Dirichlet problem (1), (2) in the more general case when the r.h.s. is an upper-Carathéodory multivalued mapping. The existence and localization result (cf. Theorem 4.1 below) is obtained by combining the bound sets approach with the continuation principle developed in [2].

The paper is organized as follows. In the second section, suitable definitions and statements which will be used in the sequel are recalled. Section 3 is devoted to studying of bound sets and Liapunov-like bounding functions for Dirichlet problems. At first, the C^1-bounding functions with locally Lipschitzian gradients are considered. Consequently, it is shown how conditions ensuring the existence of bound set change in case of C^2-bounding functions. In Section 4, the bound sets approach is combined with the continuation principle (developed in [2]) and the existence and localization result is obtained in this way for the Dirichlet problem (1), (2). Finally, an illustrating example is also supplied.

2 Preliminaries

Let us start with notations we use in the paper. If (X, d) is a metric space and $A \subset X$, by $\overline{A}$, $\operatorname{Int} A$ and ∂A, we mean the *closure*, the *interior* and the *boundary* of A, respectively. For a subset $A \subset X$ and $\varepsilon > 0$, we define the set $N_\varepsilon(A) := \{x \in X \mid \exists a \in A \colon d(x,a) < \varepsilon\}$, i.e. $N_\varepsilon(A)$ is an open neighborhood of the set A in X.

For a given compact real interval J, we denote by $C(J, \mathbb{R}^n)$ (by $C^1(J, \mathbb{R}^n)$) the set of all functions $x \colon J \to \mathbb{R}^n$ which are continuous (have continuous first derivatives) on J. By $\mathrm{AC}^1(J, \mathbb{R}^n)$, we shall mean the set of all functions $x \colon J \to \mathbb{R}^n$ with absolutely continuous first derivatives on J.

We also need following definitions and notions from multivalued theory in the sequel. Let Y be a metric space. We say that F is a *multivalued mapping* from X to Y (written $F \colon X \multimap Y$) if, for every $x \in X$, a nonempty subset $F(x)$ of Y is given. We associate with F its graph Γ_F, the subset of $X \times Y$, defined by

$$\Gamma_F := \{(x,y) \in X \times Y \mid y \in F(x)\}.$$

A multivalued mapping $F \colon X \multimap Y$ is called *upper semi-continuous* (shortly, u.s.c.) if, for each open set $U \subset Y$, the set $\{x \in X \mid F(x) \subset U\}$ is open in X.

Let Y be a metric space and $(\Omega, \mathcal{U}, \mu)$ be a *measurable space,* i.e. a nonempty set Ω equipped with a suitable σ-algebra $\mathcal{U}$ of its subsets and a countably additive measure μ on $\mathcal{U}$. A multivalued mapping $F\colon \Omega \multimap Y$ is called *measurable* if $\{\omega \in \Omega \mid F(\omega) \subset V\} \in \mathcal{U}$, for each open set $V \subset Y$.

We say that mapping $F\colon J \times \mathbb{R}^m \multimap \mathbb{R}^n$, where $J \subset \mathbb{R}$ is a compact interval, is an *upper-Carathéodory mapping* if the map $F(\cdot, x)\colon J \multimap \mathbb{R}^n$ is measurable, for all $x \in \mathbb{R}^m$, the map $F(t, \cdot)\colon \mathbb{R}^m \multimap \mathbb{R}^n$ is u.s.c., for almost all $t \in J$, and the set $F(t,x)$ is compact and convex, for all $(t,x) \in J \times \mathbb{R}^m$.

Let $X \cap Y \neq \emptyset$ and $F\colon X \multimap Y$. We say that a point $x \in X \cap Y$ is a *fixed point* of F if $x \in F(x)$. The set of all fixed points of F is denoted by $\mathrm{Fix}(F)$, i.e.

$$\mathrm{Fix}(F) := \{x \in X \mid x \in F(x)\}.$$

We employ the following selection result in the sequel.

Proposition 2.1 (cf., e.g., [3]) *Let $J \subset \mathbb{R}$ be a compact interval and $F\colon J \times \mathbb{R}^m \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping satisfying $|y| \leq r(t)(1 + |x|)$, for every $(t,x) \in J \times \mathbb{R}^m$, and every $y \in F(t,x)$, where $r\colon J \to [0,\infty)$ is an integrable function. Then the composition $F(t, q(t))$ admits, for every $q \in C(J, \mathbb{R}^m)$, a single-valued measurable selection.*

In the sequel, we will also need the following slight modification of the continuation principle developed in [2] for problems on arbitrary, possibly non-compact, intervals. The difference between the presented result and the one in [2] consists in replacement of the non-compact interval by the compact one which simplify the last, so called transversality, condition.

Proposition 2.2 (cf. [2, Theorem 3.1. and Remark 2.2.]) *Let us consider the b.v.p.*

$$\left.\begin{array}{r} \ddot{x}(t) \in F(t, x(t), \dot{x}(t)), \quad \text{for a.a. } t \in [0,T], \\ x \in S, \end{array}\right\} \tag{3}$$

where $F\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping and S is a subset of $\mathrm{AC}^1([0,T], \mathbb{R}^n)$. Let $H\colon [0,T] \times \mathbb{R}^{4n} \times [0,1] \multimap \mathbb{R}^n$ be an upper-Carathéodory mapping such that

$$H(t,c,d,c,d,1) \subset F(t,c,d), \quad \text{for all } (t,c,d) \in [0,T] \times \mathbb{R}^{2n}. \tag{4}$$

Assume that

(*i*) *there exists a retract Q of $C^1([0,T], \mathbb{R}^n)$, with $Q \setminus \partial Q \neq \emptyset$, and a closed subset S_1 of S such that the associated problem*

$$\left.\begin{array}{r} \ddot{x}(t) \in H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda), \quad \text{for a.a. } t \in [0,T], \\ x \in S_1 \end{array}\right\} \tag{5}$$

has, for each $(q, \lambda) \in Q \times [0,1]$, a non-empty, convex set of solutions, (*ii*) *there exists a nonnegative, integrable function* $\alpha\colon [0,T] \to \mathbb{R}$ *such that*

$$|H(t, x(t), \dot{x}(t), q(t), \dot{q}(t), \lambda)| \le \alpha(t)(1 + |x(t)| + |\dot{x}(t)|), \quad a.e.\ in\ [0,T],$$

for any $(q, \lambda, x) \in \Gamma_{\mathfrak{T}}$, *where* $\mathfrak{T}$ *denotes the multivalued map which assigns to any* $(q, \lambda) \in Q \times [0,1]$ *the set of solutions of* (5),

(*iii*) $\mathfrak{T}(Q \times \{0\}) \subset Q$,

(*iv*) *there exist a point* $t_0 \in [0,T]$ *and constants* $M_0 \ge 0,\ M_1 \ge 0$ *such that* $|x(t_0)| \le M_0$ *and* $|\dot{x}(t_0)| \le M_1$, *for any* $x \in \mathfrak{T}(Q \times [0,1])$,

(*v*) *the solution map* $\mathfrak{T}$ *has no fixed points on the boundary* ∂Q *of* Q, *for every* $(q, \lambda) \in Q \times [0,1]$.

Then the b.v.p. (3) has a solution in $S_1 \cap Q$.

3 Bound sets theory for Dirichlet problem

The direct verification of transversality condition (v) in Proposition 2.2 is quite complicated. Therefore, a Liapunov-like function V, usually called a *bounding function*, which can guarantee this condition will be introduced now.

Hence, let $K \subset \mathbb{R}^n$ be a nonempty, open set with $0 \in K$ and let $V\colon \mathbb{R}^n \to \mathbb{R}$ be a continuous function satisfying

(H1) $V|_{\partial K} = 0$,

(H2) $V(x) \le 0$, for all $x \in \overline{K}$.

Definition 3.1 The set K is called a *bound set for the Dirichlet problem* (1), (2) if every solution x of problem (1), (2) such that $x(t) \in \overline{K}$, for each $t \in [0,T]$, does not satisfy $x(t^*) \in \partial K$, for any $t^* \in [0,T]$.

Remark 3.1 Let us note that the existence of a bound set K for the Dirichlet problem (1), (2) does not guarantee the existence of a solution of problem (1), (2). It only ensures that if there would exist a solution lying in $\overline{K}$, then this solution did not touch the boundary of K at any point, i.e. it lies in $\operatorname{Int} K$.

At first, the sufficient conditions for the existence of a bound set for the Dirichlet problem (1), (2) in the general case will be shown in Proposition 3.1 below. Afterwards, the regularity assumptions on the bounding function V will be made more strict and the practically applicable version of Proposition 3.1 will be obtained (see Corollary 3.1 below).

Proposition 3.1 *Let* $K \subset \mathbb{R}^n$ *be a nonempty open set with* $0 \in K$ *and* $F\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *be an upper-Carathéodory multivalued mapping. Assume that there exists a function* $V \in C^1(\mathbb{R}^n, \mathbb{R})$ *with* ∇V *locally Lipschitzian and satisfying conditions (H1) and (H2). Suppose, moreover, that there exists* $\varepsilon > 0$

such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$ and $v \in \mathbb{R}^n$, at least one of the following conditions

$$\limsup_{h\to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0 \tag{6}$$

$$\limsup_{h\to 0^+} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0 \tag{7}$$

holds, for all $w \in F(t,x,v)$. Then K is a bound set for the Dirichlet problem (1), (2).

Proof We assume, by a contradiction, that K is not a bound set for the Dirichlet problem (1), (2), i.e. that there exist a solution $x\colon [0,T] \to \overline{K}$ of (1), (2) and $t^* \in [0,T]$ such that $x(t^*) \in \partial K$. The point t^* must lie in $(0,T)$, according to the boundary condition (2) and the fact that $0 \in K$.

Since ∇V is locally Lipschitzian, there exist a bounded open set $U \subset \mathbb{R}^n$ with $x(t^*) \in U$ and a constant $L > 0$ such that $\nabla V|_U$ is Lipschitzian with constant L. Let $\delta > 0$ be such that $x(t) \in U \cap N_\varepsilon(\partial K)$, for each $t \in [t^* - \delta, t^* + \delta]$.

Let us define the function $g\colon [0,T] \to \mathbb{R}$ by the formula $g(t) := V(x(t))$. According to the regularity properties of x and V, $g \in C^1([0,T],\mathbb{R})$. Since $g(t^*) = 0$ and $g(t) \le 0$, for all $t \in [0,T]$, the point t^* is a local maximum point for g and $\dot g(t^*) = 0$. Moreover, there exist points $t^{**} \in (t^* - \delta, t^*)$, $t^{***} \in (t^*, t^* + \delta)$ such that $\dot g(t^{**}) \ge 0$ and $\dot g(t^{***}) \le 0$.

Since $\dot g(t) = \langle \nabla V(x(t)), \dot x(t)\rangle$, where $\nabla V(x(t))$ is locally Lipschitzian and $\dot x(t)$ is absolutely continuous on $[t^*-\delta, t^*+\delta]$, $\ddot g(t)$ exists, for a.a. $t \in [t^*-\delta, t^*+\delta]$. Consequently,

$$0 \ge -\dot g(t^{**}) = \dot g(t^*) - \dot g(t^{**}) = \int_{t^{**}}^{t^*} \ddot g(s)\, ds \tag{8}$$

and

$$0 \ge \dot g(t^{***}) = \dot g(t^{***}) - \dot g(t^*) = \int_{t^*}^{t^{***}} \ddot g(s)\, ds. \tag{9}$$

At first, let us assume that condition (6) holds and let $t \in (t^{**}, t^*)$ be such that $\ddot g(t)$ and $\ddot x(t)$ exist. Then,

$$\lim_{h\to 0} \frac{\dot x(t+h) - \dot x(t)}{h} = \ddot x(t),$$

and therefore there exists a function $a(h)$, $a(h) \to 0$ as $h \to 0$, such that, for each h,

$$\dot x(t+h) = \dot x(t) + h[\ddot x(t) + a(h)]. \tag{10}$$

Moreover, since $x \in C^1([0,T],\mathbb{R}^n)$, there exists a function $b(h)$, $b(h) \to 0$ as $h \to 0$, such that, for each h,

$$x(t+h) = x(t) + h[\dot x(t) + b(h)]. \tag{11}$$

Consequently, we obtain

$$\ddot{g}(t) = \lim_{h\to 0} \frac{\dot{g}(t+h) - \dot{g}(t)}{h} = \limsup_{h\to 0^-} \frac{\dot{g}(t+h) - \dot{g}(t)}{h}$$

$$= \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t+h)), \dot{x}(t+h)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h}$$

$$= \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t) + h[\dot{x}(t) + b(h)]), \dot{x}(t) + h[\ddot{x}(t) + a(h)]\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h}$$

$$\geq \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h[\ddot{x}(t) + a(h)]\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h}$$

$$-L \cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t) + a(h)]|$$

$$= \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h\ddot{x}(t)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h}$$

$$-L \cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t) + a(h)]| + \langle \nabla V(x(t) + h\dot{x}(t)), a(h)\rangle .$$

Since

$$\langle \nabla V(x(t) + h\dot{x}(t)), a(h)\rangle - L \cdot |b(h)| \cdot |\dot{x}(t) + h[\ddot{x}(t) + a(h)]| \to 0 \text{ as } h \to 0$$

and since assumption (6) holds,

$$\ddot{g}(t) \geq \limsup_{h\to 0^-} \frac{\langle \nabla V(x(t) + h\dot{x}(t)), \dot{x}(t) + h\ddot{x}(t)\rangle - \langle \nabla V(x(t)), \dot{x}(t)\rangle}{h} > 0,$$

which leads to a contradiction with the inequality (8).

Secondly, let us assume that condition (7) holds and let $s \in (t^*, t^{***})$ be such that $\ddot{g}(s)$ and $\ddot{x}(s)$ exist. Then it is possible to show, using the same procedure as before, that, according to assumption (7),

$$\ddot{g}(s) \geq \limsup_{h\to 0^+} \frac{\langle \nabla V(x(s) + h\dot{x}(s)), \dot{x}(s) + h\ddot{x}(s)\rangle - \langle \nabla V(x(s)), \dot{x}(s)\rangle}{h} > 0,$$

which leads to a contradiction with the inequality (9).

Therefore, we get the contradiction in case that at least one of conditions (6), (7) hold which completes the proof. □

Definition 3.2 A function $V\colon \mathbb{R}^n \to \mathbb{R}$ from Proposition 3.1 satisfying conditions (H1), (H2) and at least one of conditions (6), (7) is called a *bounding function* for the set K relative to (1), (2).

Remark 3.2 It is obvious from the proof of Proposition 3.1 that the element $v \in \mathbb{R}^n$ in (6), (7) plays the role of solution derivative. Thus, if there would exist an integrable function $\alpha\colon [0,T] \to \mathbb{R}$ such that $|F(t, x(t), \dot{x}(t))| \leq \alpha(t)$, for a.a. $t \in [0,T]$ and all solutions $x(\cdot)$ of (1), (2), then conditions (6), (7) could be weaken.

In fact, in such a case, it holds for an arbitrary solution $x(\cdot)$ of (1), (2) that $|\dot{x}(0)| \leq \int_0^T \alpha(t)\,dt$ (see part ad (iv) in the proof of Theorem 4.1 below), and so, for a.a. $t \in [0,T]$,

$$|\dot{x}(t) - \dot{x}(0)| \leq \int_0^t |\ddot{x}(s)|\,ds \leq \int_0^t \alpha(s)\,ds.$$

Consequently,

$$|\dot{x}(t)| \leq |\dot{x}(0)| + \int_0^t \alpha(s)\,ds \leq 2\int_0^T \alpha(t)\,dt.$$

Therefore, if $|F(t,x(t),\dot{x}(t))| \leq \alpha(t)$, for a.a. $t \in [0,T]$ and all solutions $x(\cdot)$ of (1), (2), conditions (6), (7) change as follows:

There exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$ and $v \in \mathbb{R}^n$ with $|v| \leq 2\int_0^T \alpha(t)\,dt$, at least one of the following condition

$$\limsup_{h\to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0 \tag{12}$$

$$\limsup_{h\to 0^+} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0 \tag{13}$$

holds, for all $w \in F(t,x,v)$.

When the bounding function V is of class C^2, both conditions (6) and (7) can be rewritten in the same way in terms of gradients and Hessian matrices.

Corollary 3.1 *Let $K \subset \mathbb{R}^n$ be a nonempty open set with $0 \in K$ and $F\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ be an upper-Carathéodory multivalued mapping. Assume that there exists a function $V \in C^2(\mathbb{R}^n,\mathbb{R})$ satisfying conditions (H1) and (H2). Moreover, assume that there exists $\varepsilon > 0$ such that, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$ and $v \in \mathbb{R}^n$, condition*

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle > 0 \tag{14}$$

holds, for all $w \in F(t,x,v)$. Then K is a bound set for the Dirichlet problem (1), (2).

Proof The statement of Corollary 3.1 follows immediately from the fact that if $V \in C^2(\mathbb{R}^n,\mathbb{R})$, then, for all $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$, $v \in \mathbb{R}^n$ and $w \in F(t,x,v)$,

$$\begin{aligned} &\limsup_{h\to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} \\ &= \limsup_{h\to 0^+} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} \\ &= \langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle. \end{aligned}$$

□

Remark 3.3 The element $v \in \mathbb{R}^n$ in (14) plays again the role of the solution derivative. Therefore, if the mapping F satisfies the condition specified in Remark 3.2, it is sufficient to require condition (14) in Corollary 3.1 only for all $v \in \mathbb{R}^n$ with $|v| \le 2\int_0^T \alpha(t)\,dt$, and not for all $v \in \mathbb{R}^n$.

4 The existence and localization result for Dirichlet problem

In this section, we investigate the Dirichlet problem (1), (2) by combining the continuation principle from Proposition 2.2 with bound sets results developed in the previous section.

For this purpose, let us specify the general problem (3) as the Dirichlet problem (1), (2). Moreover, let $K \subset \mathbb{R}^n$ be a nonempty, open, bounded set with $0 \in K$. If we define the set Q of candidate solutions from Proposition 2.2 by formula

$$Q := \{q \in C^1([0,T],\mathbb{R}^n) \mid q(t) \in \overline{K}, \text{ for all } t \in [0,T]\} \tag{15}$$

and the associated problem (5), for each $(q,\lambda) \in Q \times [0,1]$, as follows

$$\left.\begin{array}{c} \ddot{x}(t) \in \lambda F(t,q(t),\dot{q}(t)), \text{ for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \end{array}\right\} \tag{16}$$

then we will be able to clearly verify all conditions in Proposition 2.2.

Theorem 4.1 *Let us consider the Dirichlet problem (1), (2), where* $F\colon [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *is an upper-Carathéodory multivalued mapping. Moreover, assume that*

(i) *the closure* $\overline{K}$ *of the set* K *is a retract of* $\mathbb{R}^n$,

(ii) *there exists a nonnegative, integrable function* $\alpha\colon [0,T] \to \mathbb{R}$ *such that*

$$|F(t,q(t),\dot{q}(t))| \le \alpha(t), \text{ a.e. in } [0,T],$$

for each $q \in Q$, *where* Q *is defined by formula (15),*

(iii) *there exists a function* $V \in C^1(\mathbb{R}^n,\mathbb{R})$ *with* ∇V *locally Lipschitzian and satisfying conditions (H1) and (H2),*

(iv) *there exists* $\varepsilon > 0$ *such that, for all* $x \in \overline{K} \cap N_\varepsilon(\partial K)$, $t \in (0,T)$, $\lambda \in (0,1)$ *and* $v \in \mathbb{R}^n$ *with* $|v| \le 2\int_0^T \alpha(t)\,dt$, *at least one of the following conditions*

$$\limsup_{h \to 0^-} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0 \tag{17}$$

$$\limsup_{h \to 0^+} \frac{\langle \nabla V(x+hv), v+hw\rangle - \langle \nabla V(x), v\rangle}{h} > 0 \tag{18}$$

holds, for all $w \in \lambda F(t,x,v)$.

Then the Dirichlet problem (1), (2) has a solution $x(\cdot)$ such that $x(t) \in \overline{K}$, for all $t \in [0,T]$.

Proof We will check that all the assumptions of Proposition 2.2 are satisfied.

ad (i) Since $\overline{K}$ is, according to assumption (i), a retract of $\mathbb{R}^n$, there exists a continuous function $\phi\colon \mathbb{R}^n \to \overline{K}$ satisfying $\phi(x) = x$, for each $x \in \overline{K}$. Let us define a function $\tilde{\phi}\colon C^1([0,T],\mathbb{R}^n) \to Q$ in the following way: for each $x \in C^1([0,T],\mathbb{R}^n)$, $\tilde{\phi}(x) = \tilde{x}$, where $\tilde{x}\colon [0,T] \to \overline{K}$ satisfies $\tilde{x}(t) = \phi(x(t))$, for each $t \in [0,T]$. It follows from the definition of Q and the properties of ϕ that $\tilde{\phi}(q) = q$, for each $q \in Q$, and that $\tilde{\phi}$ is continuous. Therefore, Q is a retract of $C^1([0,T],\mathbb{R}^n)$, as required.

In order to prove that all associated problems have desired topological structure, let us specify, for each $(q,\lambda) \in Q \times [0,1]$, the associated problem (5) as the fully linearized problem (16). The homogeneous problem corresponding to b.v.p. (16),

$$\left.\begin{array}{r} \ddot{x}(t) = 0, \text{ for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \end{array}\right\} \tag{19}$$

has only the trivial solution. Moreover, for each $(q,\lambda) \in Q \times [0,1]$, there exists at least one solution $x(\cdot)$ of (16) given, for a.a. $t \in [0,T]$, by $x(t) = \int_0^T G(t,s) f_{q,\lambda}(s)\,ds$, where G is the Green function associated to the homogeneous problem (19) and $f_{q,\lambda}(\cdot)$ is a measurable selection of $\lambda F(\cdot, q(\cdot), \dot{q}(\cdot))$.* Thus, for each $(q,\lambda) \in Q \times [0,1]$, the set of solutions of (16) is nonempty.

The set of solutions of (16) is, for each $(q,\lambda) \in Q \times [0,1]$. also convex (cf., e.g., the proof of Theorem 4.1 in [2], for $A(t) = B(t) \equiv 0$, $I = [0,T]$ and the closed, convex set $S_1 = \{x(\cdot) \in AC^1([0,T],\mathbb{R}^n) \mid x(0) = x(T) = 0\}$).

Furthermore, since $Q \setminus \partial Q$ is nonempty, condition (i) from Proposition 2.2 holds.

ad (ii) Condition (ii) in Proposition 2.2 is directly guaranteed by assumption (ii).

ad (iii) The fulfillment of condition (iii) in Proposition 2.2 follows immediately from the fact that, for $\lambda = 0$, all associated problems (for an arbitrary $q \in Q$) transform into the b.v.p. (19) which has only the trivial solution. Therefore, since $0 \in K$ and Q is defined by formula (15), assumption (iii) in Proposition 2.2 holds as well.

ad (iv) Let $x(\cdot)$ be a solution of the b.v.p. (16), for some $(q,\lambda) \in Q \times [0,1]$. From boundary condition $x(0) = 0$, it follows that $|x(0)| = 0$. Moreover, since also $x(T) = 0$, there exists a point $\xi \in (0,T)$ such that $\dot{x}(\xi) = 0$. Therefore,

$$|\dot{x}(0)| = |\dot{x}(\xi) - \dot{x}(0)| = \left|\int_0^\xi \ddot{x}(t)\,dt\right| \le \int_0^\xi |\ddot{x}(t)|\; dt \le \int_0^\xi \alpha(t)\; dt \le \int_0^T \alpha(t)\,dt.$$

Since $\alpha(\cdot)$ is integrable, there exists a constant M such that $\int_0^T \alpha(t)\,dt \le M$, and therefore, $|\dot{x}(0)| \le M$. Hence, condition (iv) in Proposition 2.2 is satisfied with $t_0 = 0$, $M_0 = 0$ and $M_1 = M$.

*The existence of the measurable selection $f_{q,\lambda}(\cdot)$ is guaranteed by Proposition 2.1.

ad (v) Let us assume that $q_* \in Q$ is, for some $\lambda \in [0,1]$, a solution of

$$\left.\begin{array}{c} \ddot{q}_*(t) \in \lambda F(t, q_*(t), \dot{q}_*(t)), \text{ for a.a. } t \in [0,T], \\ x(T) = x(0) = 0, \end{array}\right\} \tag{20}$$

i.e. a fixed point of solution mapping $\mathfrak{T}$ defined in Proposition 2.2.

At first, let $\lambda = 1$. If q_* is a fixed point of $\mathfrak{T}(\cdot, 1)$, the original problem (1), (2) has a solution in Q, and we are done.

Secondly, let us investigate the case when $\lambda = 0$. Then problem (20) transform into the b.v.p. (19) which has only the trivial solution. Therefore, for $\lambda = 0$, it holds that $q_* \equiv 0$ which lies in $Int\ Q$. Hence, if $\lambda = 0$, condition (v) in Proposition 2.2 is satisfied.

Finally, let us assume that $\lambda \in (0,1)$. It was shown in [10] that if $q_*(t) \in K$, for all $t \in [0,T]$, then $q_* \in Int\ Q$. Thus, if $q_* \colon [0,T] \to \overline{K}$ would lie in ∂Q, there must exist a point $t_* \in (0,T)$ such that $q_*(t_*) \in \partial K$. But then q_* can not be a solution of the Dirichlet problem (20), for any $\lambda \in (0,1)$, since hypotheses (i), (iii) and (iv) guarantee that K is a bound set for the Dirichlet problem (20), for all $\lambda \in (0,1]$.

Therefore, condition (v) from Proposition 2.2 is satisfied, for all $\lambda \in [0,1)$, which completes the proof. □

Example 4.1 As an illustrating example, let us consider the single-valued scalar Dirichlet problem with discontinuous right-hand side

$$\left.\begin{array}{r} \ddot{x} = 5 \cdot \operatorname{sgn}(x(t)) + \sin(\dot{x}(t) \cdot x(t)) \cdot \operatorname{sgn}\left(t - \frac{1}{2}\right) + \ln(x(t)^2 + 2), \\ \text{for a.a. } t \in [0,1], \\ x(1) = x(0) = 0. \end{array}\right\} \tag{21}$$

Because of discontinuity in $\operatorname{sgn}(x(t))$, we can only consider Filippov solutions of (21) which can be identified (see, e.g., [1, 5]) as Carathéodory solutions of the following b.v.p. with upper-Carathéodory r.h.s.

$$\left.\begin{array}{r} \ddot{x} \in 5 \cdot \operatorname{Sgn}(x(t)) + \sin(\dot{x}(t) \cdot x(t)) \cdot \operatorname{sgn}\left(t - \frac{1}{2}\right) + \ln(x(t)^2 + 2), \\ \text{for a.a. } t \in [0,1], \\ x(1) = x(0) = 0, \end{array}\right\} \tag{22}$$

where

$$\operatorname{Sgn} y := \begin{cases} -1, & \text{for } y < 0, \\ [-1,1], & \text{for } y = 0, \\ 1, & \text{for } y > 0. \end{cases}$$

In order to apply Theorem 4.1, let us define sets K and Q in the following way:

$$K := (-6, 6) \tag{23}$$

and

$$Q := \left\{ x \in C^1([0,1], \mathbb{R}) \mid x(t) \in \overline{K}, \text{ for all } t \in [0,1] \right\}.$$

Moreover, let us consider, for each $(q, \lambda) \in Q \times [0,1]$, the fully linearized associated problem

$$\left.\begin{array}{r} \ddot{x} \in \lambda \left(5 \cdot \mathrm{Sgn}(q(t)) + \sin(\dot{q}(t) \cdot q(t)) \cdot \mathrm{sgn}\left(t - \frac{1}{2}\right) + \ln(q(t)^2 + 2)\right), \\ \text{a.e. in } [0,1], \\ x(1) = x(0) = 0. \end{array}\right\} \quad (24)$$

Let us now verify particular assumptions of Theorem 4.1.

ad (i) The set K defined by formula (23) is a nonempty, open and bounded subset of $\mathbb{R}$ with $0 \in K$. Moreover, since $\overline{K}$ is a convex set, it is a retract of $\mathbb{R}$, and so assumption (i) is valid.

ad (ii) The fulfilment of hypothesis (ii) follows immediately from the form of right-hand side in (24) and the definition of Q.

ad (iii) Let us define the function V by formula

$$V(x) = \frac{1}{2}\left(x^2 - 36\right).$$

Since $V(x) = 0$, for all $x \in \partial K$, and $V(x) \leq 0$, for all $x \in \overline{K}$, the function V satisfies conditions (H1) and (H2). Moreover, for each $x \in \partial K$, $\nabla V(x) = x$ and $HV(x) = 1$. Furthermore, since $V \in C^2(\mathbb{R}, \mathbb{R})$, condition (iii) holds as well.

ad (iv) Let us set $\varepsilon := 1$. Then, for all $x \in [-6,-5) \cup (5,6]$, $t \in (0,1)$, $\lambda \in (0,1]$, $v \in \mathbb{R}$ and

$$w \in \lambda \left(5 \cdot \mathrm{Sgn}(x) + \sin(v \cdot x) \cdot \mathrm{sgn}\left(t - \frac{1}{2}\right) + \ln(x^2 + 2)\right),$$

the following condition

$$\langle HV(x)v, v\rangle + \langle \nabla V(x), w\rangle = v \cdot v + x \cdot w \geq \lambda(5 \cdot |x| - |x| - |x| \cdot \ln(38)) > 0 \quad (25)$$

holds. Thus, assumption (iv) of Theorem 4.1 is satisfied.

Therefore, all assumptions of Theorem 4.1 are satisfied, and hence, the Dirichlet problem (22) admits a solution $x(\cdot)$ such that $|x(t)| < 6$, for all $t \in [0,1]$. This solution represents a Filippov solution of the original problem (21).

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