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Linear Error Propagation Law and Nonlinear Functions

Lubomír KUBÁČEK [1], Eva TESAŘÍKOVÁ [2]

[1] *Department of Mathematical Analysis and Applications of Mathematics
Faculty of Science, Palacký University
17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: kubacekl@inf.upol.cz*

[2] *Department of Algebra and Geometry, Faculty of Science, Palacký University
17. listopadu 12, 771 46 Olomouc, Czech Republic
e-mail: tesariko@inf.upol.cz*

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Abstract

Linear error propagation law (LEPL) has been using frequently also for nonlinear functions. It can be adequate for an actual situation however it need not be so. It is useful to use some rule in order to recognize whether LEPL is admissible. The aim of the paper is to find such rule.

Key words: Linear error propagation law, bias, nonlinear function.

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1 Introduction

Error propagation law, mainly in its linear version (LEPL) is frequently used in practice; sometimes in serious situations. For example let us mention measurement of temperature in nuclear power station.

Effectiveness of the power station is growing with the growing temperature of the reactor. However it cannot overcome some value, since a terrible disaster can occur. The temperature is estimated as a nonlinear function of several measured quantities and the variance of estimated temperature is a starting point for a control of the temperature of the reactor. Can be used LEPL in this situation?

A practical rule for the utization of LEPL is given in the paper.

2 Notation

Let $\mathbf{f}(\cdot)\colon R^n \to R^s$, where R^n is n-dimensional linear vector space and $\mathbf{f}(\cdot)$ be a known vector function, which can be developed in the Taylor series. Let $\boldsymbol{\xi} \sim_n (\boldsymbol{\mu}, \boldsymbol{\Sigma})$ be an n-dimensional random vector with the mean value $E(\boldsymbol{\xi}) = \boldsymbol{\mu}$ and with the covariance matrix $\mathrm{Var}(\boldsymbol{\xi}) = \boldsymbol{\Sigma}$.

In the following text it is assumed that the support of the probability measure of the vector $\boldsymbol{\xi}$ is imbedded into the sphere with the radius of convergence of the function $\mathbf{f}(\cdot)$ (in more detail cf. [3]), or at least that the supporrt can be trimmed in such a way that it is imbedded into the convergence sphere and at the same time the needed statistical moments of the trimmed distribution differ only non-essentially from the original moments.

The notation

$$\boldsymbol{\varepsilon} = \boldsymbol{\xi} - \boldsymbol{\mu} = (\varepsilon_1, \dots, \varepsilon_n)'$$

is used in the following text. The notation

$$\left(\frac{\partial}{\partial \mu_1}\varepsilon_1 + \dots + \frac{\partial}{\partial \mu_n}\varepsilon_n\right) f_i(\boldsymbol{\mu}), \quad i = 1, \dots, s,$$

means

$$\frac{\partial f_i(\boldsymbol{\mu})}{\partial \mu_1}\varepsilon_1 + \dots + \frac{\partial f_i(\boldsymbol{\mu})}{\partial \mu_n}\varepsilon_n, \quad i = 1, \dots, s.$$

Analogously

$$\left(\frac{\partial}{\partial \mu_1}\varepsilon_1 + \dots + \frac{\partial}{\partial \mu_n}\varepsilon_n\right)^2 f_i(\boldsymbol{\mu}) = \sum_{j=1}^{n} \frac{\partial^2 f_i(\boldsymbol{\mu})}{\partial \mu_j^2}\varepsilon_j^2 + 2\sum_{j\neq k}^{n} \frac{\partial^2 f_i(\boldsymbol{\mu})}{\partial \mu_j \partial \mu_k}\varepsilon_j\varepsilon_k,$$

$$\left(\frac{\partial}{\partial \mu_1}\varepsilon_1 + \dots + \frac{\partial}{\partial \mu_n}\varepsilon_n\right)^3 f_i(\boldsymbol{\mu})$$

$$= \sum_{j=1}^{n} \frac{\partial^3 f_i(\boldsymbol{\mu})}{\partial \mu_j^3}\varepsilon_j^3 + \frac{3!}{2!1!}\sum_{j\neq k}^{n} \frac{\partial^3 f_i(\boldsymbol{\mu})}{\partial \mu_j \partial \mu_k^2}\varepsilon_j\varepsilon_k^2 + \frac{3!}{1!1!1!}\sum_{j\neq k\neq l}^{n} \frac{\partial^3 f_i(\boldsymbol{\mu})}{\partial \mu_j \partial \mu_k \partial \mu_l}\varepsilon_j\varepsilon_k\varepsilon_l,$$

etc.

Thus

$$\mathbf{f}(\boldsymbol{\xi}) = \mathbf{f}(\boldsymbol{\mu}) + \begin{pmatrix} \sum_{j=1}^{\infty} \frac{1}{j!}\left(\frac{\partial}{\partial \mu_1}\varepsilon_1 + \ldots + \frac{\partial}{\partial \mu_n}\varepsilon_n\right)^j f_1(\boldsymbol{\mu}) \\ \vdots \\ \sum_{j=1}^{\infty} \frac{1}{j!}\left(\frac{\partial}{\partial \mu_1}\varepsilon_1 + \ldots + \frac{\partial}{\partial \mu_n}\varepsilon_n\right)^j f_s(\boldsymbol{\mu}) \end{pmatrix}.$$

Let

$$\begin{aligned}
\mathbf{F} &= \frac{\partial \mathbf{f}(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}'} = \begin{pmatrix} \frac{\partial f_1(\boldsymbol{\mu})}{\partial \mu_1}, \dots, \frac{\partial f_1(\boldsymbol{\mu})}{\partial \mu_n} \\ \dots\dots\dots\dots\dots\dots \\ \frac{\partial f_s(\boldsymbol{\mu})}{\partial \mu_1}, \dots, \frac{\partial f_s(\boldsymbol{\mu})}{\partial \mu_n} \end{pmatrix}, \\
\mathbf{F}_i &= \frac{\partial^2 f_i(\boldsymbol{\mu})}{\partial \boldsymbol{\mu} \partial \boldsymbol{\mu}'}, \quad i = 1, \dots, s, \\
\mathbf{P}_F &= \mathbf{F}(\mathbf{F}'\mathbf{F})^{-}\mathbf{F}', \\
\mathbf{F}^{-} &\dots\ g\text{-inverse of the matrix } \mathbf{F},\ \text{i.e. } \mathbf{F}\mathbf{F}^{-}\mathbf{F} = \mathbf{F} \\
&\quad\ \text{(in more detail cf. [5])}, \\
\mathbf{M}_F &= \mathbf{I} - \mathbf{P}_F, \\
\mathbf{I}_s &\dots \text{ identical } s \times s \text{ matrix}, \\
\mathbf{A}^{+} &\dots \text{ the Moore–Penrose } g\text{-inverse of the matrix } \mathbf{A},\ \text{i.e.}, \\
&\quad\ \mathbf{A}\mathbf{A}^{+}\mathbf{A} = \mathbf{A},\ \mathbf{A}^{+}\mathbf{A}\mathbf{A}^{+} = \mathbf{A}^{+},\ \mathbf{A}\mathbf{A}^{+} = (\mathbf{A}\mathbf{A}^{+})', \\
&\quad\ \mathbf{A}^{+}\mathbf{A} = (\mathbf{A}^{+}\mathbf{A})', \\
\boldsymbol{\xi} &\sim N_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \text{ the vector } \boldsymbol{\xi} \text{ is normally distributed with the} \\
&\quad\ \text{mean value } E(\boldsymbol{\xi}) = \boldsymbol{\mu} \text{ and with the covariance matrix} \\
&\quad\ \text{Var}(\boldsymbol{\xi}) = \boldsymbol{\Sigma}, \\
\mathbf{A} \leq_L \mathbf{B} &\dots \text{ means that } \mathbf{B} - \mathbf{A} \text{ is positive semidefinite.}
\end{aligned}$$

3 Criterion for LEPL

The quadratic version of the Taylor series is

$$\mathbf{f}(\boldsymbol{\xi}) = \mathbf{f}(\boldsymbol{\mu}) + \mathbf{F}\boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\kappa}(\boldsymbol{\varepsilon}), \tag{1}$$

where

$$\begin{aligned}
\boldsymbol{\kappa}(\boldsymbol{\varepsilon}) &= (\kappa_1(\boldsymbol{\varepsilon}), \dots, \kappa_s(\boldsymbol{\varepsilon}))', \\
\kappa_j(\boldsymbol{\varepsilon}) &= \boldsymbol{\varepsilon}'\mathbf{F}_j\boldsymbol{\varepsilon}, \quad j = 1, \dots, s.
\end{aligned}$$

A function $\mathbf{f}(\cdot)$ in the following text is assumed to be approximated by (1). If such approximation is not sufficient for our purposes, then the rules given in the following text must be replaced by procedures given in [3].

Lemma 3.1 *The mean value of* $\mathbf{f}(\boldsymbol{\xi})$ *from (1) is*

$$E\big[\mathbf{f}(\boldsymbol{\xi})\big] = \mathbf{f}(\boldsymbol{\mu}) + \frac{1}{2}\begin{pmatrix} \text{Tr}(\mathbf{F}_1\boldsymbol{\Sigma}) \\ \vdots \\ \text{Tr}(\mathbf{F}_s\boldsymbol{\Sigma}) \end{pmatrix}$$

and its covariance matrix is

$$\mathrm{Var}\left[\mathbf{f}(\boldsymbol{\xi})\right] =$$

$$= \mathbf{F}\boldsymbol{\Sigma}\mathbf{F}' + \frac{1}{2}\begin{pmatrix} E(\boldsymbol{\varepsilon}'\mathbf{F}_1\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\mathbf{F}') \\ \vdots \\ E(\boldsymbol{\varepsilon}'\mathbf{F}_s\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\mathbf{F}') \end{pmatrix} + \frac{1}{2}\left[E(\boldsymbol{\varepsilon}'\mathbf{F}_1\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\mathbf{F}'), \dots, E(\boldsymbol{\varepsilon}'\mathbf{F}_s\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}'\mathbf{F}')\right]$$

$$+\frac{1}{4}\begin{pmatrix} \mathrm{Var}(\boldsymbol{\varepsilon}'\mathbf{F}_1\boldsymbol{\varepsilon}), & \mathrm{cov}(\boldsymbol{\varepsilon}'\mathbf{F}_1\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'\mathbf{F}_2\boldsymbol{\varepsilon}), & \dots, & \mathrm{cov}(\boldsymbol{\varepsilon}'\mathbf{F}_1\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'\mathbf{F}_s\boldsymbol{\varepsilon}) \\ \dots & \dots & \dots & \dots \\ \mathrm{cov}(\boldsymbol{\varepsilon}'\mathbf{F}_s\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'\mathbf{F}_1\boldsymbol{\varepsilon}), & \mathrm{cov}(\boldsymbol{\varepsilon}'\mathbf{F}_s\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'\mathbf{F}_2\boldsymbol{\varepsilon}), & \dots, & \mathrm{Var}(\boldsymbol{\varepsilon}'\mathbf{F}_s\boldsymbol{\varepsilon}) \end{pmatrix}.$$

Proof is obvious.

Corollary 3.2 *If* $\boldsymbol{\varepsilon} \sim N_n(\mathbf{0}, \boldsymbol{\Sigma})$*, then the covariance matrix of* $\mathbf{f}(\boldsymbol{\xi})$ *from (1) is*

$$\mathrm{Var}\left[\mathbf{f}(\boldsymbol{\xi})\right] =$$

$$= \mathbf{F}\boldsymbol{\Sigma}\mathbf{F}' + \frac{1}{2}\begin{pmatrix} \mathrm{Tr}(\mathbf{F}_1\boldsymbol{\Sigma}\mathbf{F}_1\boldsymbol{\Sigma}), & \mathrm{Tr}(\mathbf{F}_1\boldsymbol{\Sigma}\mathbf{F}_2\boldsymbol{\Sigma}), & \dots, & \mathrm{Tr}(\mathbf{F}_1\boldsymbol{\Sigma}\mathbf{F}_s\boldsymbol{\Sigma}) \\ \dots & \dots & \dots & \dots \\ \mathrm{Tr}(\mathbf{F}_s\boldsymbol{\Sigma}\mathbf{F}_1\boldsymbol{\Sigma}), & \mathrm{Tr}(\mathbf{F}_s\boldsymbol{\Sigma}\mathbf{F}_2\boldsymbol{\Sigma}), & \dots, & \mathrm{Tr}(\mathbf{F}_s\boldsymbol{\Sigma}\mathbf{F}_s\boldsymbol{\Sigma}) \end{pmatrix}.$$

Here the relationship $\mathrm{cov}(\boldsymbol{\varepsilon}'\mathbf{A}\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'\mathbf{B}\boldsymbol{\varepsilon}) = 2\,\mathrm{Tr}(\mathbf{A}\boldsymbol{\Sigma}\mathbf{B}\boldsymbol{\Sigma})$ *is used (cf. e.g., [2]). In the following text the second term on the right hand side is neglected. A nonlinearity of a function* $f(\cdot)$ *is in the first step expressed by the term*

$$\frac{1}{2}\begin{pmatrix} \mathrm{Tr}(\mathbf{F}_1\boldsymbol{\Sigma}) \\ \vdots \\ \mathrm{Tr}(\mathbf{F}_s\boldsymbol{\Sigma}) \end{pmatrix}$$

what is approximately the bias $E(\boldsymbol{\xi}) - \mathbf{f}(\boldsymbol{\mu})$*. The bias of the estimator can be tolerated if it is small with a comparison with a standard deviation of the estimator (cf. some analogy in definition of the intrinsic and parametric curvatures of a regression model [1] and also the nonlinearity measures in [4].) The covariance matrix of the estimator is approximately* $\mathbf{F}\boldsymbol{\Sigma}\mathbf{F}'$*. Thus the following definition can be used in order to obtain a criterion for a utilization of* LEPL.

Definition 3.3 Let $\mathbf{h} \in R^s$. Then

$$c_h(\boldsymbol{\mu}) = \frac{\left|\frac{1}{2}\mathbf{h}'\begin{pmatrix} \mathrm{Tr}(\mathbf{F}_1\boldsymbol{\Sigma}) \\ \vdots \\ \mathrm{Tr}(\mathbf{F}_s\boldsymbol{\Sigma}) \end{pmatrix}\right|}{\sqrt{\mathbf{h}'\mathbf{F}\boldsymbol{\Sigma}\mathbf{F}'\mathbf{h}}}$$

is a criterion function for the random variable $\mathbf{h}'\mathbf{f}(\boldsymbol{\xi})$.

Its meaning is obvious. If $c_h(\boldsymbol{\mu}) < \varepsilon$, then the approximate bias of the estimator $\mathbf{h}'\mathbf{f}(\boldsymbol{\xi})$ from Lemma 2.1 of the quantity $\mathbf{h}'\mathbf{f}(\boldsymbol{\mu})$ is less than ε-multiple of its approximate standard deviation. A sufficient small $\varepsilon > 0$ enables us to use $\mathrm{Var}\left[\mathbf{h}'\mathbf{f}(\boldsymbol{\xi})\right] = \mathbf{h}'\mathbf{F}\boldsymbol{\Sigma}\mathbf{F}'\mathbf{h}$ instead of the precise formula (cf. [3]), since the nonlineareity is manifested non-essentially.

Theorem 3.4 *(i) Let* $\mathbf{F\Sigma F}' >_L \mathbf{0}$. *If*

$$\frac{1}{4}\Big(\operatorname{Tr}(\mathbf{F}_1\mathbf{\Sigma}),\dots,\operatorname{Tr}(\mathbf{F}_s\mathbf{\Sigma})\Big)(\mathbf{F\Sigma F}')^{-1}\begin{pmatrix}\operatorname{Tr}(\mathbf{F}_1\mathbf{\Sigma})\\ \vdots\\ \operatorname{Tr}(\mathbf{F}_s\mathbf{\Sigma})\end{pmatrix} < \varepsilon^2,$$

then

$$\forall\{\mathbf{h}\in R^s\}\left|\frac{1}{2}\mathbf{h}'\begin{pmatrix}\operatorname{Tr}(\mathbf{F}_1\mathbf{\Sigma})\\ \vdots\\ \operatorname{Tr}(\mathbf{F}_s\mathbf{\Sigma})\end{pmatrix}\right| \leq \varepsilon\sqrt{\mathbf{h}'\mathbf{F\Sigma F}'\mathbf{h}}.$$

(ii) Let $\mathbf{F\Sigma F}'$ *be not positive definite, i.e. its spectral decomposition is* $\mathbf{F\Sigma F}' = \sum_{i=1}^r \lambda_i \mathbf{g}_i\mathbf{g}_i'$, $r < s$. *Let* $\lambda_{\max} = \max\{\lambda_1,\dots,\lambda_r\}$. *If*

$$\frac{1}{4}\Big(\operatorname{Tr}(\mathbf{F}_1\mathbf{\Sigma}),\dots,\operatorname{Tr}(\mathbf{F}_s\mathbf{\Sigma})\Big)(\mathbf{F\Sigma F}'+\lambda_{\max}\mathbf{M}_F)^{-1}\begin{pmatrix}\operatorname{Tr}(\mathbf{F}_1\mathbf{\Sigma})\\ \vdots\\ \operatorname{Tr}(\mathbf{F}_s\mathbf{\Sigma})\end{pmatrix} < \varepsilon^2,$$

then

$$\forall\{\mathbf{h}\in R^s\}\left|\frac{1}{2}\mathbf{h}'\begin{pmatrix}\operatorname{Tr}(\mathbf{F}_1\mathbf{\Sigma})\\ \vdots\\ \operatorname{Tr}(\mathbf{F}_s\mathbf{\Sigma})\end{pmatrix}\right| \leq \varepsilon\sqrt{\mathbf{h}'(\mathbf{F\Sigma F}'+\lambda_{\max}\mathbf{M}_F)\mathbf{h}}.$$

Proof The Scheffé inequality [6] is used, i.e. if $\mathbf{b}\in R^s$ is given vector and $\mathbf{W}$ is a $s\times s$ positive definite matrix, then

$$\forall\{\mathbf{h}\in R^s\}|\mathbf{h}'\mathbf{b}| \leq \varepsilon\sqrt{\mathbf{h}'\mathbf{W}\mathbf{h}} \Leftrightarrow \mathbf{b}'\mathbf{W}^{-1}\mathbf{b} \leq \varepsilon^2.$$

Now it is sufficient to consider

$$\mathbf{b} = \frac{1}{2}\begin{pmatrix}\operatorname{Tr}(\mathbf{F}_1\mathbf{\Sigma})\\ \vdots\\ \operatorname{Tr}(\mathbf{F}_s\mathbf{\Sigma})\end{pmatrix}$$

and $\mathbf{W} = \mathbf{F\Sigma F}'$ and $\mathbf{F\Sigma F}' + \lambda_{\max}\mathbf{M}_F$, respectively. □

Definition 3.5 The linearization region for a function $\mathbf{f}(\cdot)$ is
(i) in the case $\mathbf{F\Sigma F}' >_L \mathbf{0}$

$$\mathcal{L}_\varepsilon = \left\{\boldsymbol{\mu}\colon \frac{1}{4}\Big(\operatorname{Tr}(\mathbf{F}_1\mathbf{\Sigma}),\dots,\operatorname{Tr}(\mathbf{F}_s\mathbf{\Sigma})\Big)(\mathbf{F\Sigma F}')^{-1}\begin{pmatrix}\operatorname{Tr}(\mathbf{F}_1\mathbf{\Sigma})\\ \vdots\\ \operatorname{Tr}(\mathbf{F}_s\mathbf{\Sigma})\end{pmatrix} < \varepsilon^2\right\}.$$

(ii) In the case $\mathbf{F\Sigma F}'$ is not positive definite, it is

$$\mathcal{L}_\varepsilon =$$

$$= \left\{\boldsymbol{\mu}\colon \frac{1}{4}\Big(\operatorname{Tr}(\mathbf{F}_1\mathbf{\Sigma}),\dots,\operatorname{Tr}(\mathbf{F}_s\mathbf{\Sigma})\Big)(\mathbf{F\Sigma F}'+\lambda_{\max}\mathbf{M}_F)^{-1}\begin{pmatrix}\operatorname{Tr}(\mathbf{F}_1\mathbf{\Sigma})\\ \vdots\\ \operatorname{Tr}(\mathbf{F}_s\mathbf{\Sigma})\end{pmatrix} < \varepsilon^2\right\}.$$

Thus for a given $\mathbf{\Sigma}$ and if we know that $\boldsymbol{\mu} \in \mathcal{L}_\varepsilon$, we can use the approximate relation $\operatorname{Var}\left[\mathbf{f}(\boldsymbol{\xi})\right] \approx \mathbf{F}\mathbf{\Sigma}\mathbf{F}'$.

The $(1-\alpha)$-confidence region for $\boldsymbol{\mu}$, with sufficiently small α must be used for a check of $\boldsymbol{\mu} \in \mathcal{L}_\varepsilon$.

Corollary 3.6 *If $s = 1$, then*

$$\mathcal{L}_\varepsilon = \left\{ \boldsymbol{\mu} \colon \frac{\left| \frac{1}{2} \operatorname{Tr}\left(\frac{\partial^2 f(\boldsymbol{\mu})}{\partial \boldsymbol{\mu} \partial \boldsymbol{\mu}'} \mathbf{\Sigma} \right) \right|}{\sqrt{\frac{\partial f(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}'} \mathbf{\Sigma} \frac{\partial f(\boldsymbol{\mu})}{\partial \boldsymbol{\mu}}}} < \varepsilon \right\}.$$

4 Numerical examples

In the following examples estimators of the variance $\operatorname{Var}[\mathbf{f}(\widehat{\boldsymbol{\beta}})]$ for frequently occurring functions are given. Actual values of $\boldsymbol{\beta}$ are in the regions $\mathcal{L}$. The true values of variances and biases are obtained from the simulations ($n = 10\,000$) and $\sigma = \sqrt{\operatorname{Var}\left[\mathbf{f}(\boldsymbol{\xi})\right]}$ is given in the form $0.025 \pm 0.000\,056$. Here $0.000\,56$ is an estimator of the standard error of the estimator of the standard error, i.e. $0.707\sigma/\sqrt{n-1}$ in more detail cf. [7]. This enables us to compare LEPL with simulation.

1) Let

$$f(\beta) = \frac{1}{\beta}, \ \beta \neq 0, \quad \text{i.e.,} \quad \mathcal{L}_\varepsilon = \{\beta \colon \sigma < \varepsilon|\beta|\}.$$

If $\varepsilon = 0.1$, $\widehat{\beta} \sim_1 (4, (0.4)^2)$, then

$$b = E\left(\frac{1}{\widehat{\beta}}\right) - \frac{1}{\beta} = \frac{1}{10000} \sum_{i=1}^{10000} \left(\frac{1}{\widehat{\beta}^{(i)}}\right) - \frac{1}{4} = 0.002\,703,$$

$$\operatorname{Var}\left(\frac{1}{\widehat{\beta}}\right) = \frac{1}{9999} \left\{ \sum_{i=1}^{10000} \left(\frac{1}{\widehat{\beta}^{(i)}}\right)^2 - 10000 \left[\frac{1}{10000} \sum_{i=1}^{10000} \left(\frac{1}{\widehat{\beta}^{(i)}}\right)\right]^2 \right\}$$

$$= 0.000\ 634\ 186,$$

$$\sqrt{\operatorname{Var}\left(\frac{1}{\widehat{\beta}}\right)} = 0.025\,183 \pm 0.000\,18,$$

$$\text{LEPL} = 0.025,$$

$$|b/\sigma| = 0.107 (\varepsilon = 0.1).$$

2) Let

$$\mathbf{f}(\beta) = \begin{pmatrix} \cos\beta \\ \sin\beta \end{pmatrix},$$

i.e. $\mathcal{L}_\varepsilon = \left\{ \beta \colon \frac{1}{4} \left[\operatorname{Tr}(F_1\Sigma), \operatorname{Tr}(F_2\Sigma)\right] \left[\mathbf{F}\Sigma\mathbf{F}' + \lambda_{\max}\mathbf{M}_F\right]^{-1} \begin{pmatrix} \operatorname{Tr}(F_1\Sigma) \\ \operatorname{Tr}(F_2\Sigma) \end{pmatrix} < \varepsilon^2 \right\}.$

Since

$$\mathbf{F} = \begin{pmatrix} -\sin\beta \\ \cos\beta \end{pmatrix}, \quad \Sigma = \sigma^2, \quad F_1 = -\cos\beta, \quad F_2 = -\sin\beta,$$

we have

$$\mathbf{F}\Sigma\mathbf{F}' = \sigma^2 \begin{pmatrix} \sin^2\beta, & -\sin\beta\cos\beta \\ -\sin\beta\cos\beta, & \cos^2\beta \end{pmatrix} = \sigma^2 \begin{pmatrix} \sin\beta \\ -\cos\beta \end{pmatrix} (\sin\beta, -\cos\beta),$$

i.e. $\lambda_{\max} = \sigma^2$. Further

$$\mathbf{M}_F = \begin{pmatrix} \cos^2\beta, & \sin\beta\cos\beta \\ \sin\beta\cos\beta, & \sin^2\beta \end{pmatrix}.$$

Thus $\mathbf{F}\Sigma\mathbf{F}' + \lambda_{\max}\mathbf{M}_F = \sigma^2\mathbf{I}$ and

$$\mathcal{L}_\varepsilon = \left\{\beta\colon \frac{1}{4}\sigma^2 < \varepsilon^2\right\} = R^1.$$

If

$$\widehat{\beta} \sim_1 (0.174\,533(=10^o), [0.4(=22.9^\circ)]^2), \quad \varepsilon = 0.2,$$

and

$$(1,1)\mathbf{f}(\beta) = \cos\beta + \sin\beta,$$

then

$$b = \frac{1}{10000}\sum_{i=1}^{10000}(\cos\widehat{\beta}^{(i)} + \sin\widehat{\beta}^{(i)}) - 1.158\,456 = -0.086\,950,$$

$$\mathrm{Var}(\cos\widehat{\beta} + \sin\widehat{\beta})$$

$$= \frac{1}{9999}\left\{\sum_{i=1}^{10000}(\cos\widehat{\beta}^{(i)} + \sin\widehat{\beta}^{(i)})^2 - 10000\left[\frac{1}{10000}\sum_{i=1}^{10000}(\cos\widehat{\beta}^{(i)} + \sin\widehat{\beta}^{(i)})\right]^2\right\}$$

$$= 0.103\ 795,$$

$$\sqrt{\mathrm{Var}(\cos\widehat{\beta} + \sin\widehat{\beta})} = 0.322\,172 \pm 0.002\,28,$$

$$\mathrm{LEPL} = 0.324\,47,$$

$$|b|/\sigma = 0.269(\varepsilon = 0.2).$$

3) Let

$$f(\beta) = \exp(\beta), \quad \text{i.e.,} \quad \mathcal{L}_\varepsilon = \left\{\beta\colon \frac{1}{2}\sigma < \varepsilon\right\} = R^1.$$

If

$$\varepsilon = 0.2, \quad \widehat{\beta} \sim_1 (10, (2 \times 0.2)^2),$$

then

$$b = \frac{1}{10000}\sum_{i=1}^{10000}\exp(\widehat{\beta}^{(i)}) - \exp(10) = 1\,415.35,$$

$$\operatorname{Var}[\exp(\widehat{\beta})] = \frac{1}{9999}\left\{\sum_{i=1}^{10000}[\exp(\widehat{\beta}^{(i)}]^2 - 10000\left[\frac{1}{10000}\sum_{i=1}^{10000}\exp(\widehat{\beta}^{(i)}\right]^2\right\}$$

$$= 8.164\,14 \times 10^7,$$

$$\sqrt{\operatorname{Var}[\exp(\widehat{\beta})]} = 9\,035.56 \pm 63.89,$$

$$\mathrm{LEPL} = 8\,811,$$

$$|b/\sigma| = 0.157(\varepsilon = 0.2).$$

4) Let

$$f(\beta) = \ln\beta, \quad \text{i.e.,} \quad \mathcal{L}_\varepsilon = \{\beta\colon \beta > 0,\, \sigma < \varepsilon\beta\}.$$

If

$$\varepsilon = 0.2, \quad \widehat{\beta} \sim_1 (10, 2^2),$$

we have

$$b = \frac{1}{10000}\sum_{i=1}^{10000}\ln\widehat{\beta}^{(i)} - 2.302585 = -0.022\,242,$$

$$\operatorname{Var}(\ln\widehat{\beta}) = \frac{1}{9999}\left\{\sum_{i=1}^{10000}(\ln\widehat{\beta}^{(i)})^2 - 10000\left[\frac{1}{10000}\sum_{i=1}^{10000}\ln\widehat{\beta}^{(i)}\right]^2\right\}$$

$$= 0.043\ 151,$$

$$\sqrt{\operatorname{Var}(\ln\widehat{\beta})} = 0.207\,729 \pm 0.001\,5,$$

$$\mathrm{LEPL} = 0.200,$$

$$|b/\sigma| = 0.100(\varepsilon = 0.2).$$

5) Let

$$f(\beta) = \beta^k, \quad \text{i.e.,} \quad \mathcal{L}_\varepsilon = \left\{\beta\colon \sigma < \frac{2\varepsilon}{k-1}|\beta|\right\}.$$

If

$$k = 3, \quad \varepsilon = 0.2, \quad \widehat{\beta} \sim_1 (10, 2^2),$$

then

$$b = \frac{1}{10000}\sum_{i=1}^{10000}(\widehat{\beta}^{(i)})^3 - 10000 = 110.475,$$

$$\mathrm{Var}(\widehat{\beta}^3) = \frac{1}{9999}\left\{\sum_{i=1}^{10000}(\widehat{\beta}^{(i)})^6 - 10000\left[\frac{1}{10000}\sum_{i=1}^{10000}(\widehat{\beta}^{(i)})^3\right]^2\right\}$$
$$= 370\ 896,$$
$$\sqrt{\mathrm{Var}[(\widehat{\beta}^{(i)})^3]} = 609.012 \pm 4.307,$$
$$\mathrm{LEPL} = 600,$$
$$|b/\sigma| = 0.181(\varepsilon = 0.2).$$

6) Let

$$f(\beta) = \frac{1}{\beta^k}, \quad \text{i.e.,} \quad \mathcal{L}_\varepsilon = \left\{\beta : \sigma < \frac{2\varepsilon}{k+1}|\beta|\right\}.$$

If

$$k = 3, \quad \varepsilon = 0.2, \quad \widehat{\beta} \sim_1 (10, 1^2),$$

then

$$b = \frac{1}{10000}\sum_{i=1}^{10000}\left(\frac{1}{(\widehat{\beta}^{(i)})^3}\right) - \frac{1}{1000} = 7.106\,79 \times 10^{-5},$$

$$\mathrm{Var}\left(\frac{1}{\widehat{\beta}^3}\right) = \frac{1}{9999}\left\{\sum_{i=1}^{10000}\left(\frac{1}{(\widehat{\beta}^{(i)})^6}\right)^2 - 10000\left[\frac{1}{10000}\sum_{i=1}^{10000}\left(\frac{1}{(\widehat{\beta}^{(i)})^3}\right)\right]^2\right\}$$
$$= 1.193\,77 \times 10^{-7},$$
$$\sqrt{\mathrm{Var}\left(\frac{1}{\widehat{\beta}^3}\right)} = 0.000\,345\,51 \pm 0.000\,002\ 4,$$
$$\mathrm{LEPL} = 0.000\,300,$$
$$|b/\sigma| = 0.206(\varepsilon = 0.2).$$

7) Let $f(\beta_1, \beta_2) = \beta_1\beta_2$. If

$$\begin{pmatrix}\widehat{\beta}_1\\ \widehat{\beta}_2\end{pmatrix} \sim_2 \left[\begin{pmatrix}\beta_1,\\ \beta_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2, & 0\\ 0, & \sigma_2^2\end{pmatrix}\right],$$

then $\mathcal{L}_\varepsilon = R^2$, since $\mathbf{F}_1 = \begin{pmatrix}0, & 1\\ 1, & 0\end{pmatrix}$. If

$$\begin{pmatrix}\widehat{\beta}_1\\ \widehat{\beta}_2\end{pmatrix} \sim_2 \left[\begin{pmatrix}10\\ 100\end{pmatrix}, \begin{pmatrix}0.1^2, & 0\\ 0, & 0.1^2\end{pmatrix}\right],$$

then $\mathcal{L}_\varepsilon = R^2$,

$$b = \frac{1}{10000}\sum_{i=1}^{10000}(\widehat{\beta}_1^{(i)}\widehat{\beta}_2^{(i)}) - 1000 = -0.007,$$

$$\mathrm{Var}(\widehat{\beta}_1\widehat{\beta}_2) = \frac{1}{9999}\left\{\sum_{i=1}^{10000}[(\widehat{\beta}_1^{(i)}\widehat{\beta}_2^{(i)})^2 - 10000\left[\frac{1}{10000}\sum_{i=1}^{10000}(\widehat{\beta}_1^{(i)}\widehat{\beta}_2^{(i)})\right]^2\right\}$$

$$= 96.916\,9,$$

$$\sqrt{\mathrm{Var}(\widehat{\beta}_1\widehat{\beta}_2)} = 9.844 \pm 0.070,$$

$$\mathrm{LEPL} = 10.05,$$

$$|b/\sigma| = 0.000\,7.$$

8) Let

$$f(\beta_1,\beta_2) = \beta_1 \tan\beta_2, \quad \text{i.e.,} \quad \mathcal{L}_\varepsilon = \left\{\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}: \frac{\sigma_2^2\beta_1\tan\beta_2}{\sqrt{\sigma_1^2\sin^2\beta_2\cos^2\beta_2 + \sigma_2^2\beta_1^2}} < \varepsilon\right\}.$$

The values of the function

$$g(\beta_1,\beta_2) = \frac{\sigma_2^2\beta_1\tan\beta_2}{\sqrt{\sigma_1^2\sin^2\beta_2\cos^2\beta_2 + \sigma_2^2\beta_1^2}}$$

for $\sigma_1^2 = (0.1\ \mathrm{m})^2$ and $\sigma_2^2 = (48.48 \times 10^{-6} = 10'')^2$ are given in the following table

Table
Values of the function $g(\cdot,\cdot\cdot) \times 10^6$

25°	1.143	1.712	2.277	2.839	3.395
20°	1.063	1.591	2.114	2.632	3.143
15°	1.005	1.501	1.991	2.473	2.944
10°	0.963	1.433	1.891	2.332	2.753
5°	0.924	1.347	1.730	2.068	2.361
0°	0.000	0.000	0.000	0.000	0.000
	40 m	60 m	80 m	100 m	120 m

If

$$\begin{pmatrix}\widehat{\beta}_1\\ \widehat{\beta}_2\end{pmatrix} \sim_2 \left[\begin{pmatrix}80\ \mathrm{m}\\ 0.349\,066(=20^\circ)\end{pmatrix}, \begin{pmatrix}\sigma_1^2, & 0\\ 0, & \sigma_2^2\end{pmatrix}\right],$$

then

$$b = \frac{1}{10000} \sum_{i=1}^{10000} \widehat{\beta}_1^{(i)} \tan(\widehat{\beta}_2^{(i)}) - 29.118\,\mathrm{m} = -0.000\,5\,\mathrm{m},$$

$$\mathrm{Var}(\widehat{\beta}_1 \tan(\widehat{\beta}_2))$$

$$= \frac{1}{9999} \left\{ \sum_{i=1}^{10000} (\widehat{\beta}_1^{(i)} \tan \widehat{\beta}_2^{(i)})^2 - 10000 \left[\frac{1}{10000} \sum_{i=1}^{10000} \widehat{\beta}_1^{(i)} \tan \widehat{\beta}_2^{(i)} \right]^2 \right\}$$

$$= 0.000\,975\,4\,\mathrm{m}^2,$$

$$\sqrt{\mathrm{Var}(\widehat{\beta}_1 \tan \widehat{\beta}_2)} = (0.031\,23 \pm 0.000\,22)\,\mathrm{m},$$

$$\mathrm{LEPL} = 0.036\,7\,\mathrm{m},$$

$$|b/\sigma| = 0.016\,0.$$

9) Let 2D coordinates x, y of the point P be transformed into another coordinate system ξ, η by the transformation

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \cos\beta_3, & \sin\beta_3 \\ -\sin\beta_3, & \cos\beta_3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(frequent problem in cartography and geodesy). Let

$$(\beta_1, \beta_2, \beta_3, x, y) = (100\,\mathrm{m}, 100\,\mathrm{m}, 0.785\,397 (= 45^0), 300\,\mathrm{m}, 400\,\mathrm{m}).$$

The uncertainty of this vector is given by the covariance matrix

$$\mathbf{\Sigma} = \mathrm{Diag}\left[(0.1\,\mathrm{m})^2, (-0.1\,\mathrm{m})^2, (0.017453 = 1^0)^2, (0.1\,\mathrm{m})^2, (0.1\,\mathrm{m})^2\right]$$

(the uncertainties are extremaly large in order to show how the rule works). The aim is to determine $\mathrm{Var}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$. The utilization of LEPL is possible if the value

$$\frac{1}{4}\left[\mathrm{Tr}(\mathbf{F}_1\mathbf{\Sigma}), \mathrm{Tr}(\mathbf{F}_2\mathbf{\Sigma})\right](\mathbf{F}\mathbf{\Sigma}\mathbf{F}')^{-1} \begin{pmatrix} \mathrm{Tr}(\mathbf{F}_1\mathbf{\Sigma}) \\ \mathrm{Tr}(\mathbf{F}_2\mathbf{\Sigma}) \end{pmatrix}$$

is sufficiently small. In our case

$$\mathbf{F} = \begin{pmatrix} 1, & 0, & -x\sin\beta_3 + y\cos\beta_3, & \cos\beta_3, & \sin\beta_3 \\ 0, & 1, & -x\cos\beta_3 - y\sin\beta_3, & -\sin\beta_3, & \cos\beta_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1, & 0, & 70.710\,700, & 0.707\,107, & 0.707\,107 \\ 0, & 1, & -494.974\,900, & -0.707\,107, & 0.707\,107 \end{pmatrix},$$

$$\mathbf{F}_1 = \begin{pmatrix} 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & -x\cos\beta_3 - y\sin\beta_3, & -\sin\beta_3, & \cos\beta_3 \\ 0, & 0, & -\sin\beta_3, & 0, & 0 \\ 0, & 0, & \cos\beta_3, 0, & 0 & \end{pmatrix},$$

$$\mathbf{F}_2 = \begin{pmatrix} 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & 0, & 0, & 0 \\ 0, & 0, & x\cos\beta_3 - y\sin\beta_3, & -\cos\beta_3, & -\sin\beta_3 \\ 0, & 0, & -\cos\beta_3, & 0, & 0 \\ 0, & 0, & -\sin\beta_3, 0, & 0 & \end{pmatrix},$$

$$\mathbf{F\Sigma F}' = \begin{pmatrix} 1.543\,038, & -10.661\,264 \\ -10.661\,264, & 74.648\,851 \end{pmatrix},$$

$$\frac{1}{4}\big[\,\mathrm{Tr}(\mathbf{F}_1\mathbf{\Sigma}), \mathrm{Tr}(\mathbf{F}_2\mathbf{\Sigma})\big](\mathbf{F\Sigma F}')^{-1}\begin{pmatrix} \mathrm{Tr}(\mathbf{F}_1\mathbf{\Sigma}) \\ \mathrm{Tr}(\ \mathbf{F}_2\mathbf{\Sigma}) \end{pmatrix} = 0.289\,950.$$

The value $\varepsilon = \sqrt{0.289\,950} = 0.538\,472$ indicates that LEPL for a determination of $\mathrm{Var}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ is not suitable in this case. The reason is too large uncertainty of the values β_1, β_2, β_3, x, y.

Let the covariance matrix be

$$\begin{aligned} \mathbf{\Sigma} &= \\ &= \frac{1}{5.3847^2}\,\mathrm{Diag}\left[(0.1\,\mathrm{m})^2, (-0.1\,\mathrm{m})^2, (0.017453)^2, (0.1\,\mathrm{m})^2, (0.1\,\mathrm{m})^2\right] \\ &= \mathrm{Diag}\left[(0.019\,\mathrm{m})^2, (0.019\,\mathrm{m})^2, (0.003241 = 11.142')^2, (0.019\,\mathrm{m})^2, (0.019\,\mathrm{m})^2\right], \end{aligned}$$

where $5.3847^2 = (10\sqrt{0.289\,950})^2$. In this case

$$\frac{1}{4}\big[\,\mathrm{Tr}(\mathbf{F}_1\mathbf{\Sigma}), \mathrm{Tr}(\mathbf{F}_2\mathbf{\Sigma})\big](\mathbf{F\Sigma F}')^{-1}\begin{pmatrix} \mathrm{Tr}(\mathbf{F}_1\mathbf{\Sigma}) \\ \mathrm{Tr}(\ \mathbf{F}_2\mathbf{\Sigma}) \end{pmatrix} = 0.01 (= \varepsilon^2),$$

what is sufficiently small value ε^2 for a utilization of LEPL. The approximate $\mathrm{Var}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$ is

$$\mathrm{Var}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \mathbf{F\Sigma F}' = \begin{pmatrix} 0.053\,217, & -0.367693 \\ -0.367\,693, & 2.574\,542 \end{pmatrix}$$

and

$$\sqrt{\mathrm{Var}(\xi)} = 0.231\,\mathrm{m}, \quad \sqrt{\mathrm{Var}(\eta)} = 1.605\,\mathrm{m}.$$

The values obtained by simulation are

$$\mathbf{b} = \frac{1}{10000}\sum_{i=1}^{10000}\begin{pmatrix} \xi^{(i)} \\ \eta^{(i)} \end{pmatrix} - \begin{pmatrix} 594.975 \\ 170.711 \end{pmatrix} = \begin{pmatrix} -0.003\,29 \\ 0.004\,76 \end{pmatrix},$$

$$\begin{aligned} \mathbf{\Sigma} &= \frac{1}{9999}\sum_{i=1}^{10000}\left[\begin{pmatrix} \xi^{(i)} \\ \eta^{(i)} \end{pmatrix} - \frac{1}{10000}\sum_{i=1}^{10000}\begin{pmatrix} \xi^{(i)} \\ \eta^{(i)} \end{pmatrix}\right] \\ &\times \left[\begin{pmatrix} \xi^{(i)} \\ \eta^{(i)} \end{pmatrix} - \frac{1}{10000}\sum_{i=1}^{10000}\begin{pmatrix} \xi^{(i)} \\ \eta^{(i)} \end{pmatrix}\right]' = \begin{pmatrix} 0.050\,2, & -0.343\,8 \\ -0.343\,8, & 2.380\,7 \end{pmatrix}, \end{aligned}$$

and

$$\sqrt{\mathrm{Var}(\xi)} = (0.223 \pm 0.002)\ \mathrm{m}, \quad \sqrt{\mathrm{Var}(\eta)} = (1.543 \pm 0.010)\ \mathrm{m},$$
$$|b_\xi/\sigma_\xi| = 0.014\,8, \quad |b_\eta/\sigma_\eta| = 0.003\,1.$$

10) Let in the plane the coordinates of two points P_1, P_2 be known. At the point P_1 the horizontal angle Θ_1 between the directions to the point P_2 and to the point A with unknown coordinates is measured with the standard deviation σ_Θ. Analogously at the point P_2 the angle Θ_2 is measured. Let the distance between the points P_1, P_2 be s and the direction from P_2 to P_1 be given by the angle α. Then the x coordinate x_A of the point A is given by the relationship

$$x_A = x_{P_2} + s\frac{\sin\Theta_1\cos(\alpha - \Theta_2)}{\sin(\Theta_1 + \Theta_2)}.$$

Since the measurement of the angles Θ_1, Θ_2 is stochastically independent, LEPL for x_A is

$$\mathrm{Var}(\widehat{x}_A) = \left(\frac{\partial x_A}{\partial \Theta_1}\right)^2 \mathrm{Var}(\widehat{\Theta_1}) + \left(\frac{\partial x_A}{\partial \Theta_2}\right)^2 \mathrm{Var}(\widehat{\Theta_2}),$$

where

$$\frac{\partial x_A}{\partial \Theta_1} = s\frac{\cos(\alpha - \Theta_2)\sin\Theta_2}{\sin^2(\Theta_1 + \Theta 2)}, \quad \frac{\partial x_A}{\partial \Theta_2} = -s\frac{\sigma\Theta_1\cos(\alpha - \Theta_1)}{\sin^2(\Theta_1 + \Theta_2)}.$$

Since $\mathrm{Var}\begin{pmatrix}\widehat{\Theta}_1\\ \widehat{\Theta}_2\end{pmatrix} = \sigma_\Theta^2 \mathbf{I}$, we shall need in the expression

$$\mathrm{Tr}\left(\frac{\partial^2 x_A}{\partial\begin{pmatrix}\Theta_1\\ \Theta_2\end{pmatrix}\partial(\Theta_1, \Theta_2)}\mathrm{Var}\begin{pmatrix}\widehat{\Theta}_1\\ \widehat{\Theta}_2\end{pmatrix}\right)$$

the second derivatives $\frac{\partial^2 x_A}{\partial \Theta_1^2}$, $\frac{\partial^2 x_A}{\partial \Theta_2^2}$ only, i.e.

$$\frac{\partial^2 x_A}{\partial \Theta_1^2} = -2s\frac{\cos(\alpha - \Theta_2)\cos(\Theta_1 + \Theta_2)\sin\Theta_2}{\sin^3(\Theta_1 + \Theta_2)},$$
$$\frac{\partial^2 x_A}{\partial \Theta_2^2} = -2s\frac{\sin\Theta_1\cos(\alpha - \Theta_1)\cos(\Theta_1 + \Theta_2)}{\sin^3(\Theta_1 + \Theta_2)}.$$

For the sake of simplicity let $\Theta_1 = \Theta_2 = \frac{\pi}{4}$. Then

$$\mathrm{Tr}\left(\frac{\partial^2 x_A}{\partial\begin{pmatrix}\Theta_1\\ \Theta_2\end{pmatrix}\partial(\Theta_1, \Theta_2)}\mathrm{Var}\begin{pmatrix}\widehat{\Theta}_1\\ \widehat{\Theta}_2\end{pmatrix}\right) = 0,$$

thus $\mathcal{L} = R^2$ and

$$\text{LEPL} = \text{Var}(\widehat{x}_A) = \sigma_\Theta^2 s^2 \left[\cos^2\left(\alpha - \frac{\pi}{4}\right)\sin^2\frac{\pi}{4} + \sin^2\frac{\pi}{4}\cos^2\left(\alpha - \frac{\pi}{4}\right)\right].$$

Let $\alpha - \frac{\pi}{4} = \frac{\pi}{4}$, $s = 1000\,\text{m}$ and $\sigma_\Theta^2 = \left(\frac{600}{206265}\right)^2$, i.e. $\sigma_\Theta = 10'$ (in sexagesimal degrees). Then

$$\text{LEPL} = \sqrt{\text{Var}(\widehat{x}_A)} = \sigma_\Theta s\sqrt{0.5} = 2.057\,\text{m}.$$

Values of $\sqrt{\text{Var}(\widehat{x}_A)}$ given by simulation ($n = 10\,000$) are between (2.022 ± 0.014) m and (2.075 ± 0.015) m, what means very good agreement LEPL with simulation.

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