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Implication and Equivalential Reducts of Basic Algebras*

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Abstract

A term operation implication is introduced in a given basic algebra $\mathcal{A}$ and properties of the implication reduct of $\mathcal{A}$ are treated. We characterize such implication basic algebras and get congruence properties of the variety of these algebras. A term operation equivalence is introduced later and properties of this operation are described. It is shown how this operation is related with the induced partial order of $\mathcal{A}$ and, if this partial order is linear, the algebra $\mathcal{A}$ can be reconstructed by means of its equivalential reduct.

Key words: Basic algebra, implication algebra, implication reduct, equivalential algebra, equivalential reduct.

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1 Preliminaries

The concept of basic algebra was introduced by the first author, see e.g. [3] for details. Recall that by a *basic algebra* we mean an algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ of type $\langle 2, 1, 0\rangle$ satisfying the following identities

(BA1) $x \oplus 0 = x$,

(BA2) $\neg\neg x = x$,

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(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$,

(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$,

where $1 = \neg 0$. Let us note that this axiom system is from [4], the original one from [3] contains two more identities which can be derived by means of (BA1)–(BA4).

A basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ is called *commutative* if it satisfies the identity $x \oplus y = y \oplus x$.

The following lemma is known (see [4, 3]).

Lemma 1 *Every basic algebra satisfies the identities*

(a) $0 \oplus x = x$,

(b) $x \oplus 1 = 1 \oplus x = 1$,

(c) $x \oplus \neg x = 1 = \neg x \oplus x$.

As shown e.g. in [3], every basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ can be considered as an ordered set with the least element 0 and the greatest element 1, where

$$x \leq y \quad \text{if and only if} \quad \neg x \oplus y = 1. \tag{$*$}$$

Moreover, it is a lattice, where

$$x \vee y = \neg(\neg x \oplus y) \oplus y \quad \text{and} \quad x \wedge y = \neg(\neg(x \oplus \neg y) \oplus \neg y).$$

If $x \leq y$ or $y \leq x$ for each two elements x, y of A then $\mathcal{A}$ will be called a *chain basic algebra*.

Since basic algebras are of the same type as MV-algebras and differ from them only in the fact that associativity and commutativity of the operation $\oplus$ is not asked, we can define the connectives implication "$\to$" and equivalence "$\leftrightarrow$" in the same way, i.e. they are term operations

$$x \to y := \neg x \oplus y \quad \text{and} \quad x \leftrightarrow y := (x \to y) \wedge (y \to x).$$

To reveal the properties of $\to$ and $\leftrightarrow$ we will study these connectives without relations to other operations, i.e. we are focused on the implication or equivalential reducts of basic algebras.

2 Implication basic algebras

Basic algebras form an important class of algebras used in several non-classical logics due to the fact that this class contains e.g. orthomodular lattices $\mathcal{L} = (L; \vee, \wedge, {}^{\perp}, 0, 1)$, where $x \oplus y = (x \wedge y^{\perp}) \vee y$ and $\neg x = x^{\perp}$, which form an axiomatization of the logic of quantum mechanics as well as MV-algebras (see e.g. [5]), which get an axiomatization of many-valued Łukasiewicz logics. Let us note that similar analysis of axioms of implication quantum algebras were studied also by J. C. Abbott [1] and by N. D. Megill and M. Pavičić [7].

Since the connective implication plays a crucial role in the all above mentioned logics, we would like to characterize this operation also in basic algebras. Therefore, we introduce the following concept:

Definition 1 An algebra $(A;\circ)$ of type $\langle 2\rangle$ is called an *implication basic algebra* if it satisfies the following identities

(I1) $(x\circ x)\circ x = x$,

(I2) $(x\circ y)\circ y = (y\circ x)\circ x$,

(I3) $(((x\circ y)\circ y)\circ z)\circ(x\circ z) = x\circ x$.

Lemma 2 *Let $(A;\circ)$ be an implication basic algebra. Then there exists an element $1\in A$ which is an algebraic constant and $(A;\circ)$ satisfies the identities*

(i) $x\circ x = 1$,

(ii) $x\circ 1 = 1$,

(iii) $1\circ x = x$,

(iv) $((x\circ y)\circ y)\circ y = x\circ y$,

(v) $y\circ(x\circ y) = 1$.

Proof Substituting z by y and y by x in (I3) and applying (I1) we get

$$x\circ x = (((x\circ x)\circ x)\circ y)\circ(x\circ y) = (x\circ y)\circ(x\circ y).$$

When x is now substituted by $x\circ y$, we derive

$$((x\circ y)\circ y)\circ((x\circ y)\circ y) = (x\circ y)\circ(x\circ y)$$

and hence $((x\circ y)\circ y)\circ((x\circ y)\circ y) = x\circ x$. Applying (I2) we infer

$$y\circ y = ((y\circ x)\circ x)\circ((y\circ x)\circ x) = ((x\circ y)\circ y)\circ((x\circ y)\circ y) = x\circ x,$$

thus $(A;\circ)$ satisfies the identity

$$x\circ x = y\circ y.$$

This means that $(A;\circ)$ contains an algebraic constant which will be denoted by 1 and hence it satisfies the identity $x\circ x = 1$, which is (i). Using this, (I1) can be reformulated as

$$1\circ x = x,$$

which is (iii). By (i) and (I3) we get

$$(((x\circ y)\circ y)\circ z)\circ(x\circ z) = 1$$

and due to (I2), we derive easily also

$$(((x\circ y)\circ y)\circ z)\circ(y\circ z) = 1.$$

Substituting $x\circ y$ instead of x and z we get

$$((((x\circ y)\circ y)\circ y)\circ(x\circ y))\circ(y\circ(x\circ y)) = 1.$$

By (I3) and (iii) we conclude

$$y \circ (x \circ y) = 1,$$

which is (v). For $y = x$ we obtain (ii) immediately.

It remains to prove (iv). Using (iii) and (v), we have

$$(y \circ (x \circ y)) \circ (x \circ y) = 1 \circ (x \circ y) = x \circ y.$$

Due to (I2), $(y \circ (x \circ y)) \circ (x \circ y) = ((x \circ y) \circ y) \circ y$ whence (iv) is evident. □

Theorem 1 *The identities (I1), (I2), (I3) are independent.*

Proof Consider a two element groupoid $\mathcal{A} = (\{0,1\}, \circ)$, where $\circ$ is defined by the table

$$\begin{array}{c|cc|} \circ & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 1 & 1 \\ \hline \end{array}.$$

Then $\mathcal{A}$ satisfies (I1), (I3), but not (I2) since

$$(0 \circ 1) \circ 1 = 0 \neq 1 = (1 \circ 0) \circ 0.$$

If $\circ$ is defined by the table

$$\begin{array}{c|cc|} \circ & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 1 \\ \hline \end{array},$$

then $\mathcal{A}$ satisfies (I1), (I2), but not (I3) since

$$(((0 \circ 1) \circ 1) \circ 1) \circ (0 \circ 1) = 1 \neq 0 = 0 \circ 0.$$

If $\circ$ is defined as the constant operation $x \circ y = 1$ for every $x, y \in \{0,1\}$ then $\mathcal{A}$ satisfies (I2), (I3), but not (I1) since

$$(0 \circ 0) \circ 0 = 1 \neq 0.$$

□

The connection between basic algebras and implication basic algebras is established by the following:

Theorem 2 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra. Define $x \circ y = \neg x \oplus y$. Then $(A; \circ)$ is an implication basic algebra.*

Proof Applying (BA1)–(BA4) and Lemma 1, we can easily check the identities (I1)–(I3) as follows

(I1): $(x \circ x) \circ x = \neg(\neg x \oplus x) \oplus x = \neg 1 \oplus x = 0 \oplus x = x$;

(I2): $(x \circ y) \circ y = \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x = (y \circ x) \circ x$;

(I3): $(((x \circ y) \circ y) \circ z) \circ (x \circ z) = \neg(\neg(\neg(\neg x \oplus y) \oplus y) \oplus z) \oplus (\neg x \oplus z) = 1 = \neg x \oplus x = x \circ x$. □

Remark 1 Since basic algebras serve as an algebraic axiomatization of certain many-valued logic, where $\oplus$ is considered as a disjunction and $\neg$ as a negation, the term function $\neg x \oplus y$ can be recognized as an implication (formally the same construction as in the classical propositional calculus). This motivated us to call $(A; \circ)$ an implication basic algebras due to the relation given by Theorem 2.

To reveal the structure of implication basic algebras we introduce a partial order relation.

Lemma 3 *Let $(A; \circ)$ be an implication basic algebra. Define a binary relation $\leq$ on A as follows*

$$x \leq y \quad \textit{if and only if} \quad x \circ y = 1.$$

Then $\leq$ is a partial order on A such that $x \leq 1$ for each $x \in A$. Moreover,

$$z \leq x \circ z \quad \textit{and} \quad x \leq y \quad \textit{implies} \quad y \circ z \leq x \circ z$$

for all $x, y, z \in A$.

Proof By (i) of Lemma 2 we have that $\leq$ is reflexive. Assume $x \leq y$ and $y \leq x$. Then $x \circ y = 1$, $y \circ x = 1$ and by (I2) and (I1)

$$x = 1 \circ x = (y \circ x) \circ x = (x \circ y) \circ y = 1 \circ y = y,$$

which is proving antisymmetry of $\leq$.

If $x \leq y$ and $y \leq z$ then $x \circ y = 1$, $y \circ z = 1$ and, due to (I3) and Lemma 2 we get

$$\begin{aligned} 1 &= (((x \circ y) \circ y) \circ z) \circ (x \circ z) = ((1 \circ y) \circ z) \circ (x \circ z) \\ &= (y \circ z) \circ (x \circ z) = 1 \circ (x \circ z) = x \circ z \end{aligned}$$

thus also $x \leq z$ proving transitivity of $\leq$. Hence $\leq$ is a partial order on A and due to (ii) of Lemma 2, $x \leq 1$ for each $x \in A$.

Further, if $x \leq y$ and $z \in A$ then $x \circ y = 1$ and, by (I3),

$$1 = (((x \circ y) \circ y) \circ z) \circ (x \circ z) = ((1 \circ y) \circ z) \circ (x \circ z) = (y \circ z) \circ (x \circ z)$$

getting $y \circ z \leq x \circ z$. Putting here $y = 1$ we obtain $z = 1 \circ z \leq x \circ z$. □

The partial order $\leq$ introduced in Lemma 3 will be called the *induced partial order* of the implication basic algebra $(A; \circ)$.

Remark 2 Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and $x \circ y = \neg x \oplus y$. Then the induced partial order of the implication basic algebra $(A; \circ)$ coincides with the partial order of $\mathcal{A}$ defined by $(*)$ in Preliminaries.

Theorem 3 *Let $(A; \circ)$ be an implication basic algebra and $\leq$ its induced partial order. Then $(A; \leq)$ is a join-semilattice with the greatest element 1 where $x \vee y = (x \circ y) \circ y$.*

Proof By Lemma 3 and (I2) we infer $y \leq (x \circ y) \circ y$ and $x \leq (y \circ x) \circ x = (x \circ y) \circ y$ thus $(x \circ y) \circ y$ is a common upper bound of x, y. Assume $x, y \leq z$. Then by double using of the Lemma 3 we have

$$(x \circ y) \circ y \leq (z \circ y) \circ y = (y \circ z) \circ z = 1 \circ z = z,$$

thus $(x \circ y) \circ y$ is the least upper bound of x, y, i.e.

$$x \vee y = (x \circ y) \circ y$$

is the supremum of x, y. □

Let $(A; \circ)$ be an implication basic algebra. The semilattice $(A; \vee)$ derived in Theorem 3 will be called the *induced semilattice* of $(A; \circ)$.

Theorem 4 *Let $(A; \circ)$ be an implication basic algebra and $(A; \vee)$ its induced semilattice. For each $p \in A$, the interval $[p, 1]$ is a lattice $([p, 1]; \vee, \wedge_p, {}^p)$ with an antitone involution $x \mapsto x^p$ where*

$$x^p = x \circ p \quad \textit{and} \quad x \wedge_p y = ((x \circ p) \vee (y \circ p)) \circ p$$

for all $x, y \in [p, 1]$.

Proof Assume $x \in [p, 1]$. By Lemma 3, $x \mapsto x^p$ is a partial order reversing mapping and moreover we have $x^p = x \circ p \geq p$, thus $x \mapsto x^p$ is a mapping of $[p, 1]$ into itself. By Theorem 3, $x^{pp} = (x \circ p) \circ p = x \vee p = x$ and hence it is an involution of $[p, 1]$. This yields that we can apply De Morgan laws to show that

$$(x^p \vee y^p)^p = ((x \circ p) \vee (y \circ p)) \circ p = x \wedge_p y$$

is the infimum of $x, y \in [p, 1]$ and hence $([p, 1]; \vee, \wedge_p, {}^p)$ is a lattice with an antitone involution. □

Corollary 1 *Let $(A; \circ)$ be an implication basic algebra and $\leq$ its induced partial order. Then $(A; \leq)$ is a join-semilattice with the greatest element 1 such that for each $p \in A$ the interval $[p, 1]$ is a basic algebra $([p, 1]; \oplus_p, \neg_p, p)$ where $x \oplus_p y = (x \circ p) \circ y$ and $\neg_p x = x \circ p$ for all $x, y \in [p, 1]$.*

In what follows, $([p, 1]; \oplus_p, \neg_p, p)$ will be called an *interval basic algebra.* Theorem 4 describes the semilattice structure of an implication basic algebra. We are going to show that this description is complete, i.e. that the converse of Theorem 4 holds.

Theorem 5 *Let $(A; \vee, 1)$ be a join-semilattice with the greatest element 1 such that for each $p \in A$ the interval $[p, 1]$ is a lattice with an antitone involution $x \mapsto x^p$. Define $x \circ y = (x \vee y)^y$. Then $(A; \circ)$ is an implication basic algebra.*

Proof Since $x \vee y \in [y,1]$ for every $x, y \in A$, the operation $\circ$ is well-defined. We are going to check the identities (I1), (I2), (I3).
(I1): $(x \circ x) \circ x = ((x \vee x)^x \vee x)^x = x^{xx} = x$;
(I2): $(x \circ y) \circ y = ((x \vee y)^y \vee y)^y = (x \vee y)^{yy} = x \vee y = y \vee x = (y \vee x)^{xx} = ((y \vee x)^x \vee x)^x = (y \circ x) \circ x$;
(I3): $(((x \circ y) \circ y) \circ z) \circ (x \circ z) = ((x \vee y) \vee z)^z \circ (x \vee z)^z = 1 = (x \vee x)^x = x \circ x$ since $((x \vee y) \vee z)^z \leq (x \vee z)^z$. □

We say that $(A; \circ)$ is an *implication basic algebra with the least element* if there exists an element $0 \in A$ such that $0 \leq a$ for each $a \in A$ (where $\leq$ is the induced partial order). By Lemma 3 the identity

$$0 \circ x = 1$$

holds in any implication basic algebra with the least element 0.

The following result shows that our implication basic algebra really catches all the properties of implication $x \to y := \neg x \oplus y$ in any basic algebra.

Theorem 6 *Let $(A; \circ)$ be an implication basic algebra with the least element 0. Define the term operations $\neg x = x \circ 0$ and $x \oplus y = (x \circ 0) \circ y$. Then $(A; \oplus, \neg, 0)$ is a basic algebra and $x \circ y = \neg x \oplus y$.*

Proof We need to check the axioms (BA1)–(BA4) of basic algebras.
(BA1) and (BA2): $x \oplus 0 = (x \circ 0) \circ 0 = x \vee 0 = x$; $\neg\neg x = (x \circ 0) \circ 0 = x$.
For (BA3) and (BA4) we use the fact that

$$\neg x \oplus y = ((x \circ 0) \circ 0) \circ y = (x \vee 0) \circ y = x \circ y.$$

(BA3): $\neg(\neg x \oplus y) \oplus y = (x \circ y) \circ y = (y \circ x) \circ x = \neg(\neg y \oplus x) \oplus x$ by (I2).
(BA4): $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = ((((x \circ 0) \circ y) \circ y) \circ z) \circ ((x \circ 0) \circ z) = 1$ by (I3).
By Theorem 3, $(x \circ 0) \circ 0 = x \vee 0 = x$ and hence $x \circ y = ((x \circ 0) \circ 0) \circ y = (x \circ 0) \oplus y = \neg x \oplus y$. □

Let us note that the induced partial order of an implication algebra $(A; \circ)$ coincides with that of $(A; \oplus, \neg, 0)$ defined by $(*)$.

An implication basic algebra $(A; \circ)$ is called *commutative* if $(x \circ p) \circ y = (y \circ p) \circ x$ for all $x, y \in [p, 1]$. By Corollary 1, if $(A; \circ)$ is commutative then for each $p \in A$, $x \oplus_p y = y \oplus_p x$ for all $x, y \in [p, 1]$ in the interval basic algebra $([p, 1]; \oplus_p, \neg_p, p)$. Applying Theorem 8.5.9 from [3], we can infer the following:

Corollary 2 *Let $(A; \circ)$ be a commutative implication basic algebra and $(A; \vee)$ its induced semilattice. Then*

(a) for each $p \in A$ the interval basic algebra $([p, 1]; \oplus_p, \neg_p, p)$ is a commutative basic algebra;

(b) for each $p \in A$ the interval lattice $([p, 1], \vee, \wedge_p)$ is distributive.

In what follows, we can check several important congruence conditions of implication basic algebras. Denote by $\mathcal{IB}$ the variety of implication basic algebras.

Recall that an algebra $\mathcal{A}$ with a constant 1 is *weakly regular* (see e.g. [2]) if every congruence Θ on $\mathcal{A}$ is determined by its 1-class $[1]_\Theta$, in other words, if for each $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$

$$[1]_\Theta = [1]_\Phi \quad \text{implies} \quad \Theta = \Phi.$$

An algebra $\mathcal{A}$ is *congruence 3-permutable* if

$$\Theta \circ \Phi \circ \Theta = \Phi \circ \Theta \circ \Phi$$

for each $\Theta, \Phi \in \mathrm{Con}\mathcal{A}$. An algebra $\mathcal{A}$ is *congruence distributive* if

$$\Theta \wedge (\Phi \vee \Psi) = (\Theta \wedge \Phi) \vee (\Theta \wedge \Psi)$$

for all $\Theta, \Phi, \Psi \in \mathrm{Con}\mathcal{A}$. An algebra $\mathcal{A}$ with a constant 1 is *distributive at 1* if

$$[1]_{\Theta\wedge(\Phi\vee\Psi)} = [1]_{(\Theta\wedge\Phi)\vee(\Theta\wedge\Psi)}$$

for all $\Theta, \Phi, \Psi \in \mathrm{Con}\mathcal{A}$.

It is evident that if an algebra $\mathcal{A}$ with a constant 1 is weakly regular and distributive at 1 then it is congruence distributive.

Theorem 7 *The variety $\mathcal{IB}$ is weakly regular, congruence 3-permutable and congruence distributive.*

Proof By the theorem of Csákány (see e.g. Theorem 6.4.3 in [2]), a variety is weakly regular if and only if there exist binary terms $t_1(x, y), \ldots, t_n(x, y)$ $(n \geq 1)$ such that $t_1(x, y) = \cdots = t_n(x, y) = 1$ if and only if $x = y$. In $\mathcal{IB}$ we can take $n = 2$ and $t_1(x, y) = x \circ y$, $t_2(x, y) = y \circ x$. Then clearly $t_1(x, x) = t_2(x, x) = x \circ x = 1$ and, if $t_1(x, y) = 1$ and $t_2(x, y) = 1$ then $x \leq y$ and $y \leq x$ whence $x = y$.

To prove distributivity at 1, by Theorem 8.3.2 in [2] we need only to find a binary term $t(x, y)$ in $\mathcal{IB}$ satisfying the identities

$$t(x, x) = t(1, x) = 1 \quad \text{and} \quad t(x, 1) = x.$$

By Definition 1 and Lemma 2, we can take $t(x, y) = y \circ x$. Using the fact that $\mathcal{IB}$ is weakly regular and distributive at 1, we conclude that $\mathcal{IB}$ is congruence distributive.

To prove 3-permutability of $\mathcal{IB}$, we need to find ternary terms $p_1(x, y, z)$, $p_2(x, y, z)$ such that

$$x = p_1(x, z, z), \quad p_1(x, x, z) = p_2(x, z, z), \quad p_2(x, x, z) = z$$

(see e.g. Theorem 3.1.18 in [2]). For this, we can take $p_1(x, y, z) = (z \circ y) \circ x$ and $p_2(x, y, z) = (x \circ y) \circ z$. Then $p_1(x, z, z) = (z \circ z) \circ x = 1 \circ x = x$, $p_1(x, x, z) = (z \circ x) \circ x = (x \circ z) \circ z = p_2(x, z, z)$ and $p_2(x, x, z) = (x \circ x) \circ z = 1 \circ z = z$. $\square$

Remark 3 Congruence distributivity of the variety $\mathcal{IB}$ can be shown also directly by using Jónsson terms. We can pick up $n = 3$ and $t_0(x, y, z) = x$, $t_1(x, y, z) = ((z \circ y) \circ (z \circ x)) \circ x$, $t_2(x, y, z) = ((y \circ z) \circ (x \circ z)) \circ z$ and $t_3(x, y, z) = z$. It is an easy exercise to verify the corresponding Maltsev condition.

3 Derived equivalential algebras

Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a basic algebra and $1 = \neg 0$. For $x, y \in A$ we define

$$x \sqcup y = (x \circ y) \wedge (y \circ x) = (\neg x \oplus y) \wedge (\neg y \oplus x).$$

The algebra $(A; \sqcup, 0)$ will be called the *derived equivalential algebra* of $\mathcal{A}$.

The concept of equivalential algebra was introduced formerly for the equivalential reducts of Heyting algebras in [9], see e.g. [8] for the complex setting. It was shown in [6] that this algebra can be described by three axioms:

(E1) $(x \cdot x) \cdot y = y$,

(E2) $((x \cdot y) \cdot z) \cdot z = (x \cdot z) \cdot (y \cdot z)$,

(E3) $((x \cdot y) \cdot ((x \cdot z) \cdot z)) \cdot ((x \cdot z) \cdot z) = x \cdot y$.

Unfortunately, if we consider our derived equivalential algebra defined above, the axioms (E2), (E3) are violated as it can be shown in the following example.

Example 1 Let us consider the four element chain basic algebra $(A; \oplus, \neg, 0)$, where $A = \{0, a, b, 1\}$ with $0 < b < a < 1$ and the operations $\oplus$ and $\neg$ are given by the tables

$\oplus$	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	1	1
b	b	1	1	a

$\neg$	0	1	a	b
	1	0	b	a

.

Then for the operation $\sqcup$ we have

$\sqcup$	0	1	a	b
0	1	0	b	a
1	0	1	a	b
a	b	a	1	a
b	a	b	a	1

and hence

$$(0 \sqcup a) \sqcup (a \sqcup a) = b \sqcup 1 = b \neq 1 = a \sqcup a = (b \sqcup a) \sqcup a = ((0 \sqcup a) \sqcup a) \sqcup a.$$

Thus (E2) does not hold in A.
Similarly,

$$\begin{aligned} 0 \sqcup 1 = 0 \neq a = b \sqcup a = (0 \sqcup a) \sqcup a = (0 \sqcup (b \sqcup a)) \sqcup (b \sqcup a) \\ = ((0 \sqcup 1) \sqcup ((0 \sqcup a) \sqcup a)) \sqcup ((0 \sqcup a) \sqcup a), \end{aligned}$$

thus (E3) is also violated.

Lemma 4 *Let $(A; \oplus, \neg, 0)$ be a basic algebra and $(A; \square, 0)$ its derived equivalential algebra. If $x, y \in A$ such that $x \leq y$ then $(x \square y) \square x = y$.*

Proof Let $x, y \in A$ such that $x \leq y$. Then by Lemma 3 $x \circ y = 1$ and hence

$$x \square y = (x \circ y) \wedge (y \circ x) = y \circ x.$$

Since $x \leq y \circ x$ by Lemma 3, we have $x \circ (y \circ x) = 1$. By Theorem 3

$$(x \square y) \square x = ((y \circ x) \circ x) \wedge (x \circ (y \circ x)) = (y \vee x) \wedge 1 = y \wedge 1 = y.$$

□

Let us note that the converse of Lemma 4 does not hold in general as it is shown in Example 2 below.

Now we are going to describe basic properties of the operation $\square$.

Lemma 5 *Let $(A; \oplus, \neg, 0)$ be a basic algebra, $(A; \square, 0)$ its derived equivalential algebra and $x, y, z \in A$. Then*

(a) $x \square y = y \square x$,

(b) $x \square 0 = \neg x$,

(c) $(0 \square x) \square 0 = x$,

(d) $x \square 1 = x$,

(e) $x \square x = 1$,

(f) *if* $z \leq x \leq y$ *then* $y \square z \leq x \square z$,

where $1 = \neg 0$.

Proof (a): Obviously by the definition of $\square$ and commutativity of $\wedge$.
(b): $x \square 0 = (\neg x \oplus 0) \wedge (\neg 0 \oplus x) = \neg x \wedge (1 \oplus x) = \neg x \wedge 1 = \neg x$.
(c): $(0 \square x) \square 0 = \neg x \square 0 = \neg\neg x = x$.
(d): $x \square 1 = (\neg x \oplus 1) \wedge (\neg 1 \oplus x) = 1 \wedge x = x$.
(e): By Lemma 1, $x \square x = \neg x \oplus x = 1$.
(f): If $z \leq x \leq y$ then $z \circ x = 1$ and $z \circ y = 1$ and therefore $x \square z = x \circ z$, $y \square z = y \circ z$. Using Lemma 3 we obtain $y \square z = y \circ z \leq x \circ z = x \square z$. □

Remark 4 Consider a chain basic algebra $(A; \oplus, \neg, 0)$ and elements $x, y \in A$. We have either $x \leq y$ or $y \leq x$, thus either $x \circ y = 1$ or $y \circ x = 1$ and hence $x \square y = y \circ x$ in the first case and $x \square y = x \circ y$ in the second one.

Theorem 8 *Let $(A; \oplus, \neg, 0)$ be a chain basic algebra and $(A; \square, 0)$ its derived equivalential algebra and $x, y \in A$. Then*

(i) $x = 1$ *if and only if* $x \square x = x$.

(ii) *if* $x \neq 1$ *then* $x \leq y$ *if and only if* $(x \square y) \square x = y$.

Proof By (e) of Lemma 5 we infer (i). At first, let $x \neq 1$ and assume $x \circ y = y$. Then $x \vee y = (x \circ y) \circ y = y \circ y = 1$ and, due to the fact that $(A; \leq)$ is a chain, we conclude $y = 1$. For (ii), by Lemma 4 it is sufficient to prove that for $x \neq 1$, the implication $(x \mathbin{\square} y) \mathbin{\square} x = y \Rightarrow x \leq y$ holds.

Assume that $x \neq 1$ and $x \nleq y$, i.e. $y < x$. If $x \circ y = y$, then $y = 1$ as shown above, a contradiction with $y < x$. Hence $x \circ y \neq y$. According to Remark 4, $x \mathbin{\square} y = x \circ y$. Hence

$$(x \mathbin{\square} y) \mathbin{\square} x = (x \circ y) \mathbin{\square} x = ((x \circ y) \circ x) \wedge (x \circ (x \circ y)).$$

Then either $x \leq x \circ y$ or $x \circ y \leq x$. In the first case, $x \circ (x \circ y) = 1$ and by Lemma 3

$$(x \mathbin{\square} y) \mathbin{\square} x = (x \circ y) \circ x \geq x > y,$$

so $(x \mathbin{\square} y) \mathbin{\square} x \neq y$. In the second case, $(x \circ y) \circ x = 1$. Since $x \neq 1$, thus by Lemma 3 $(x \mathbin{\square} y) \mathbin{\square} x = x \circ (x \circ y) \geq x \circ y > y$. □

Remark 5 (a) Let us note that if $x = 1$ then by Lemma 5 $(1 \mathbin{\square} y) \mathbin{\square} 1 = y$ for any $y \in A$. Hence, the assumption $x \neq 1$ cannot be avoided in (ii) of Theorem 8.
(b) Theorem 8 shows that for a chain basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ we are able to reconstruct the induced partial partial order of $\mathcal{A}$ from the derived equivalential algebra $(A; \square, 0)$. The element 1 is then the greatest one in $(A; \leq)$ and the partial order of other elements is described by (ii) of Theorem 8.
(c) The result of Theorem 8 cannot be reformulated for a basic algebra which is a direct (or a subdirect) product of chain basic algebras, see the following example.

Example 2 Consider a basic algebra $\mathcal{A} = (\{0, a, b, \neg a, \neg b, 1\}; \oplus, \neg, 0)$ as shown in Fig. 1 which is the direct product of chain basic algebras $\underline{\mathbf{3}} \times \underline{\mathbf{2}}$.

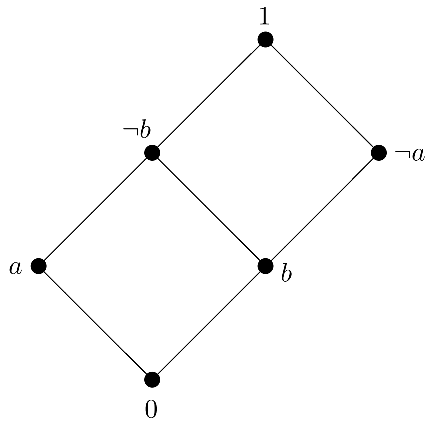

Fig. 1

The operation $\square$ in the derived equivalential algebra $(A; \square, 0)$ is given by the table

$\square$	0	a	b	$\neg b$	$\neg a$	1
0	1	$\neg a$	$\neg b$	b	a	0
a	$\neg a$	1	b	$\neg b$	0	a
b	$\neg b$	b	1	$\neg a$	$\neg b$	b
$\neg b$	b	$\neg b$	$\neg a$	1	b	$\neg b$
$\neg a$	a	0	$\neg b$	b	1	$\neg a$
1	0	a	b	$\neg b$	$\neg a$	1

We can see that $a \neq 1$ and $(a \square b) \square a = b \square a = b$, but $a \not\leq b$. It is a consequence of the fact that the representation of a in $\underline{\mathbf{3}} \times \underline{\mathbf{2}}$ is $(0,1)$, so in the second coordinate the assumption $x \neq 1$ is violated.

Lemma 6 *Let $\mathcal{A} = (A; \oplus, \neg, 0)$ be a chain basic algebra, $(A; \square, 0)$ its derived equivalential algebra and $1 = \neg 0$. Then $(A; \square, 0)$ satisfies:*

(g) if $x \neq 1$, $y \neq 1$, $(x \square y) \square x = y$ and $(y \square z) \square y = z$ then $(x \square z) \square x = z$;

(h) if $x \neq 1$, $y \neq 1$, $(x \square y) \square x = y$ and $(y \square x) \square y = x$ then $x = y$.

Proof To prove (g) we use Theorem 8, so $(x \square y) \square x = y$ and $(y \square z) \square y = z$ means that $x \leq y$ and $y \leq z$ thus $x \leq z$, i.e. $(x \square z) \square x = z$. Analogously for (h), $(x \square y) \square x = y$ and $(y \square x) \square y = x$ means $x \leq y$ and $y \leq x$, so $x = y$. $\square$

According to the properties of derived equivalential algebras as exhibited above we can introduce the following concept.

Definition 2 An algebra $\mathcal{E} = (E; \square, 0)$ of type $\langle 2, 0 \rangle$ satisfying:

(i) $(x \square x) \square y = y$;

(ii) if $x \neq 0 \square 0 \neq y$, $(x \square y) \square x = y$ and $(y \square z) \square y = z$ then $(x \square z) \square x = z$;

(iii) if $x \neq 0 \square 0 \neq y$, $(x \square y) \square x = y$ and $(y \square x) \square y = x$ then $x = y$;

(iv) $x \square y = y \square x$;

(v) $(0 \square x) \square 0 = x$;

(vi) $x \square x = y \square y$;

(vii) if $z \leq x \leq y$ then $y \square z \leq x \square z$;

will be called a *b-equivalential algebra*.

Remark 6 Due to Lemma 5 and 6 for any chain basic algebra $\mathcal{A} = (A; \oplus, \neg, 0)$ the derived equivalential algebra of $\mathcal{A}$ is a b-equivalential algebra.

Theorem 9 *Let $\mathcal{E} = (E; \square, 0)$ be a b-equivalential algebra. Define a binary relation $\leq$ on E as follows:*

(A) $x \leq 0 \square 0$ for each $x \in E$;

(B) if $0 \square 0 \leq x$ then $x = 0 \square 0$;

(C) if $x \neq 0 \square 0$ then $x \leq y$ if and only if $(x \square y) \square x = y$.

Then $\leq$ is a partial order on E.

Proof First, we check reflexivity of $\leq$. For $x \neq 0 \sqcap 0$ using (i) we obtain $(x \sqcap x) \sqcap x = x$, which by (C) means $x \leq x$. For $x = 0 \sqcap 0$ we have by (A) $0 \sqcap 0 \leq 0 \sqcap 0$.

Now, to show antisymmetry of $\leq$ consider two cases. For $x \neq 0 \sqcap 0 \neq y$ such that $x \leq y$ and $y \leq x$ we have by (C) $(x \sqcap y) \sqcap x = y$ and $(y \sqcap x) \sqcap y = x$, thus by (iii) $x = y$. If $x = 0 \sqcap 0$ and $x \leq y$ and $y \leq x$ (which by (A) holds for each $y \in E$) then $y = 0 \sqcap 0$ by (B) thus also $x = y$.

To check transitivity of the relation we consider three cases. First, if $x \neq 0 \sqcap 0 \neq y$ and $x \leq y$ and $y \leq z$ then by (C) $(x \sqcap y) \sqcap x = y$ and $(y \sqcap z) \sqcap y = z$. Using (ii) we get $(x \sqcap z) \sqcap x = z$, so by (C) $x \leq z$. If $x = 0 \sqcap 0$ and $x \leq y$ and $y \leq z$, we obtain by double using of (B) that $y = z = 0 \sqcap 0$, which by (A) means that $x \leq z$. And the last, if $x \neq 0 \sqcap 0 = y$ and $x \leq y$ and $y \leq z$ then analogously by (B) we get $z = 0 \sqcap 0$ and by (A) $x \leq z$.

Altogether, the binary relation $\leq$ is a partial order on E. $\square$

In what follows, $\leq$ will be called the *induced partial order* of a b-equivalential algebra $\mathcal{E} = (E; \sqcap, 0)$. We show some properties of the induced partial order of $\mathcal{E}$.

Remark 7 Let us note that if part of (C) trivially holds even without the condition $x \neq 0 \sqcap 0 = 1$ in a non-trivial b-equivalential algebra (i.e. if $0 \neq 0 \sqcap 0$).

We can prove the following

Lemma 7 *Let $\mathcal{E} = (E; \sqcap, 0)$ be a b-equivalential algebra. Then 0 is its least element, the element $0 \sqcap 0$ is the greatest one and, moreover, the following holds:*

$$\text{if} \quad x \leq y \quad \text{then} \quad x \leq x \sqcap y.$$

Proof By Theorem 9 (C) and Definition 2 (v), 0 is the least and by Theorem 9 (A), 1 is the greatest element of $\mathcal{E}$. If $\mathcal{E}$ is a trivial algebra, i.e. $0 = 0 \sqcap 0$ then $x = 0 \sqcap 0$ and hence $x \leq y$ implies $y = x = x \sqcap y = 0 \sqcap 0$. In the non-trivial case we use Remark 7 and for $x, y \in E$ such that $x \leq y$ we compute

$$(x \sqcap (x \sqcap y)) \sqcap x = ((x \sqcap y) \sqcap x) \sqcap x = y \sqcap x = x \sqcap y,$$

which means that $x \leq x \sqcap y$. $\square$

In the following we denote by 1 the greatest element $0 \sqcap 0$ of a b-equivalential algebra $\mathcal{E} = (E; \sqcap, 0)$. Now we demonstrate how to reconstruct a chain basic algebra from a given b-equivalential algebra.

Theorem 10 *Let $\mathcal{E} = (E; \sqcap, 0)$ be a b-equivalential algebra and $\leq$ be its induced partial order. If this partial order is linear (i.e. $x \leq y$ or $y \leq x$ for every $x, y \in A$) then $\mathcal{E}$ can be converted into a chain basic algebra $\mathcal{A}(E) = (E; \oplus, \neg, 0)$, where $\neg x = x \sqcap 0$ and $\oplus$ is defined as follows*

$$x \oplus y := \begin{cases} \neg x \sqcap y, & \text{if } x \leq \neg y, \\ 1, & \text{if } \neg y \leq x. \end{cases}$$

Moreover, $\mathcal{E}$ is the derived equivalential algebra of $\mathcal{A}(E)$.

Proof Let $x \in E$. By (iv) and (v), we obtain

$$\neg\neg x = (x \sqcap 0) \sqcap 0 = (0 \sqcap x) \sqcap 0 = x,$$

which is (BA2). For (BA1) we compute $x \oplus 0 = \neg x \sqcap 0 = \neg\neg x = x$. Putting $z = 0$ in (vii), we obtain:

$$x \leq y \quad \Longrightarrow \quad \neg y \leq \neg x. \tag{$**$}$$

To check the axiom (BA3) let us consider two possible cases for $x, y \in E$.

(3.1) $y \leq x$

Then $\neg x \leq \neg y$, and hence $\neg x \oplus y = \neg\neg x \sqcap y = x \sqcap y$, therefore

$$\neg(\neg x \oplus y) \oplus y = \neg(x \sqcap y) \oplus y.$$

We can use the fact that for $y \leq x$ we have $y \leq x \sqcap y$ by Lemma 7. Hence $\neg(x \sqcap y) \leq \neg y$, thus

$$\neg(\neg x \oplus y) \oplus y = \neg\neg(x \sqcap y) \sqcap y = (x \sqcap y) \sqcap y = (y \sqcap x) \sqcap y = x.$$

Since $\neg y \oplus x = 1$ and hence

$$\neg(\neg y \oplus x) \oplus x = \neg 1 \oplus x = (1 \sqcap 0) \oplus x = 0 \oplus x = \neg 0 \sqcap x = 1 \sqcap x = x.$$

Together we conclude

$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x.$$

(3.2) $x \leq y$

By symmetry we compute analogously as in (3.1)

$$\neg(\neg x \oplus y) \oplus y = y = \neg(\neg y \oplus x) \oplus x,$$

which means that (BA3) holds in $\mathcal{A}(E)$.

It remains to check the identity (BA4). Let us consider two possibilities for elements $x, y, z \in E$.

(4.1) $x \leq \neg y$

The condition is equivalent to $y \leq \neg x$ by $(**)$, from which (using Lemma 7 and (iv)) we get $y \leq \neg x \sqcap y$ and further, using $(**)$, $\neg(\neg x \sqcap y) \leq \neg y$. Then

$$\begin{gathered}\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = \neg(\neg(\neg(\neg x \sqcap y) \oplus y) \oplus z) \oplus (x \oplus z) \\ = \neg(\neg(\neg\neg(\neg x \sqcap y) \sqcap y) \oplus z) \oplus (x \oplus z) = \neg(\neg((\neg x \sqcap y) \sqcap y) \oplus z) \oplus (x \oplus z) \\ = \neg(\neg((y \sqcap \neg x) \sqcap y) \oplus z) \oplus (x \oplus z) = \neg(\neg\neg x \oplus z) \oplus (x \oplus z) \\ = \neg(x \oplus z) \oplus (x \oplus z) = 1\end{gathered}$$

by the definition of $\oplus$.

(4.2) $\neg y \leq x$

Then $x \oplus y = 1$ and

$$\begin{aligned}\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) &= \neg(\neg(\neg 1 \oplus y) \oplus z) \oplus (x \oplus z)\\ &= \neg(\neg(0 \oplus y) \oplus z) \oplus (x \oplus z) = \neg(\neg y \oplus z) \oplus (x \oplus z).\end{aligned}$$

Now we need to discuss two subcases.

(4.2a) $x \leq \neg z$

That means $\neg y \leq x \leq \neg z$, thus

$$\neg y \oplus z = \neg\neg y \sqcap z = y \sqcap z$$

and

$$x \oplus z = \neg x \sqcap z.$$

Using $(**)$, we can rewrite the condition of (4.2a) as $z \leq \neg x \leq y$. By (vii) we obtain

$$y \sqcap z \leq \neg x \sqcap z,$$

thus

$$\neg(\neg x \sqcap z) \leq \neg(y \sqcap z).$$

We conclude

$$\begin{aligned}\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) &= \neg(\neg y \oplus z) \oplus (x \oplus z)\\ &= \neg(y \sqcap z) \oplus (\neg x \sqcap z) = 1.\end{aligned}$$

(4.2b) $\neg z \leq x$

Then we get $x \oplus z = 1$ and

$$\begin{aligned}\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) &= \neg(\neg y \oplus z) \oplus (x \oplus z)\\ &= \neg(y \sqcap z) \oplus 1 = 1.\end{aligned}$$

In both the cases we can see that (BA4) holds, thus $\mathcal{A}(E) = (A; \oplus, \neg, 0)$ is a basic algebra. Since the induced partial order of $(E; \sqcap, 0)$ is linear, $\mathcal{A}(E)$ is a chain basic algebra. Moreover, if $x \leq y$ or equivalently $\neg y \leq \neg x$, we have

$$(\neg x \oplus y) \wedge (\neg y \oplus x) = 1 \wedge (\neg\neg y \sqcap x) = y \sqcap x = x \sqcap y.$$

If $y \leq x$ then analogously

$$(\neg x \oplus y) \wedge (\neg y \oplus x) = (\neg\neg x \sqcap y) \wedge 1 = x \sqcap y.$$

Thus $\mathcal{E}$ is the derived equivalential algebra of $\mathcal{A}(E)$. $\square$

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