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Existence Results for First Order Impulsive Functional Differential Equations with State-Dependent Delay

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Abstract

In this paper we study the existence of solutions for impulsive differential equations with state dependent delay. Our results are based on the Leray–Schauder nonlinear alternative and Burton–Kirk fixed point theorem for the sum of two operators.

Key words: Differential equation, state-dependent delay, fixed point, impulses, infinite delay.

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1 Introduction

This paper deals with the existence of solutions to the initial value problems (IVP for short) for the impulsive differential equations of the form,

$$y'(t) = f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J = [0, b],\ t \neq t_k,\ k = 1, \dots, m, \tag{1}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, \dots, m, \tag{2}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{3}$$

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where $f\colon J \times \mathcal{B} \to \mathbb{R}$, $\rho\colon J \times \mathcal{B} \to \mathbb{R}$, $\phi \in \mathcal{B}$ are given functions, $I_k\colon \mathbb{R} \to \mathbb{R}$, $k = 1, \ldots, m$ are continuous functions, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$,

$$y(t_k^+) = \lim_{h \to 0^+} y(t_k + h) \quad \text{and} \quad y(t_k^-) = \lim_{h \to 0^-} y(t_k + h)$$

represent the right and left hand limits of $y(t)$ at $t = t_k$, $k = 1, \ldots, m$, and $\mathcal{B}$ is an abstract *phase space*, to be specified later. For any function y and any $t \in [0, b]$, we denote by y_t the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t + \theta)$ for $\theta \in (-\infty, 0]$. We assume that the histories y_t belong to $\mathcal{B}$.

Impulsive differential equations appear frequently in applications because many evolutionary process from fields as physics, aeronautic, economics, engineering, populations dynamics, etc. In this way they makes changes of states at certain moments of time. Such changes can be reasonably well approximated as being instantaneous changes of this state which we will represented by impulses in our work and then these processes are modeled by impulsive differential equations and for this reason the study of this type of equations has received great attention in the last years. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Bainov and Simeonov [7], Benchohra et al. [8], Lakshmikantham et al. [23], and Samoilenko and Perestyuk [27]. Other works for impulsive differential equations with state dependent delay are [1, 6, 20]. On other hand, there exists a extensive literature devoted to the case where the impulses are absent (i.e. $I_k = 0$, $k = 1, \ldots, m$), see for instance [2, 3, 4, 5, 10, 11, 12, 14, 15, 16, 17, 18, 21, 22, 23, 24, 25, 28, 29].

The study of partial differential equations with state dependent delay have been initiated recently, and concerning this matter we cite the pioneer works Rezounenko et al. [26].

This paper is organized as follows: in Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3 we give two results, the first one is based on Leray–Schauder's alternative and the second one is based on a fixed point theorem of Burton and Kirk [9] for the sum of a contraction map and a completely continuous map. Finally in Section 4 we give an example to illustrate the theory presented in the previous sections.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)|\colon t \in J\}.$$

For $\psi \in \mathcal{B}$ the norm of ψ is defined by

$$\|\psi\|_{\mathcal{B}} = \sup\{|\psi(\theta)|\colon \theta \in (-\infty, 0]\}.$$

$L^1(J,\mathbb{R})$ denotes the Banach space of measurable functions $y\colon J \longrightarrow \mathbb{R}$ which are Lebesgue integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

$AC([a,b],\mathbb{R})$ denotes the space of absolutely continuous functions $y\colon [a,b] \to \mathbb{R}$.

(A_1) If $y\colon (-\infty, b) \to \mathbb{R}, b > 0$, $y_0 \in \mathcal{B}$, and $y(t_k^-)$ and $y(t_k^+)$, $k = 1,\dots,m$ exist with $y(t_k^-) = y(t_k)$, $k = 1,\dots,m$ then for every $t \in [0,b)$ the following conditions hold:
(i) $y_t \in \mathcal{B}$;
(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
(iii) There exist two functions $K(\cdot), M(\cdot)\colon\ R_+ \to\ R_+$, independent of y, with K continuous and M locally bounded such that:

$$\|y_t\|_{\mathcal{B}} \leq K(t)\sup\{\ |y(s)|\colon 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

(A_2) The space $\mathcal{B}$ is complete.

Denote

$$K_b = \sup\{K(t)\colon t \in [0,b]\}$$

and

$$M_b = \sup\{M(t)\colon t \in [0,b]\}.$$

3 Existence of solutions

Consider the following space

$$PC(J,\mathbb{R}) = \Big\{y\colon [0,b] \to \mathbb{R}\colon y \text{ is continuous at } t \neq t_k,\ y(t_k^-) = y(t_k) \text{ and } y(t_k^+) \text{ exists, for all } k = 1,\dots,m\Big\}$$

$PC(J,\mathbb{R})$ is a Banach space with norm

$$\|y\|_{PC} = \sup_{t\in J}|y(t)|.$$

Set

$$\mathcal{B}_b = \{y\colon (-\infty,b] \to \mathbb{R}\colon y|_{(-\infty,0]} \in \mathcal{B},\ y|_J \in PC(J,\mathbb{R})\},$$

and let $\|.\|_b$ the seminorm in $\mathcal{B}_b$ defined by

$$\|y\|_b = \|y_0\|_{\mathcal{B}} + \sup\{|y(t)|\colon 0 \leq t \leq b\}, y \in \mathcal{B}_b.$$

Set

$$J' := J\backslash\{t_1, t_2, \dots, t_m\}.$$

We define a solution to the problem (1)–(3) as follows:

Definition 3.1 A function $y \in \mathcal{B}_b$ is called a solution for (1)–(3) if y satisfies (1)–(3).

We have the following result which is useful in what follows.

Lemma 3.1 *Let $h\colon J \to \mathbb{R}$ be continuous. A function y is a solution of the integral equation*

$$y(t) = \begin{cases} \phi(t) & \text{if } t \in (-\infty, 0], \\ \phi(0) + \int_0^t h(s)ds + \sum_{0<t_k<t} I_k(y(t_k^-)), & \text{if } t \in (0, b], \end{cases} \tag{4}$$

if and only if y is a solution of the fractional IVP

$$y'(t) = h(t), \text{ for each, } t \in J', \tag{5}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \; k = 1, \dots, m, \tag{6}$$

$$y(0) = y_0. \tag{7}$$

We will need to introduce the following hypotheses

(Hϕ) The function $t \to \phi_t$ is continuous from

$$\mathcal{R}(\rho^-) = \{\rho(s, \varphi)\colon (s, \varphi) \in J \times \mathcal{B}, \rho(s, \varphi) \leq 0\}$$

into $\mathcal{B}$ and there exists a continuous and bounded function $L^\phi\colon \mathcal{R}(\rho^-) \to (0, \infty)$ such that $\|\phi_t\|_\mathcal{B} \leq L^\phi(t)\|\phi\|_\mathcal{B}$ for every $t \in \mathcal{R}(\rho^-)$.

(H1) The function $f\colon J \times \mathcal{B} \to \mathbb{R}$ is of Carathéodory's type;

(H2) There exist $p \in L^1(J, \mathbb{R}_+)$ and $\psi\colon [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$|f(t, u)| \leq p(t)\psi(\|u\|_\mathcal{B}) \text{ for each } t \in J \text{ and all } u \in \mathcal{B}.$$

(H3) The functions $I_k, \; k = 1, \dots, m$ are continuous and there exists $\psi_1\colon [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$|I_k(u)| \leq \psi_1(|u|) \text{ for each } u \in \mathbb{R}.$$

(H4) There is a constant $M > 0$ such that

$$\frac{M}{K_b\psi(M)\int_0^b p(s)ds + m\psi_1(M) + M_b\|\phi\|_\mathcal{B} + K_b|\phi(0)|} > 1.$$

The next result is a consequence of the phase space axioms.

Lemma 3.2 ([19, Lemma 2.1]) *If* $y\colon(-\infty,b]\to\mathbb{R}$ *is a function such that* $y_0=\phi$ *and* $y|_J\in PC(J,\mathbb{R})$*, then*

$$\|y_s\|_{\mathcal{B}}\le (M_b+L^{\phi})\|\phi\|_{\mathcal{B}}+K_b\sup\{\|y(\theta)\|;\ \theta\in[0,\max\{0,s\}]\},\quad s\in\mathcal{R}(\rho^-)\cup J,$$

where

$$L^{\phi}=\sup_{t\in\mathcal{R}(\rho^-)}L^{\phi}(t).$$

Remark 3.1 We remark that condition (H_ϕ) is satisfied by functions which are continuous and bounded. In fact, if the space $\mathcal{B}$ satisfies axiom C_2 in [21] then there exists a constant $L>0$ such that $\|\phi\|_{\mathcal{B}}\le L\sup\{\|\phi(\theta)\|\colon\theta\in[-\infty,0]\}$ for every $\phi\in\mathcal{B}$ that is continuous and bounded (see [21] Proposition 7.1.1) for details. Consequently,

$$\|\phi_t\|_{\mathcal{B}}\le L\frac{\sup_{\theta\le 0}\|\phi(\theta)\|}{\|\phi\|_{\mathcal{B}}}\|\phi\|_{\mathcal{B}},\quad \text{for every } \phi\in\mathcal{B}\backslash\{0\}.$$

Theorem 3.1 *Assume that the hypotheses (H1)–(H3) and* (H_ϕ) *hold. Then the problem (1)–(3) has at least one solution on* $(-\infty,b]$*.*

Proof The proof will be given in several steps.

Define the operator $N\colon\mathcal{B}_b\to\mathcal{B}_b$ by:

$$N(y)(t)=\begin{cases}\phi(t), & \text{if } t\in(-\infty,0],\\ \phi(0)+\int_0^t f(s,y_{\rho(s,y_s)})\,ds+\sum_{0<t_k<t}I_k(y(t_k^-)), & \text{if } t\in(0,b].\end{cases}\tag{8}$$

Let $x(.)\colon(-\infty,b]\to\mathbb{R}$ be the function defined by

$$x(t)=\begin{cases}\phi(t), & \text{if } t\in(-\infty,0],\\ \phi(0), & \text{if } t\in(0,b].\end{cases}$$

Then $x_0=\phi$. For each $z\in\mathcal{B}_b$ with $z_0=0$, we denote by $\overline{z}$ the function defined by

$$\overline{z}(t)=\begin{cases}0, & \text{if } t\in(-\infty,0],\\ z(t), & \text{if } t\in(0,b].\end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t)=\phi(0)+\int_0^t f(s,y_{\rho(s,y_s)})\,ds+\sum_{0<t_k<t}I_i(y(t_k^-)),$$

we can decompose $y(.)$ into $y(t)=\overline{z}(t)+x(t)$, $0\le t\le b$, which implies $y_t=\overline{z}_t+x_t$, for every $t\in[0,b]$, and the function $z(\cdot)$ satisfies

$$z(t)=\int_0^t f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})\,ds+\sum_{0<t_k<t}I_k(z(t_k^-)+x(t_k^-)).$$

Set

$$C=\{z\in\mathcal{B}_b\colon z_0=0\}.$$

Let $\|.\|_0$ be the norm in C defined by

$$\|z\|_0=\|z_0\|_{\mathcal{B}}+\sup\{|z(s)|\colon 0\le s\le b\}=\sup\{|z(s)|\colon 0\le s\le b\}.$$

We define the operator $P\colon C\to C$ by

$$P(z)(t)=\int_0^t f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})\,ds+\sum_{0<t_k<t} I_k(z(t_k^-)+x(t_k^-)).$$

Obviously the operator N has a fixed point is equivalent to P has one, so we need to prove that P has a fixed point. We shall use the nonlinear alternative of Leray–Schauder type [13].

Step 1: *P is continuous*
Let $\{z_n\}$ be a sequence such that $z_n\to z$ in C. Then

$$\begin{aligned}|P(z_n)(t)-P(z)(t)|\le\int_0^t &|f(s,\overline{z_n}_{\rho(s,\overline{z_n}_s+x_s)}+x_{\rho(s,\overline{z_n}_s+x_s)})\\ &-f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})|\,ds\\ &+\sum_{0<t_k<t}|I_k(z_n(t_k^-)+x(t_k^-))-I_k(z(t_k^-)+\phi(0))|.\end{aligned}$$

Since I_k, $k=1,\dots,m$ are continous and f is a Carathéodory function, we have by the Lebesgue dominated convergence theorem

$$\|P(z_n)-P(z)\|_0\to 0\ \text{ as } n\to\infty.$$

Step 2: *P maps bounded sets into bounded sets in C.*
Indeed, it is enough to show that for any $\eta>0$, there exists a positive constant ℓ such that for each $z\in B_\eta=\{z\in C\colon \|z\|_0\le\eta\}$, by (H2) we have for each $t\in[0,b]$,

$$\begin{aligned}&\|P(z)(t)\|\le\\ &\le\int_0^t\|f(s,\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)})\|\,ds+\sum_{0<t_k<t}|I_k(z(t_k^-)+\phi(0))|\\ &\le\int_0^t p(s)\psi(\|\overline{z}_{\rho(s,\overline{z}_s+x_s)}+x_{\rho(s,\overline{z}_s+x_s)}\|)\,ds+m\psi_1(\eta+\phi(0))\\ &\le\psi(K_b\eta+K_b|\phi(0)|+M_b\|\phi\|_{\mathcal{B}})\int_0^t p(s)ds+m\psi_1(\eta+\phi(0))\\ &\le\psi(K_b\eta+K_b|\phi(0)|+M_b\|\phi\|_{\mathcal{B}})\int_0^b p(s)ds+m\psi_1(\eta+\phi(0))\\ &=l.\end{aligned}$$

Step 3: *N maps bounded sets into equicontinuous sets of C_0.*

Let $l_1, l_2 \in [0,b]$, $l_1 < l_2$, let B_η a bounded set of C as in Claim 2, and let $z \in B_\eta$. Then,

$$\begin{aligned}
&|P(z)(l_2) - P(z)(l_1)| \leq \\
&\leq \int_{l_1}^{l_2} |f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})|\, ds + \sum_{0<t_k<l_2-l_1} |I_k(z(t_k^-) + \phi(0))| \\
&\leq \int_{l_1}^{l_2} |f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})|\, ds + \sum_{0<t_k<l_2-l_1} \psi_1(|z(t_k^-) + \phi(0))| \\
&\leq \psi(K_b\eta + K_b|\phi(0)| + M_b\|\phi\|_{\mathcal{B}})) \int_{l_1}^{l_2} p(s)\, ds + \sum_{0<t_k<l_2-l_1} \psi_1(|z(t_k^-) + \phi(0))|.
\end{aligned}$$

As $l_1 \to l_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claims 1 to 3 together with the Arzelá–Ascoli theorem, we can conclude that P is continuous and completely continuous.

Step 4: *A priori bounds.*

Let z be a possible solution of the equation $z = \lambda P(z)$ for some $\lambda \in (0,1)$. Then for each $t \in [0,b]$, we have

$$|z(t)| \leq \int_0^t p(s)\psi(\|\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}})ds + \sum_{0<t_k<t} \psi_1(|z(t_k^-) + \phi(0))|.$$

But

$$\begin{aligned}
\|\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} &\leq \|\overline{z}_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} + \|x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}} \\
&\leq K(t)\sup\{|z(s)|\colon 0 \leq s \leq t\} + M(t)\|z_0\|_{\mathcal{B}} \\
&\quad + K(t)\sup\{|x(s)|\colon 0 \leq s \leq t\} + M(t)\|x_0\|_{\mathcal{B}} \\
&\leq K_b \sup\{|z(s)|\colon 0 \leq s \leq t\} + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|.
\end{aligned}$$

$$\begin{aligned}
|z(t)| &\leq \int_0^t P(s)\psi(\|\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}})\, ds + \sum_{0<t_k<t} \psi_1(|z(t_k^-) + \phi(0))| \\
&\leq \int_0^t P(s)\psi(K_b \sup\{|z(s)|\colon 0 \leq s \leq t\} + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|)\, ds + m\psi_1(\mu(t)).
\end{aligned}$$

thus

$$\begin{aligned}
&K_b|z(s)| + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)| \\
&\leq K_b \int_0^t P(s)\psi(K_b \sup_{0\leqslant s\leqslant t}\{|z(s)|) + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|)\, ds \\
&\quad + m\psi_1(\mu(t)) + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|.
\end{aligned}$$

We consider the function μ defined by

$$\mu(t) = \sup\{K_b|z(s)| + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|\colon 0 \leq s \leq t\}, \quad 0 \leq t \leq b.$$

$t^* \in [0, t]$ be such that

$$\mu(t) = K_b|z(t^*)| + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|.$$

By the previous inequality we have for $t \in [0, b]$

$$\mu(t) \leq K_b \int_0^t p(s)\psi(\mu(s))ds + m\psi_1(\mu(t)) + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|.$$

Thus

$$\frac{\|\mu\|_0}{K_b\psi(\|\mu\|_0)\int_0^b p(s)ds + m\psi_1(\|\mu\|_0) + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|} \leq 1. \tag{9}$$

From (9) and (H4) we have

$$\|\mu\|_0 \neq M.$$

Set

$$U = \{y \in C\colon \|y\|_0 < M + 1\}.$$

From the choice of U, there is no $y \in \partial U$ such that $y = \lambda P(y)$ for some $\lambda \in [0, 1]$. The nonlinear alternative of Leray–Schauder type implies that P has a fixed point, hence N has a fixed point which is a solution of problem (1)–(3). □

Our main result in this section is based upon the following fixed point theorem due to Burton and Kirk [9].

Theorem 3.2 *Let X be a Banach space, and $\mathcal{A}$, $\mathcal{D}\colon X \to X$ two operators satisfying:*

(i) $\mathcal{A}$ is a contraction, and

(ii) $\mathcal{D}$ is completely continuous.

Then either

(a) the operator equation $y = \mathcal{A}(y) + \mathcal{D}(y)$ has a solution, or

(b) the set

$$\mathcal{E} = \left\{u \in X : \lambda\mathcal{A}\left(\frac{u}{\lambda}\right) + \lambda\mathcal{D}(u) = u\right\}$$

is unbounded for $\lambda \in (0, 1)$.

Set

$$\mathcal{R}(\rho^-) = \{\rho(s, \varphi)\colon (s, \varphi) \in J \times \mathcal{B}, \rho(s, \varphi) \leq 0\}.$$

We always assume that $\rho\colon I \times \mathcal{B} \to (-\infty, b]$ is continuous. Additionally, we introduce following hypotheses:

(H5) There exist constants $d_k > 0,\ k = 1, \ldots, m$ with

$$K_b \sum_{k=1}^{m} d_k < 1 \quad \text{and} \quad \sum_{k=1}^{m} d_k < 1$$

such that

$$|I_k\,(y) - I_k\,(x)\,| \leq d_k \|y - x\|_{\mathcal{B}}, \quad \text{for each } y, x \in \mathcal{B}$$

$$\int_{c_1}^{\infty} \frac{du}{\psi(u)} > c_2 \int_0^b p\,(s)\,ds, \tag{10}$$

where

$$c_1 = \frac{K_b \sum\limits_{k=1}^{m} |I_k(0)| + M_b \|\phi\|_{\mathcal{B}} + K_b |\phi(0)|}{1 - K_b \sum\limits_{k=1}^{m} d_k} \tag{11}$$

and

$$c_2 = \frac{K_b}{1 - K_b \sum\limits_{k=1}^{m} d_k}. \tag{12}$$

Theorem 3.3 *Assume that (Hϕ), (H1), (H2) and (H5) hold. Then the problem (1)–(3) has at least one solution on $(-\infty, b]$.*

Proof Transform the problem (1)–(3) into a fixed point problem. Consider the operator N defined in the proof of Theorem 3.1. Let

$$\mathcal{B}_b^0 = \{x \in \mathcal{B}_b \colon x_0 = 0 \in \mathcal{B}\}.$$

For any $x \in \mathcal{B}_b^0$ we have

$$\|x\|_b = \|x_0\|_{\mathcal{B}} + \sup\{|x(s)| \colon 0 \leq s \leq b\} = \sup\{|x(s)| \colon 0 \leq s \leq b\}.$$

Thus $(\mathcal{B}_b^0, \|\cdot\|_b)$ is a Banach space. and define the operators $\mathcal{A}, \mathcal{D} \colon \mathcal{B}_b^0 \to \mathcal{B}_b^0$ by:

$$\mathcal{D}(z)(t) = \int_0^t f\left(s, \overline{z}_{\rho(s, z_s + x_s)} + z_{\rho(s, \overline{z}_s + x_s)}\right) ds, \quad t \in J \tag{13}$$

and

$$\mathcal{A}(z)(t) = \sum_{0 < t_k < t} I_k(z(t_k) + x(t_k)), \quad t \in J. \tag{14}$$

Obviously the operator N has a fixed point is equivalent to $\mathcal{A} + \mathcal{D}$ has one, so it turns to prove that $\mathcal{A} + \mathcal{D}$ has a fixed point. We shall show that the operators $\mathcal{A}$ and $\mathcal{D}$ satisfies all the conditions of Theorem 3.2. For better readability, we break the proof into a sequence of steps.

Step 1: *$\mathcal{D}$ is continuous.*

Let $\{z_n\}$ be a sequence such that $z_n \to z$ in $\mathcal{B}_b^0$. At first, we study the convergence of the sequences $(z^n_{\rho(s,z^n_s)})_{n\in\mathbb{N}}$, $s \in J$. If $s \in J$ is such that $\rho(s,z_s) > 0$ for every $n > N$. In the case, for $n > N$ we see that

$$\begin{aligned}\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} &\le \|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z^n_s)}\|_{\mathcal{B}} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} \\ &\le K_b\|z_n - z\|_{\mathcal{B}} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}},\end{aligned}$$

which prove that $z^n_{\rho(s,z^n_s)} \to x_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s,z_s) > 0$. Similarly, if $\rho(s,z_s) < 0$ and $n \in \mathbb{N}$ is such that $\rho(s,z^n_s) < 0$ for every $n > N$, we get

$$\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} = \|\phi_{\rho(s,z^n_s)} - \phi_{\rho(s,z_s)}\|_{\mathcal{B}} = 0$$

which also shows that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s,z_s) < 0$. Combining the previous arguments, we can prove that $z^n_{\rho(s,z^n_s)} \to \phi$ for every $s \in J$ such that $\rho(s,z_s) = 0$. Finally,

$$\begin{aligned}&|\mathcal{D}(z_n)(t) - \mathcal{D}(z)(t)| \\ &= \left|\int_0^t [f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+z_s)})]\, ds\right| \\ &\le \int_0^t \left|f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+z_s)})\right| ds \\ &\le \int_0^t \left|f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)})\right| ds \\ &+ \int_0^t \left|f(s, z_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+z_s)})\right| ds.\end{aligned}$$

We infer that $f(s, z^n_{\rho(s,z^n_s)}) \to f(s, z_{\rho(s,x_s)})$ as $n \to \infty$, for every $s \in J$. An application of the Lebesgue dominated convergence theorem implies that

$$\|\mathcal{D}(z_n) - \mathcal{D}(z)\|_b \to 0 \; as \; n \to \infty.$$

Thus $\mathcal{D}$ is continuous.

Step 2: *$\mathcal{D}$ maps bounded sets into bounded sets in $\mathcal{B}_b^0$.*

It is enough to show that for any $\eta > 0$ there exists a positive constant l such that for each $x \in B_\eta = \{z \in \mathcal{B}_b^0 \colon \|z\|_b \le \eta\}$ we have $\|\mathcal{D}\,(y)\|_b \le l$. So choose $z \in B_\eta$, then from Lemma 3.2 it follows that

$$\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}} \le K_b\eta + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)| = r_*.$$

Then we have for each $t \in J$

$$\begin{aligned}|\mathcal{D}(z)(t)| &= \left|\int_0^t f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})ds\right| \\ &\le \int_0^b p(s)\psi(\|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}).\end{aligned}$$

Then we have

$$\|\mathcal{D}(z)\|_b \leq \psi(r_*)\int_0^b p(s)\,ds := l.$$

Step 3: *$\mathcal{D}$ maps bounded sets into equicontinuous sets of $\mathcal{B}_b^0$.*

We consider B_η as in Step 2 and let $l_1, l_2 \in J\backslash\{t_1,\ldots,t_m\}$, $l_1 < l_2$.

$$\begin{aligned} &|\mathcal{D}(z)(l_2) - \mathcal{D}(z)(l_1)| \\ \leq &\left|\int_{l_1}^{l_2} f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\,ds\right| \leq \psi(r_*)\int_{l_1}^{l_2} p(s)\,ds \end{aligned}$$

As $\tau_1 \to \tau_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá–Ascoli theorem, we can conclude that $\mathcal{D}$ is continuous and completely continuous.

Step 4: *$\mathcal{A}$ is a contraction.*

Let $z_1, z_2 \in \mathcal{B}_b^0$. Then for $t \in J$

$$\begin{aligned} |\mathcal{A}(z_1)(t) - \mathcal{A}(z_2)(t)| &= \left|\sum_{0<t_k<t}(I_k(z_1(t_k)+x(t_k)) - I_k(z_2(t_k)+x(t_k))\right| \\ &\leq \sum_{k=1}^m d_k|z_1(t_k) - z_2(t_k)|. \end{aligned}$$

Then

$$\|\mathcal{A}(z_1) - \mathcal{A}(z_2)\|_b \leq \left(\sum_{k=1}^m d_k\right)\|z_1 - z_2\|_b\,.$$

Hence $\mathcal{A}$ is a contraction.

Step 5: *A priori bounds.*

Now it remains to show that the set

$$\mathcal{E} = \left\{z \in \mathcal{B}_b^0 : z = \lambda\mathcal{D}(z) + \lambda\mathcal{A}\left(\frac{z}{\lambda}\right) \quad \text{for some } 0 < \lambda < 1\right\}$$

is bounded.

Let $z \in \mathcal{E}$, then $z = \lambda\mathcal{D}(z) + \lambda\mathcal{A}\left(\frac{z}{\lambda}\right)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$,

$$z(t) = \lambda\int_0^t f(s, \overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)})\,ds + \lambda\sum_{0<t_k<t} I_k\left(\frac{z(t_k)}{\lambda} + x(t_k)\right).$$

This implies by (H2), (H5) that, for each $t \in J$, we have

$$
\begin{aligned}
|z(t)| &\leq \lambda \int_0^t p(s)\psi(\|\overline{z}_{\rho(s,\overline{z}_s+x_s)} + x_{\rho(s,\overline{z}_s+x_s)}\|_{\mathcal{B}})\, ds \\
&+ \lambda \sum_{k=1}^{m} \left| I_k\left(\frac{z(t_k)}{\lambda} + x(t_k)\right)\right| \\
&\leq \lambda \int_0^t p(s)\psi\left(K_b \sup\{|z(s)| \colon 0 \leq s \leq t\} + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|\right) ds \\
&+ \lambda \sum_{k=1}^{m} \left| I_k\left(\frac{z(t_k)}{\lambda} + \phi(0)\right) - I_k(0)\right| + \lambda \sum_{k=1}^{m} |I_k(0)| \\
&\leq \int_0^t p(s)\psi\left(K_b \sup\{|z(s)| \colon 0 \leq s \leq t\} + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|\right) ds \\
&+ \lambda \sum_{k=1}^{m} |I_k(0)| + \lambda \sum_{k=1}^{m} d_k\left(\frac{z(t_k)}{\lambda} + \phi(0)\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
&K_b|z(s)| + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)| \\
&\leq K_b \int_0^t p(s)\psi\left(K_b \sup\{|z(s)| \colon 0 \leq s \leq t\} + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|\right) ds \\
&+ K_b\lambda \sum_{k=1}^{m} |I_k(0)| + K_b\lambda \sum_{k=1}^{m} d_k\left(\frac{z(t_k)}{\lambda} + \phi(0)\right) + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|.
\end{aligned}
$$

We consider the function μ defined by

$$
\mu(t) = \sup\{K_b|z(s)| + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)| \colon 0 \leq s \leq t\},\ 0 \leq t \leq b.
$$

Let $t^* \in [0,t]$ be such that

$$
\mu(t) = K_b|z(t^*)| + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|.
$$

By the previous inequality we have for $t \in [0,b]$

$$
\begin{aligned}
\mu(t) &\leq K_b \int_0^t p(s)\psi(\mu(s))\, ds \\
&+ K_b \sum_{k=1}^{m} |I_k(0)| + K_b \sum_{k=1}^{m} d_k\left(\mu(t)\right) + M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|. \qquad (15)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(1 - K_b \sum_{k=1}^{m} d_k)\mu(t) &\leq K_b \int_0^t p(s)\psi(\mu(s))\, ds + K_b \sum_{k=1}^{m} |I_k(0)| \\
&+ M_b\|\phi\|_{\mathcal{B}} + K_b|\phi(0)|.
\end{aligned}
$$

Thus

$$\mu(t) \leq c_1 + c_2 \int_0^t p(s)\psi(\mu(s))\, ds. \tag{16}$$

Let us take the right hand-side of (16) as $v(t)$. Then we have

$$\mu(t) \leq v(t) \quad \text{for all } t \in J,$$

$$v(0) = c_1,$$

and

$$v'(t) = c_2 p(t)\psi(\mu(t)), \quad a.e.\ t \in J.$$

Using the nondecreasing character of ψ we get

$$v'(t) \leq c_2 p(t)\psi(v(t)), \quad a.e.\ t \in J,$$

that is

$$\frac{v'(t)}{\psi(v(t))} \leq c_2 p\,(t)\,, \quad a.e.\ t \in J.$$

Integrating from 0 to t we get

$$\int_0^t \frac{v'(s)}{\psi(v(s))}\, ds \leq c_2 \int_0^t p\,(s)\; ds.$$

By a change of variable and (10) we get

$$\int_{v(0)}^{v(t)} \frac{du}{\psi(u)} \leq c_2 \int_0^b p\,(s)\, ds < \int_{c_1}^{\infty} \frac{du}{\psi(u)}.$$

Hence there exists a constant N such that

$$\mu(t) \leq v(t) \leq N \quad \text{for all } t \in J.$$

Now from the definition of μ it follows that

$$\|z\|_b \leq N^* \quad \text{for all } x \in \mathcal{E}.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of Theorem 3.2 we deduce that $\mathcal{A} + \mathcal{D}$ has a fixed point which is a solution of (1)–(3). □

4 An Example

To apply our results, we consider the functional differential equation with state dependent delay of the form

$$y'(t) = p(t)b(y(t - \sigma(y(t)))), \quad t \in [0, b], \tag{17}$$

$$y(t) = \phi(t), \quad t \in (-\infty, 0], \tag{18}$$

$$\Delta y(t_i) = \int_{-\infty}^{t_i} \gamma_i(t_i - s)y(s)\,ds, \tag{19}$$

where $\gamma_i \in C([0,\infty),\mathbb{R})$, $\sigma \in C(\mathbb{R},[0,\infty))$, $0 < t_1 < t_2 < \ldots < t_n < b$, $p\colon [0,b] \to \mathbb{R}^+$, $a\colon \mathbb{R} \to \mathbb{R}$, and we assume the existence of positive constants b_1, b_2 such that $|b(t)| \le b_1|t| + b_2$ for every $t \in \mathbb{R}$.

Set $\gamma > 0$. For the phase space, we choose $\mathcal{B}$ to be defined by

$$\mathcal{B} = PC^\gamma = \{\phi \in PC((-\infty,0],\mathbb{R})\colon \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists}\}$$

with the norm

$$\|\phi\|_\gamma = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}|\phi(\theta)|,\ \phi \in PC^\gamma.$$

Set

$$\rho(t,\varphi) = t - \sigma(\varphi(0)),\quad (t,\varphi) \in J \times \mathcal{B},$$
$$f(t,\varphi) = p(t)b(\varphi(0)),\quad (t,\varphi) \in J \times \mathcal{B},$$
$$I_k(y(t_k)) = \int_{-\infty}^{t_i} \gamma_i(t_i - s)y(s)\,ds.$$

We can represent system (17)–(19) by the Cauchy problem (1)–(3). It is clear that (H1) and (H2) are satisfied with

$$|f(t,\varphi)| \le p(t)[b_1\|\varphi\|_{\mathcal{B}} + b_2] \quad \text{for all } (t,\varphi) \in I \times \mathcal{B}.$$

Theorem 4.1 *Let $\varphi \in \mathcal{B}$ be such that H_φ is valid and $t \to \varphi_t$ is continuous on $\mathcal{R}(\rho^-)$. Then there exists a solution of (17)–(19).*

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References

[1] Abada, N., Agarwal, R. P., Benchohra, M., Hammouche, H.: *Existence results for nondensely defined impulsive semilinear functional differential equations with state-dependent delay.* Asian-Eur. J. Math. **1**, 4 (2008), 449–468.

[2] Adimy, M., Bouzahir, H., Ezzinbi, K.: *Local existence and stability for some partial functional differential equations with unbounded delay.* Nonlinear Anal. **48** (2002), 323–348.

[3] Adimy, M, Ezzinbi, K.: *A class of linear partial neutral functional-differential equations with nondense domain.* J. Differential Equations **147** (1998), 285–332.

[4] Ait Dads, E., Ezzinbi, K.: *Boundedness and almost periodicity for some state-dependent delay differential equations.* Electron. J. Differential Equations **2002**, 67 (2002), 1–13.

[5] Aiello, W. G., Freedman, H. I., Wu, J.: *Analysis of a model representing stage-structured population growth with state-dependent time delay.* SIAM J. Appl. Math. **52**, 3 (1992), 855–869.

[6] Anguraj, A., Arjunan, M. M., Hernàndez, E. M.: *Existence results for an impulsive neutral functional differential equation with state-dependent delay.* Appl. Anal. **86**, 7 (2007), 861–872.

[7] Bainov, D. D., Simeonov, P. S.: Systems with Impulsive effect. *Horwood*, Chichister, 1989.

[8] Benchohra, M., Henderson, J., Ntouyas, S. K.: Impulsive Differential Equations and Inclusions. *Hindawi Publishing Corporation*, Vol 2, New York, 2006.

[9] Burton, T. A., Kirk, C.: *A fixed point theorem of Krasnoselskiii–Schaefer type*. Math. Nachr. **189** (1998), 23–31.

[10] Cao, Y., Fan, J., Gard, T. C.: *The effects of state-dependent time delay on a stage-structured population growth model.* Nonlinear Anal. **19**, 2 (1992), 95–105.

[11] Corduneanu, C., Lakshmikantham, V.: *Equations with unbounded delay.* Nonlinear Anal. **4** (1980), 831–877.

[12] Domoshnitsky, A., Drakhlin, M., Litsyn, E.: *On equations with delay depending on solution.* Nonlinear Anal. **49**, 5 (2002), 689–701.

[13] Granas, A., Dugundji, J.: Fixed Point Theory. *Springer*, New York, 2003.

[14] Hale, J., Kato, J.: *Phase space for retarded equations with infinite delay.* Funkc. Ekvac. **21** (1978), 11–41.

[15] Hale, J. K., Verduyn Lunel, S. M.: Introduction to Functional Differential Equations. Applied Mathematical Sciences 99, *Springer*, New York, 1993.

[16] Hartung, F.: *Linearized stability in periodic functional differential equations with state-dependent delays.* J. Comput. Appl. Math. **174** (2005), 201–211.

[17] Hartung, F.: *Parameter estimation by quasilinearization in functional differential equations with state-dependent delays: a numerical study.* Proceedings of the Third World Congress of Nonlinear Analysts, Part 7 (Catania, 2000), Nonlinear Anal. **47** (2001), 4557–4566.

[18] Hartung, F., Turi, J.: *Identification of parameters in delay equations with state-dependent delays.* Nonlinear Anal. **29** (1997), 1303–1318.

[19] Hernández, E., Prokopczyk, A., Ladeira, L.: *A note on partial functional differential equations with state-dependent delay.* Nonlinear Anal. Real World Appl. **7** (2006), 510–519.

[20] Hernández, E., Sakthivel, R., Tanaka Aki, S.: *Existence results for impulsive evolution differential equations with state-dependent delay.* Electron. J. Differential Equations **2008**, 28 (2008), 1–11.

[21] Hino, Y., Murakami, S., Naito, T.: Functional Differential Equations with Unbounded Delay. *Springer-Verlag*, Berlin, 1991.

[22] Kolmanovskii, V., Myshkis, A.: Introduction to the Theory and Applications of Functional-Differential Equations. *Kluwer Academic Publishers*, Dordrecht, 1999.

[23] Lakshmikantham, V., Bainov D. D., Simeonov, P. S.: Theory of Impulsive Differntial Equations. *World Scientific*, Singapore, 1989.

[24] Lakshmikantham, V., Wen, L., Zhang, B.: Theory of Differential Equations with Unbounded Delay. Mathematics and its Applications, *Kluwer Academic Publishers*, Dordrecht, 1994.

[25] Liang, J., Xiao, Ti-jun: *The Cauchy problem for non linear abstract fuctionnal differential with infinte delay.* Comput. Math. Appl. **40** (2000), 693–707.

[26] Rezounenko, A. V., Wu, J.: *A non-local PDE model for population dynamics with state-selective delay: Local theory and global attractors.* J. Comput. Appl. Math. **190** (2006), 99–113.

[27] Samoilenko, A. M., Perestyuk, N. A.: Impulsive Differential Equations. *World Scientific*, Singapore, 1995.

[28] Willé, D. R., Baker, C. T. H.: *Stepsize control and continuity consistency for state-dependent delay-differential equations.* J. Comput. Appl. Math. **53** (1994), 163–170.

[29] Wu, J.: Theory and Applications of Partial Functional Differential Equations. *Springer*, New York, 1996.