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New Result on the Ultimate Boundedness of Solutions of Certain Third-order Vector Differential Equations*

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Abstract

Sufficient conditions are established for ultimate boundedness of solutions of certain nonlinear vector differential equations of third-order. Our result improves on Tunc's [C. Tunc, On the stability and boundedness of solutions of nonlinear vector differential equations of third order].

Key words: Ultimate boundedness, Lyapunov function, differential equation of third order.

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1 Introduction

For over four decades much attention have been drawn to the ultimate boundedness of solutions of ordinary scaler and vector nonlinear differential equations of third-order. See [1–6,11–20] and the references cited therein for a comprehensive treatment of the subject. Throughout, the results presented in the book of Reissig et al. [14], Lyapunov's second (direct) method has been used as a basic tool to verify the results established in these works.

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Recently, Tunc [19] discussed the stability and boundedness results of the nonlinear vector differential equation

$$\dddot{X} + \Psi(\dot{X})\ddot{X} + B\dot{X} + cX = P(t) \tag{1.1}$$

or its equivalent system form

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= Z \\ \dot{Z} &= -\Psi(Y)Z - BY - cX + P(t), \end{aligned} \tag{1.2}$$

obtained as usual by setting $\dot{X} = Y$, $\ddot{X} = Z$ in (1.1), where $t \in \mathbb{R}^+ = (0, \infty)$, $X \in \mathbb{R}^n$, c is a positive constant and B is an $n \times n$-constant symmetric matrix, Ψ is an $n \times n$-continuous symmetric matrix function for the argument displayed explicitly and the dots indicate differentiation with respect to $t, P \colon \mathbb{R}^+ \to \mathbb{R}^n$. It is also assumed that P is continuous for the argument displayed explicitly. Moreover, the existence and the uniqueness of the solutions of (1.1) will be assumed (see Picard-Lindelof theorem in Rao [13]). Let $J(\Psi(Y)Y|Y)$ denote the linear operator from the vector $\Psi(Y)Y$ to the matrix

$$J(\Psi(Y)Y|Y) = \left(\frac{\partial}{\partial y_j}\sum_{k=1}^{n}\Psi_{ik}y_k\right) = \Psi(Y) + \left(\sum_{k=1}^{n}\frac{\partial \Psi_{ik}}{\partial y_j}y_k\right),$$

$(i, j = 1, 2, \ldots, n)$, where $(y_i, y_2, \ldots, y_n)$ and (Ψ_{ik}) are components of Y and Ψ, respectively. Besides, it is also assumed as basic throughout this paper that $J(\Psi(Y)Y|Y)$ exists, symmetric and continuous. Finally, it is assumed that $n \times n$-symmetric matrix B and $n \times n$-continuous symmetric matrix function Ψ commute with each other. Our motivation comes from the paper of Tunc [19], who obtained boundedness criteria for the solutions of (1.1). The boundedness criteria obtained by Tunc [19] is of the type in which the bounding constant depends on the solution in question (see [17]). This is because the Lyapunov function used in the proof of the boundedness result is not complete (see [4,12]).

Our aim in this paper is to further study the boundedness of solutions of Eq. (1.1). In the next section, we establish a criteria for the ultimate boundedness of solutions of Eq. (1.1), which improves on Tunc [19].

2 Main results

Before stating our main result, we give a well-known algebraic result required in the proof.

Lemma 1 *Let A be a real symmetric $n \times n$-matrix. Then for any $X \in \mathbb{R}^n$,*

$$\delta_a\|X\|^2 \le \langle AX, X\rangle \le \Delta_a\|X\|^2,$$

where δ_a and Δ_a are, respectively, the least and greatest eigenvalues of the matrix A.

Proof See [20].

Lemma 2

$$\frac{d}{dt}\int_0^1 \langle \sigma c\Psi(\sigma Y)Y, Y\rangle d\sigma = \langle c\Psi(Y)Y, Z\rangle.$$

Proof See [19].

Theorem 1 *In addition to the basic assumptions imposed on $\Psi(Y), B$ and c that appeared in the system (1.2), we suppose that there exist positive constants $\delta_0, \varepsilon, a_0, a_1, b_0, b_1$ such that the following conditions are satisfied,*

(i) $n \times n-$symmetric matrices B and Ψ commute with each other and

$$a_0b_0 - c > 0, \quad b_0 \le \lambda_i(B) \le b_1, \quad a_0 + \varepsilon \le \lambda_i(\Psi(Y)) \le a_1$$

for all $Y \in \mathbb{R}^n$,

(ii) $\|P(t)\| \le \delta_0$ for all $t \ge 0$.

Then, there exists a constant $d > 0$ such that any solution $(X(t), Y(t), Z(t))$ of the system (1.2) determined by

$$X(0) = X_0, \quad Y(0) = Y_0, \quad Z(0) = Z_0$$

ultimately satisfies

$$\|X(t)\|^2 + \|Y(t)\|^2 + \|Z(t)\|^2 \le d$$

for all $t \in \mathbb{R}^+$.

Proof Our main tool in the proof of the result is the Lyapunov function $V = V(X, Y, Z)$ defined for any $X, Y, Z \in \mathbb{R}^n$, by

$$\begin{aligned} 2V \;=\; & a_0c\langle X, X\rangle + a_0\int_0^1 \langle \sigma\Psi(\sigma Y)Y, Y\rangle d\sigma + \alpha a_0 b_0^2\langle X, X\rangle \\ & + \langle BY, Y\rangle + \langle Z, Z\rangle + 2\alpha b_0 a_0^2\langle X, Y\rangle + 2\alpha a_0 b_0\langle X, Z\rangle \\ & + 2a_0\langle Y, Z\rangle + 2c\langle X, Y\rangle - \alpha a_0 b_0\langle Y, Y\rangle, \end{aligned} \tag{2.1}$$

where

$$0 < \alpha < \min\left\{\frac{1}{a_0}, \frac{a_0}{b_0}, \frac{a_0b_0 - c}{a_0b_0[a_0 + c^{-1}(b_1 - b_0)^2]}, \frac{c}{a_0b_0(a_1 - a_0)}\right\}, \tag{2.2}$$

and $a_1 > a_0$, $b_1 \ne b_0$. This function, after re-arrangements, can be rewritten as

$$\begin{aligned} 2V \;=\; & a_0b_0\|a_0^{-\frac{1}{2}}Y + a_0^{-\frac{1}{2}}b_0^{-1}cX\|^2 + \|Z + a_0Y + \alpha a_0b_0X\|^2 \\ & + a_0\int_0^1 \langle \sigma\Psi(\sigma Y)Y, Y\rangle d\sigma - 2a_0^2\langle Y, Y\rangle + \langle (B - b_0I)Y, Y\rangle \\ & + \alpha a_0 b_0^2(1 - \alpha a_0)\langle X, X\rangle + c(a_0 - cb_0^{-1})\langle X, X\rangle + a_0(a_0 - \alpha b_0)\langle Y, Y\rangle. \end{aligned} \tag{2.3}$$

We can now verify the properties of this function. First, it is clear from (2.3) that

$$V(0,0,0)=0.$$

Next, in view of the assumptions of the Theorem and Lemma 1, respectively, it follows that

$$a_0\int_0^1\langle\sigma\Psi(\sigma Y)Y,Y\rangle d\sigma-2a_0^2\langle Y,Y\rangle=a_0\int_0^1\langle\sigma(\Psi(\sigma Y)-a_0I)Y,Y\rangle d\sigma\geq\varepsilon a_0\|Y\|^2,$$

and $\langle(B-b_0I)Y,Y\rangle\geq 0$. Also, in addition,

$$\alpha a_0b_0^2(1-\alpha a_0)\langle X,X\rangle=\mu_1\|X\|^2,\quad c(a_0-cb_0^{-1})\langle X,X\rangle=\mu_2\|X\|^2,$$

and

$$a_0(a_0-\alpha b_0)\langle Y,Y\rangle=\mu_3\|Y\|^2,$$

where

$$\mu_1=\alpha a_0b_0^2(1-\alpha a_0)>0,\quad \mu_2=c(a_0-cb_0^{-1})>0$$

and

$$\mu_3=a_0(a_0-\alpha b_0)>0$$

in view of (2.2).

Hence one can get from (2.3) that

$$\begin{aligned}V\ &\geq\ \frac{1}{2}a_0b_0\|a_0^{-\frac{1}{2}}Y+a_0^{-\frac{1}{2}}b_0^{-1}cX\|^2+\|Z+a_0Y+\alpha a_0b_0X\|^2\\&\quad+\frac{1}{2}(\mu_1+\mu_2)\|X\|^2+\frac{1}{2}\mu_3\|Y\|^2+\frac{1}{2}a_0\varepsilon\|Z\|^2\\&\geq\frac{1}{2}(\mu_1+\mu_2)\|X\|^2+\frac{1}{2}\mu_3\|Y\|^2+\frac{1}{2}a_0\varepsilon\|Z\|^2.\end{aligned}\tag{2.4}$$

Thus, it is evident from the terms contained in (2.4) that there exists d_1, sufficiently small enough, such that

$$V\geq d_1(\|X\|^2+\|Y\|^2+\|Z\|^2)\tag{2.5}$$

where $d_1=\frac{1}{2}\min\{\mu_1+\mu_2,\mu_3,a_0\varepsilon\}$.

Now, let $(X,Y,Z)=(X(t),Y(t),Z(t))$ be any solution of differential system (1.2). Differentiating the function $V=V(X(t),Y(t),Z(t))$ with respect to t along system (1.2) and using Lemma 2, we have

$$\begin{aligned}\dot{V}\ &=\ -\alpha a_0b_0c\langle X,X\rangle-\langle(a_0B-cI-\alpha a_0^2b_0I)Y,Y\rangle\\&\quad-\langle(\Psi(Y)-a_0I)Z,Z\rangle-\alpha a_0b_0\langle(\Psi(Y)-a_0I)X,Z\rangle\\&\quad-\alpha a_0b_0\langle(B-b_0I)X,Y\rangle+\langle\alpha a_0b_0X+a_0Y+Z,P(t)\rangle.\end{aligned}\tag{2.5}$$

This we can rewrite as

$$\begin{aligned}\dot{V} \;=\; & -\frac{1}{2}\alpha a_0 b_0 c\langle X, X\rangle - \langle (a_0 B - cI - \alpha a_0^2 b_0 I)Y, Y\rangle \\ & - \langle (\Psi(Y) - a_0 I)Z, Z\rangle + \langle \alpha a_0 b_0 X + a_0 Y + Z, P(t)\rangle \\ & - \frac{1}{4}\alpha a_0 b_0 \left(\langle cX, X\rangle + 4\langle (\Psi(Y) - a_0 I)X, Z\rangle\right) \\ & - \frac{1}{4}\alpha a_0 b_0 c \left(\langle cX, X\rangle + 4\langle (B - b_0 I)X, Y\rangle\right).\end{aligned}$$

Since

$$\begin{aligned}& \langle cX, X\rangle + 4\langle (\Psi(Y) - a_0 I)X, Z\rangle \\ &= \|c^{\frac{1}{2}}X + 2c^{-\frac{1}{2}}(\Psi(Y) - a_0 I)Z\|^2 - \|2c^{-\frac{1}{2}}(\Psi(Y) - a_0 I)Z\|^2\end{aligned}$$

and

$$\begin{aligned}& \langle cX, X\rangle + 4\langle (B - b_0 I)X, Y\rangle \\ &= \|c^{\frac{1}{2}}X + 2c^{-\frac{1}{2}}(B - b_0 I)Y\|^2 - \|2c^{-\frac{1}{2}}(B - b_0 I)Y\|^2,\end{aligned}$$

it follows that

$$\begin{aligned}\dot{V} \;=\; & -\frac{1}{2}\alpha a_0 b_0 c\langle X, X\rangle - \langle (a_0 B - cI - \alpha a_0^2 b_0 I)Y, Y\rangle \\ & - \langle (\Psi(Y) - a_0 I)Z, Z\rangle + \frac{1}{4}\alpha a_0 b_0 \|2c^{-\frac{1}{2}}(B - b_0 I)Y\|^2 \\ & + \frac{1}{4}\alpha a_0 b_0 \|2c^{-\frac{1}{2}}(\Psi(Y) - a_0 I)Z\|^2 + \langle \alpha a_0 b_0 X + a_0 Y + Z, P(t)\rangle.\end{aligned}$$

Using the fact that

$$\|2c^{-\frac{1}{2}}(B - b_0 I)Y\|^2 = 4\langle c^{-1}(B - b_0 I)Y, (B - b_0 I)Y\rangle$$

and

$$\|2c^{-\frac{1}{2}}(\Psi(Y) - a_0 I)Z\|^2 = 4\langle c^{-1}(\Psi(Y) - a_0 I)Z, (\Psi(Y) - a_0 I)Z\rangle,$$

we have that

$$\begin{aligned}& \dot{V} = -\frac{1}{2}\alpha a_0 b_0 c\langle X, X\rangle - \left\langle \left(a_0 B - cI - \alpha a_0 b_0 [a_0 I + c^{-1}(B - b_0)^2]\right) Y, Y\right\rangle \\ & -\left\langle \left((\Psi(Y) - a_0 I)[I - \alpha a_0 b_0 c^{-1}(\Psi(Y) - a_0 I)]\right) Z, Z\right\rangle + \langle \alpha a_0 b_0 X + a_0 Y + Z, P(t)\rangle.\end{aligned}$$

Next, in view of the assumptions of Theorem and Lemma 1, respectively, it follows that

$$\begin{aligned}\dot{V} \;\leq\; & -\frac{1}{2}\alpha a_0 b_0 c\|X\|^2 - \left((a_0 b_0 - c) - \alpha a_0 b_0 [a_0 + c^{-1}(b_1 - b_0)^2]\right) \|Y\|^2 \\ & - \varepsilon \left(1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)\right) \|Z\|^2 + (\alpha a_0 b_0 \|X\| + a_0 \|Y\| + \|Z\|)\|P(t)\| \\ \leq\; & -2d_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + \delta_0(\alpha a_0 b_0 \|X\| + a_0 \|Y\| + \|Z\|),\end{aligned}$$

where

$$d_2 = \frac{1}{2}\min\bigl\{\alpha a_0 b_0 c; 2[(a_0 b_0 - c) - \alpha a_0 b_0 (a_0 + c^{-1}(b_1 - b_0)^2)]; \\ 2\varepsilon[1 - \alpha a_0 b_0 c^{-1}(a_1 - a_0)]\bigr\} > 0$$

by (2.2). Furthermore,

$$\dot{V} \leq -2d_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + d_3(\|X\| + \|Y\| + \|Z\|)$$

where $d_3 = \delta_0 \max\{1, a_0, \alpha a_0 b_0\}$. Thus, by Schwarz's inequality,

$$\dot{V} \leq -2d_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2) + d_4(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}}$$

where $d_4 = 3^{\frac{1}{2}} d_3$.

If we choose

$$(\|X\|^2 + \|Y\|^2 + \|Z\|^2)^{\frac{1}{2}} \geq d_5 = D_4 d_2^{-1},$$

we have that

$$\dot{V} \leq -d_2(\|X\|^2 + \|Y\|^2 + \|Z\|^2). \tag{2.7}$$

Thus, there exists d_6 such that

$$\dot{V} \leq -1 \text{ if } \|X\|^2 + \|Y\|^2 + \|Z\|^2 \geq d_6^2.$$

The remainder of the proof of Theorem may now be obtained by use of the estimates (2.5) and (2.7) and an obvious adaptation of the Yoshizawa type reasoning in [12]. □

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