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On Common Fixed Point Theorems for Three and Four Self Mappings Satisfying Contractive Conditions

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Abstract

In this contribution, we discuss some unique common fixed point theorems for three and four occasionally weakly compatible mappings satisfying different types of contractive condition.

Key words: Occasionally weakly compatible mappings, contractive modulus function, common fixed point theorems.

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1 Introduction

In 1982, Sessa [8] introduced the concept of weakly commuting mappings which extends the notion of commuting mappings.

After four years, Jungck [3] defined compatible mappings as an extension of weakly commuting mappings.

Later on, the same author with Murthy and Cho [4] gave another extension of weakly commuting mappings under the name of compatible mappings of type (A).

Again, Pathak and Khan [7] extended compatible of type (A) mappings to compatible mappings of type (B).

On this direction, Pathak et al. [6] introduced the new concept i.e. compatible type of (C) as another extension of compatible type of (A) and proved a common fixed point theorem in a Banach space.

In their paper [5], Jungck and Rhoades defined the notion of weakly compatible mappings as an extension of all above notions.

And recently, Al-Thagafi and Shahzad [2] gave the concept of occasionally weakly compatible mappings which is more general than weakly compatible mappings and all above notions.

So on this way we prove some common fixed point theorems for three and four occasionally weakly compatible mappings satisfying different types of contractive conditions which improve the results given by Aage and Salunke [1].

2 Preliminaries

Definition 2.1 Self mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be weakly commuting pair if, for all $x \in \mathcal{X}$

$$d(fgx, gfx) \leq d(fx, gx).$$

Definition 2.2 Self mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be (1) compatible if,

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0,$$

(2) compatible of type (A) if,

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(gfx_n, f^2x_n) = 0,$$

(3) compatible of type (B) if,

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n)\right]$$

and

$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n)\right],$$

(4) compatible of type (C) if,

$$\lim_{n\to\infty} d(fgx_n, g^2x_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(fgx_n, ft) + \lim_{n\to\infty} d(ft, f^2x_n) + \lim_{n\to\infty} d(ft, g^2x_n)\right]$$

and

$$\lim_{n\to\infty} d(gfx_n, f^2x_n) \leq \frac{1}{3}\left[\lim_{n\to\infty} d(gfx_n, gt) + \lim_{n\to\infty} d(gt, g^2x_n) + \lim_{n\to\infty} d(gt, f^2x_n)\right],$$

whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$$

for some $t \in \mathcal{X}$.

Definition 2.3 Self mappings f and g of a metric space $(\mathcal{X}, d)$ are said to be weakly compatible if they commute at their coincidence points.

Definition 2.4 Two self mappings f and g of a set $\mathcal{X}$ are occasionally weakly compatible (shortly (owc)) iff, there is a point t in $\mathcal{X}$ which is a coincidence point of f and g at which f and g commute.

Definition 2.5 A function $\Phi\colon [0,\infty) \to [0,\infty)$ is said to be a contractive modulus if $\Phi(0) = 0$ and $\Phi(t) < t$ for $t > 0$.

Definition 2.6 A real valued function Φ defined on $\mathcal{X} \subset \mathbb{R}$ is said to be upper semi-continuous if $\lim\limits_{n\to\infty} \Phi(t_n) \le \Phi(t)$, for every sequence $\{t_n\}$ in $\mathcal{X}$ with $t_n \to t$ as $n \to \infty$.

In their paper [1] Aage and Salunke proved the following results:

Theorem 2.7 *Let f, g and h be self mappings of a complete metric space $(\mathcal{X}, d)$ and Φ is a contractive modulus function satisfying:*

(i) $f(\mathcal{X}) \cup g(\mathcal{X}) \subset h(\mathcal{X})$.

(ii)
$$d^2(fx, gy) \le \max\{\Phi(d(hx,hy))\Phi(d(hx,fx)), \Phi(d(hx,hy))\Phi(d(hy,fx)), \Phi(d(hx,hy))\Phi(d(hy,gy)), \Phi(d(hx,fx))\Phi(d(hy,gy)), \Phi(d(hx,gy))\Phi(d(hy,fx))\},$$

for all x, $y \in \mathcal{X}$,

(iii) *the pair (f,h) or (g,h) is compatible of type (A).*

(iv) *If h is continuous.*

Then f, g and h have a unique common fixed point.

Theorem 2.8 *Suppose f, g, h and k are four self mappings of a complete metric space $(\mathcal{X}, d)$ satisfying the conditions*

(i) $f(\mathcal{X}) \subset k(\mathcal{X})$, $g(\mathcal{X}) \subset h(\mathcal{X})$.

(ii)
$$d^2(fx, gy) \le \max\{\Phi(d(hx,ky))\Phi(d(hx,fx)), \Phi(d(hx,ky))\Phi(d(ky,gy)), \Phi(d(hx,fx))\Phi(d(ky,gy)), \Phi(d(hx,gy))\Phi(d(ky,fx))\},$$

for all x, $y \in \mathcal{X}$.

(iii) *Φ is a contractive modulus.*

(iv) *One of f, g, h and k is continuous.*

And if

(v) *the pairs (f,h) and (g,k) are compatible of type (A).*

Then f, g, h and k have a unique common fixed point.

The following example shows that occasionally weakly compatible mappings are not compatible of type (A) in general.

Example 2.9 Let $\mathcal{X} = [0, \infty[$ with the usual metric. Define $f, g\colon \mathcal{X} \to \mathcal{X}$ by:

$$fx = \begin{cases} 4 & \text{if } x \in [0,1[\\ x^4 & \text{if } x \in [1,\infty[, \end{cases} \qquad gx = \begin{cases} 3 & \text{if } x \in [0,1[\\ \frac{1}{x^4} & \text{if } x \in [1,\infty[. \end{cases}$$

We have $f(1) = g(1) = 1$ and $fg(1) = 1 = gf(1)$; that is, f and g are owc.

Now, consider the sequence $x_n = 1 + \frac{1}{n}$ for $n \in \{1, 2, \ldots\}$.

We have $fx_n = x_n^4 \to 1$ and $gx_n = \frac{1}{x_n^4} \to 1$ as $n \to \infty$. But,

$$d(fgx_n, ggx_n) \to 1 \neq 0.$$

Therefore, f and g are not compatible of type (A).

The next definition will be needed.

Definition 2.10 Let $\mathcal{X}$ be a set. A symmetric on $\mathcal{X}$ is a mapping $d\colon \mathcal{X} \times \mathcal{X} \to [0, \infty)$ such that $d(x, y) = 0$ iff $x = y$, and $d(x, y) = d(y, x)$ for x, y in $\mathcal{X}$.

3 Main Results

3.1 Common fixed point for three mappings

Theorem 3.1 *Let $\mathcal{X}$ be a set with a symmetric d. Let f, g and h be three self mappings of $(\mathcal{X}, d)$ and Φ is a contractive modulus function satisfying:*

$$\begin{aligned} d^2(fx, gy) \leq \max\{&\Phi(d(hx,hy))\Phi(d(hx,fx)), \\ &\Phi(d(hx,hy))\Phi(d(hy,fx)), \Phi(d(hx,hy))\Phi(d(hy,gy)), \\ &\Phi(d(hx,fx))\Phi(d(hy,gy)), \Phi(d(hx,gy))\Phi(d(hy,fx))\}, \end{aligned} \quad (3.1)$$

for all x, $y \in \mathcal{X}$,

$$\textit{the pair } (f, h) \textit{ or } (g, h) \textit{ is owc.} \quad (3.2)$$

Then f, g and h have a unique common fixed point.

Proof Suppose that f and h are owc, then, there is an element $u \in \mathcal{X}$ such that $fu = hu$ and $fhu = hfu$.

First, we prove that $gu = fu = hu$. Indeed, by inequality (3.1), we get

$$\begin{aligned} d^2(fu, gu) \leq \max\{&\Phi(d(hu,hu))\Phi(d(hu,fu)), \Phi(d(hu,hu))\Phi(d(hu,fu)), \\ &\Phi(d(hu,hu))\Phi(d(hu,gu)), \Phi(d(hu,fu))\Phi(d(hu,gu)), \\ &\Phi(d(hu,gu))\Phi(d(hu,fu))\}; \end{aligned}$$

i.e., $d^2(fu, gu) \leq 0$, which implies that $d(fu, gu) = 0$ i.e. $fu = gu = hu$.

Now, suppose that $gfu \neq ffu$. By (3.1) we get

$$\begin{aligned} d^2(ffu, gfu) \leq \max\{&\Phi(d(hfu,hfu))\Phi(d(hfu,ffu)), \\ &\Phi(d(hfu,hfu))\Phi(d(hfu,ffu)), \\ &\Phi(d(hfu,hfu))\Phi(d(hfu,gfu)), \\ &\Phi(d(hfu,ffu))\Phi(d(hfu,gfu)), \\ &\Phi(d(hfu,gfu))\Phi(d(hfu,ffu))\}; \end{aligned}$$

that is, $d^2(ffu, gfu) \leq 0$, which implies that $d(ffu, gfu) = 0$ i.e. $gfu = ffu = fhu = hfu$. Again, we have

$$\begin{aligned} d^2(ffu, gu) \leq\ & \max\{\Phi(d(hfu, hu))\Phi(d(hfu, ffu)), \\ & \Phi(d(hfu, hu))\Phi(d(hu, ffu)), \Phi(d(hfu, hu))\Phi(d(hu, gu)), \\ & \Phi(d(hfu, ffu))\Phi(d(hu, gu)), \Phi(d(hfu, gu))\Phi(d(hu, ffu))\}; \end{aligned}$$

i.e.,

$$d^2(ffu, fu) \leq [\Phi(d(ffu, fu))]^2 < d^2(ffu, fu)$$

the above contradiction implies that $ffu = fhu = hfu = gfu = fu$. Put $fu = gu = hu = t$, so, t is a common fixed point of mappings f, g and h.

Now, let t and z be two distinct common fixed points of mappings f, g and h; i.e., $ft = gt = ht = t$ and $fz = gz = hz = z$. From condition (3.1) we have

$$\begin{aligned} d^2(ft, gz) \leq\ & \max\{\Phi(d(ht, hz))\Phi(d(ht, ft)), \Phi(d(ht, hz))\Phi(d(hz, ft)), \\ & \Phi(d(ht, hz))\Phi(d(hz, gz)), \Phi(d(ht, ft))\Phi(d(hz, gz)), \\ & \Phi(d(ht, gz))\Phi(d(hz, ft))\}; \end{aligned}$$

i.e.,

$$d^2(t, z) \leq \Phi^2(d(t, z)) < d^2(t, z)$$

which implies that $t = z$. □

3.2 Common fixed point for four mappings

Now, we give our second main result.

Theorem 3.2 *Let $\mathcal{X}$ be a set endowed with a symmetric d. Suppose f, g, h and k are four self mappings of $(\mathcal{X}, d)$ satisfying the conditions:*

$$\begin{aligned} d^2(fx, gy) \leq\ & \max\{\Phi(d(hx, ky))\Phi(d(hx, fx)), \Phi(d(hx, ky))\Phi(d(ky, gy)), \\ & \Phi(d(hx, fx))\Phi(d(ky, gy)), \Phi(d(hx, gy))\Phi(d(ky, fx))\}, \end{aligned} \tag{3.3}$$

for all x, $y \in \mathcal{X}$, where Φ is contractive modulus,

$$\textit{the pairs } (f, h) \textit{ and } (g, k) \textit{ are owc.} \tag{3.4}$$

Then f, g, h and k have a unique common fixed point.

Proof Since pairs of mappings (f, h) and (g, k) are owc, then, there exist two elements u and v in $\mathcal{X}$ such that $fu = hu$ and $fhu = hfu$, $gv = kv$ and $gkv = kgv$.

First, we prove that $fu = gv$. Indeed, by inequality (3.3) we get

$$\begin{aligned} d^2(fu, gv) &\le \max\{\Phi(d(hu, kv))\Phi(d(hu, fu)), \Phi(d(hu, kv))\Phi(d(kv, gv)), \\ &\quad \Phi(d(hu, fu))\Phi(d(kv, gv)), \Phi(d(hu, gv))\Phi(d(kv, fu))\} \\ &= \Phi^2(d(fu, gv)); \end{aligned}$$

i.e.,

$$d^2(fu, gv) \le \Phi^2(d(fu, gv)) < d^2(fu, gv),$$

which is a contradiction, hence, $fu = hu = gv = kv$.

Now, suppose that $ffu \ne fu$. By using inequality (3.3) we obtain

$$\begin{aligned} d^2(ffu, gv) &\le \max\{\Phi(d(hfu, kv))\Phi(d(hfu, ffu)), \\ &\quad \Phi(d(hfu, kv))\Phi(d(kv, gv)), \Phi(d(hfu, ffu))\Phi(d(kv, gv)), \\ &\quad \Phi(d(hfu, gv))\Phi(d(kv, ffu))\} \\ &= \Phi^2(d(ffu, gv)); \end{aligned}$$

that is,

$$d^2(ffu, fu) \le \Phi^2(d(ffu, fu)) < d^2(ffu, fu),$$

this contradiction implies that $ffu = fu = hfu$.

Similarly $gfu = kfu = fu$. Therefore $fu = hu = gv = kv$ is a common fixed point of mappings f, g, h and k.

Put $fu = hu = gv = kv = t$, then, $ft = ht = gt = kt = t$.

Now, let t and z be two common fixed points of mappings f, g, h and k such that $z \ne t$, so $t = ft = gt = ht = kt$ and $z = fz = gz = hz = kz$. From condition (3.3), we have

$$\begin{aligned} d^2(t, z) &= d^2(ft, gz) \\ &\le \max\{\Phi(d(ht, kz))\Phi(d(ht, ft)), \Phi(d(ht, kz))\Phi(d(kz, gz)), \\ &\quad \Phi(d(ht, ft))\Phi(d(kz, gz)), \Phi(d(ht, gz))\Phi(d(kz, ft))\} \\ &= \Phi^2(d(t, z)); \end{aligned}$$

i.e.,

$$d^2(t, z) \le \Phi^2(d(t, z)) < d^2(t, z),$$

which is a contradiction. Thus, $z = t$. □

Remark 3.3 Truly, our results improve those of Aage and Salunke because we are removed the inclusions between the images of the mappings, and we are weaken several conditions on the space $(\mathcal{X}, d)$, the contractive modulus function and all the mappings.

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