On Weakly and Pseudo Concircular Symmetric Structures on a Riemannian Manifold

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Abstract

In this paper, we examine the properties of hypersurfaces of weakly and pseudo concircular symmetric manifolds and we give an example for these manifolds.

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1 Introduction

Firstly, Tamassy and Binh introduced weakly symmetric manifolds, [1].

A non-flat Riemannian manifold \((M_n, g)\), \((n > 2)\) whose the curvature tensor satisfies the following relation is called weakly symmetric

\[
\nabla_i R_{hijk} = A_i R_{hijk} + B_h R_{ihjk} + D_i R_{hijk} + E_j R_{hilk} + F_k R_{hijl} \quad (1.1)
\]

where \(A, B, D, E, F\) are non-zero 1-forms and \(\nabla\) denotes the covariant differentiation with respect to the metric tensor of the manifold. These 1-forms are called the associated 1-forms of the manifold and an \(n\)-dimensional manifold of this kind is denoted by \((WS)_n\). It may be mentioned in this connection that
although the definition of a \((WS)_n\) is similar to that of a generalized pseudo-symmetric space studied by Chaki and Mondal, [2], the defining condition of a \((WS)_n\) is weaker than that of a generalized pseudo-symmetric manifold. De and Bandyopadhyay, [3], proved that 1-forms of \((WS)_n\) can not be all different. Then the equation (1.1) reduces to the form

\[
\nabla_l R_{hijk} = A_l R_{hijk} + B_h R_{lkij} + B_i R_{hljk} + D_j R_{hilk} + D_k R_{hijl}
\]

(1.2)

Let us consider a subspace \(V_m\) immersed in a Riemannian manifold \(V_n\) whose parametric representation is \(u^\lambda = u^\lambda(u^1, u^2, \ldots, u^n)\) where \(u^\lambda\) and \(u^i\) \((i, j, k, \ldots = 1, 2, \ldots, m)\) denote the coordinate systems of \(V_n\) and \(V_m\), respectively. A conformal transformation \(\bar{g}_{ij} = \rho^2 g_{ij}\) of the fundamental tensor of \(V_n\), being a concircular one with the function \(\rho\) satisfying the equations

\[
\rho_{ij} = \nabla_j \rho_i - \rho_i \rho_j + \frac{1}{2} g^{\alpha\beta} \rho_i \rho_j g_{ij} = \phi g_{ij}, \quad \rho_j = \frac{\partial}{\partial u^j} \ln \rho
\]

(1.3)

this transformation is called concircular transformation where \(\phi\) is a function of \(u^i\).

The present paper deals with non-concircular flat Riemannian manifold \((M_n, g)\) whose concircular curvature tensor \(Z_{hijk}\) satisfies the condition \((n > 2)\)

\[
\nabla_l Z_{hijk} = A_l Z_{hijk} + B_h Z_{lkij} + B_i Z_{hljk} + D_j Z_{hilk} + D_k Z_{hijl}
\]

(1.4)

where

\[ Z_{hijk} = R_{hijk} - \frac{R}{n(n-1)} (g_{hk} g_{ij} - g_{hj} g_{ik}) \]

\(R_{hijk}\) is the curvature tensor and \(R\) is the scalar curvature. Such a manifold will be called a weakly concircular symmetric manifold and denoted by \((WZS)_n\), [4]. It was shown that, in [5], \(Z^n_{hijk}\) is invariant under a concircular transformation.

Desa and Amur studied the concircular recurrent Riemannian manifold, [6]. The authors proved that the defining condition of a \((WZS)_n\) can always be expressed in the following form, [4]

\[
\nabla_l Z_{hijk} = A_l Z_{hijk} + B_h Z_{lkij} + B_i Z_{hljk} + D_j Z_{hilk} + D_k Z_{hijl}
\]

(1.4)

where \(A, B, D\) 1-forms (non-zero simultaneously).

From the first Bianchi identity, we get

\[
R_{hijk} + R_{hjki} + R_{hkij} = 0
\]

(1.5)

The second Bianchi identity for a Riemannian manifold is

\[
\nabla_s R_{hijk} + \nabla_j R_{hiks} + \nabla_k R_{hisj} = 0
\]

(1.6)

Let \((\hat{M}, \hat{g})\) be an \((n + 1)\)-dimensional Riemannian manifold covered by a system of coordinate neighborhoods \(\{U, y^\alpha\}\). Let \((M, g)\) be a hypersurface of \((\hat{M}, \hat{g})\) defined via a system of parametric equation \(y^\alpha = y^\alpha(x^i)\), where Greek
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indices take the values 1, 2, . . . , n + 1 and Latin indices take the values 1, 2, . . . , n
a locally coordinate system. Then, we have

\[ g_{ij} = \bar{g}_{\alpha\beta} y_i^\alpha y_j^\beta \] (1.7)

Let \( n^\alpha \) be a local unit normal to \( (M, g) \). Thus, we obtain \( \bar{g}_{\alpha\beta} n^\alpha y_i^\beta = 0 \),
\( g_{\alpha\beta} n^\alpha n^\beta = 1 \) and it is easily seen that there are the following conditions between
the contrary metric tensors of the hypersurface \( (M, g) \) and \( (\bar{M}, \bar{g}) \)

\[ g^{\alpha\beta} = g^{ij} y_i^\alpha y_j^\beta + n^\alpha n^\beta, \quad y_i^\alpha = \frac{\partial y^\alpha}{\partial x^i}, \quad (i, j = 1, 2, . . . , n; \alpha = \beta = 1, 2, . . . , n + 1) \] (1.8)

A point of a hypersurface, at which the principal directions of the curvature
are indeterminate, is called an umbilical point. In order that the lines of cur-
vature may be indeterminate at every point of the hypersurface, it is necessary
and sufficient that \( \Omega_{ij} = \omega g_{ij} \), where \( \omega \) is an invariant. According to [7],

\[ M = \Omega_{ij} g^{ij} = n\omega \] (1.9)

where the scalar \( M \) is called the mean curvature of such a hypersurface, so that
the conditions for indeterminate lines of curvature are expressible as

\[ \Omega_{ij} = \frac{M}{n} g_{ij} \] (1.10)

If all the geodesics of a hypersurface \( (M, g) \) are also geodesics of \( (\bar{M}, \bar{g}) \),
the former is called a totally geodesic hypersurface of the latter. Such hypersurfaces
are generalizations of planes in ordinary space. A necessary and sufficient condi-
tion that \( (M, g) \) be a totally geodesic hypersurface is that the normal curvature
should vanish for all directions in \( (M, g) \), and at every point. This requires

\[ \Omega_{ij} = 0 \] (1.11)

Consequently,

\[ M = 0 \] (1.12)

and (1.10) is satisfied.

The structure equations of Gauss and Mainardi–Codazzi, [8]

\[ R_{ijkl} = \bar{R}_{\alpha\beta\gamma\theta} B_{ijkl}^{\alpha\beta\gamma\theta} + \Omega_{ijkl} \]

and

\[ \nabla_k \Omega_{ij} - \nabla_j \Omega_{ik} + \bar{R}_{\beta\gamma\delta\theta} n^\beta B_{ijkl}^{\gamma\delta\theta} = 0 \]

where \( \Omega_{ijkl} = \Omega_{ijl} \Omega_{ik} - \Omega_{ij} \Omega_{lk} \).

From (1.9), the above equations reduce to the following forms

\[ R_{ijkl} = \bar{R}_{\alpha\beta\gamma\theta} B_{ijkl}^{\alpha\beta\gamma\theta} + \frac{M^2}{n^2} (g_{ij} g_{ik} - g_{ij} g_{jk}) \] (1.13)
and
\[ \tilde{R}_{\alpha\gamma\delta\theta}n^\alpha B_{ij}^{\gamma\delta\theta} = \frac{1}{n} (g_{ik} \nabla_j M - g_{ij} \nabla_k M) \] (1.14)
respectively, where \( R_{ijkl} \) and \( \tilde{R}_{\alpha\beta\gamma\delta} \) are the curvature tensors \((M, g)\) and \((\tilde{M}, \tilde{g})\), and \( B_{ijkl}^{\alpha\beta\gamma\delta} = B_i^\alpha B_j^\beta B_k^\gamma B_l^\delta, \) \( B_i^\alpha = y_i^\alpha \).

From the Gauss equation, we get
\[
\tilde{R} = R + 2 \tilde{R}_{\alpha\beta\gamma\delta} n^\alpha n^\beta - \Omega_{ijkl} g^{il} g^{jk} \] (1.15)

The concircular curvature tensors of \((M, g)\) and \((\tilde{M}, \tilde{g})\) can be written in the form
\[
Z_{hijk} = R_{hijk} + \frac{R}{n(n - 1)} G_{hijk} \] (1.16)
and
\[
\tilde{Z}_{\alpha\beta\gamma\theta} = \tilde{R}_{\alpha\beta\gamma\theta} + \frac{\tilde{R}}{n(n + 1)} G_{\alpha\beta\gamma\theta} \] (1.17)
where \( G_{hijk} = g_{hj} g_{ik} - g_{hk} g_{ij} \) and \( G_{\alpha\beta\gamma\theta} = \tilde{g}_{\alpha\gamma} \tilde{g}_{\beta\theta} - \tilde{g}_{\alpha\theta} \tilde{g}_{\beta\gamma} \). On account of (1.7), (1.13), (1.16) and (1.17), we get
\[
Z_{hijk} = \tilde{Z}_{\alpha\beta\gamma\theta} B_{hijk}^{\alpha\beta\gamma\theta} + \frac{M^2}{n^2} G_{hijk} + \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\tilde{R}}{n + 1} \right) G_{hijk} \] (1.18)

2 Totally umbilical hypersurface of a weakly concircular symmetric manifold

Now, we consider an \((n + 1)\)-dimensional weakly concircular symmetric Riemannian manifold and we denote this manifold by \((WZS)_{n+1}\). For a \((WZS)_{n+1}\), we have
\[
\nabla_c \tilde{Z}_{abcd} = A_c \tilde{Z}_{abcd} + B_a \tilde{Z}_{ebcd} + B_b \tilde{Z}_{acde} + D_c \tilde{Z}_{abed} + D_d \tilde{Z}_{abce} \] (2.1)

Using (1.17), we obtain
\[
\tilde{Z}_{abcd} n^a B_{ijkl}^{abcd} = \tilde{R}_{abcd} n^a B_{ijkl}^{abcd} \] (2.2)

We assume that the scalar curvature of \((WZS)_n\) is not constant and \((WZS)_n\) is a totally umbilical hypersurface. In this case, we find that
\[
\nabla_s Z_{hijk} = A_s \tilde{Z}_{abcd} B_{hijkl}^{abcd} + B_h \tilde{Z}_{ebcd} P^{ebcd}_{sijkl} + B_i \tilde{Z}_{acde} B_{sijh}^{acde} + D_j \tilde{Z}_{abed} P^{abcd}_{hsijk} + D_k \tilde{Z}_{abce} B_{sijh}^{abce} + \frac{1}{n^2} G_{hijk} \nabla_s M^2 \\
+ \frac{1}{n} G_{hijk} \nabla_s \left( \frac{R}{n - 1} - \frac{\tilde{R}}{n + 1} \right) + \frac{M}{n} \left( g_{ks} \tilde{R}_{abcd} P^{abcd}_{ijkl} n^a + g_{ks} \tilde{R}_{abcd} B_{sijh}^{abcd} n^a + g_{ks} \tilde{R}_{abcd} B_{sijh}^{abcd} n^d \right) \] (2.3)
By the aid of the Gauss equation, (2.3) can be written as

\[ \nabla_s Z_{hijk} = A_s \left( Z_{hijk} - \frac{M^2}{n^2}G_{hijk} - \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right)G_{hijk} \right) \]

\[ + B_h \left( Z_{sijk} - \frac{M^2}{n^2}G_{sijk} - \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right)G_{sijk} \right) \]

\[ + B_i \left( Z_{hsjk} - \frac{M^2}{n^2}G_{hsjk} - \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right)G_{hsjk} \right) \]

\[ + D_j \left( Z_{hisk} - \frac{M^2}{n^2}G_{hisk} - \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right)G_{hisk} \right) \]

\[ + D_k \left( Z_{hij} - \frac{M^2}{n^2}G_{hij} - \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right)G_{hij} \right) \]

\[ + \frac{1}{n} G_{hijk} \nabla_s M^2 + \frac{1}{n} G_{hijk} \nabla_s \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) \]

\[ + \frac{M}{n^2} \left[ (ghs, gik - gisghk) \nabla_j M + (gisghj - gijsghs) \nabla_k M \right] \]

\[ + \frac{M}{n^2} \left[ (gjs, gik - gijsghk) \nabla_h M + (gksghj - gj, sgkh) \nabla_i M \right] \]

\[ = 0 \quad (2.4) \]

Now, we suppose that \((M, g) = (WZS)_n\).

By the aid of (1.4) and (2.4), we have

\[ \left[ \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) \right] (A_s G_{hijk} + B_h G_{sijk} + B_i G_{hsjk} + D_j G_{hisk} + D_k G_{hij}) \]

\[ - G_{hijk} \nabla_s \left( \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) \right) \]

\[ - \frac{M}{n^2} (G_{hisk} \nabla_j M + G_{ihsk} \nabla_k M + G_{sijk} \nabla_h M + G_{k, sh} \nabla_i M) = 0 \quad (2.5) \]

Multiplying (2.5) by \(g^{hk}g^{ij}\), we can obtain

\[ \left( \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) \right) (2B_s + 2D_s + nA_s) \]

\[ - \frac{(n + 2)}{n^2} \nabla_s M^2 - \nabla_s \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) = 0 \quad (2.6) \]

Similarly, multiplying (2.5) by \(g^{ik}g^{hs}\), it is easily obtained that

\[ \left( \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) \right) (B_s + A_s + (n - 1)D_s) \]

\[ - \frac{(n + 2)}{2n^2} \nabla_s M^2 - \frac{1}{n} \nabla_s \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) = 0 \quad (2.7) \]

Let us suppose that

\[ R = (1 - \frac{2}{n + 1})\bar{R} \quad (2.8) \]
where the scalar curvature $R$ is not constant.

From (2.6) and (2.7), we get

$$A_s = 2D_s \quad \text{or} \quad M = 0 \quad (2.9)$$

We assume that $A_s = 2D_s$. Transvecting (1.4) with $g^{hk}$ and $g^{ij}$, we get

$$g^{ks} \nabla_s G_{hk} = (A_k - B_k + D_k)G_{hs}g^s_k \quad (2.10)$$

where $G_{hk} = R_{hk} - \frac{k}{n} g_{hk}$ ($n > 2$) is the Einstein tensor.

Similarly, transvecting (1.4) with $g^{hk}$ and $g^{ij}$, we have

$$(B_k + D_k)G_{hs}g^s_k = 0 \quad (2.11)$$

Hence, using the equations (2.9)1 and (2.10), it can be obtained that

$$(A_k + 2B_k)G_{hs}g^s_k = 0 \quad (2.12)$$

Now, multiplying the equation (1.4) by $g_{hl}$ and $g^{ij}$ and using the result

$$\nabla_s R_{h} = \frac{1}{2} \nabla_h R, \quad \text{we obtain} \quad R \equiv \text{const.} \quad \text{In the beginning, we suppose that} \quad R \neq \text{const.} \quad \text{Thus,} \quad A_s \neq 2D_s. \quad \text{From (2.9), we have} \quad M = 0, \quad \text{i.e., the hypersurface is totally geodesic. Thus, we can state the following theorem:}$$

**Theorem 2.1** In the totally umbilical hypersurface $(WZS)_n$ of $(WZS)_{n+1}$, if the expression $R = (1 - \frac{n}{n+1}) \bar{R}$, $(R \neq \text{const.})$ is satisfied then the hypersurface is totally geodesic.

**Theorem 2.2** If the totally umbilical hypersurface $(WZS)_n$ of a $(WZS)_{n+1}$ satisfies the condition

$$\frac{\bar{R}}{n+1} - \frac{R}{n-1} = c \quad (c < 0, \text{const})$$

then either the mean curvature or the scalar curvature of this hypersurface is constant.

**Proof** We assume that the totally umbilical hypersurface $(WZS)_n$ of $(WZS)_{n+1}$ satisfies the condition

$$\frac{\bar{R}}{n+1} + \frac{R}{n-1} = c \quad (2.13)$$

From (2.5) and (2.13), we obtain

$$\left(\frac{M^2}{n^2} + \frac{c}{n}\right) (A_sG_{hijk} + B_sG_{sijk} + B_tG_{hsjk} + D_jG_{hisk} + D_kG_{hijs})$$

$$- \frac{1}{n^2} G_{hijk} \nabla_s M^2 - \frac{M}{n^2} (G_{hisk} \nabla_j M)$$

$$+ G_{ihsj} \nabla_k M + G_{sijk} \nabla_h M + G_{kjs} \nabla_i M) = 0 \quad (2.14)$$

Multiplying (2.14) by $g^{hk}g^{ij}$, we find that

$$\left(\frac{M^2}{n^2} + \frac{c}{n}\right) (2B_s + 2D_s + nA_s) - \frac{(n+2)}{n^2} \nabla_s M^2 = 0 \quad (2.15)$$
Similarly, multiplying (2.14) by $g^i_k g^{hs}$, we can easily obtain that
\[
\left( \frac{M^2}{n^2} + \frac{c}{n} \right) (B_s + A_s + (n - 1)D_s) - \frac{(n + 2)}{2n^2} \nabla_s M^2 = 0 \tag{2.16}
\]
Using (2.15) and (2.16), we get
\[
M^2 = -cn \quad \text{or} \quad A_s = 2D_s \tag{2.17}
\]
On the other hand, from (1.4), we have
\[
\nabla_l Z_{hijk} = A_l Z_{hijk} + B_h Z_{lij} + B_i Z_{hjk} + D_j Z_{hik} + D_k Z_{hjl} \tag{2.18}
\]
Permutating $j, k$ and $l$ by cyclic in (2.18), adding the three equations and using the expression (1.5) and the first Bianchi Identity, we obtain
\[
(A_l - 2D_l)Z_{hijk} + (A_j - 2D_j)Z_{hikl} + (A_k - 2D_k)Z_{hilj} - \frac{1}{n(n-1)} (G_{hijk} \nabla_l R + G_{hikl} \nabla_j R + G_{hilj} \nabla_k R) \tag{2.19}
\]
Transvecting (2.19) with $g^{ij} g^{hk}$, we can obtain
\[
2(A_k - 2D_k) g^{hk} G_{hl} = \frac{(n - 2)}{n} \nabla_l R \tag{2.20}
\]
If $A_k = 2D_k$, from (2.20), then we say that the scalar curvature of this hypersurface is constant. If $A_k \neq 2D_k$, from (2.17), the mean curvature of this hypersurface must be constant. If $c = 0$ then it is clear that this hypersurface is totally geodesic. Thus, the proof is completed.

**Theorem 2.3** If a totally geodesic hypersurface of a $(WZS)_{n+1}$ satisfies the condition $R = (1 - \frac{2}{n+1}) \bar{R}$ then this hypersurface is $(WZS)_n$.

**Proof** From (1.4) and (2.4), the proof is easily seen that.

3 **Totally umbilical hypersurface of a pseudo concircular symmetric manifold**

We consider a non-concircular flat Riemannian manifold $(M, g)$ whose concircular curvature tensor $Z_{hijk}$ satisfies the condition
\[
\nabla_l Z_{hijk} = 2\lambda_l Z_{hijk} + \lambda_h Z_{lij} + \lambda_i Z_{hjk} + \lambda_j Z_{hik} + \lambda_k Z_{hjl} \tag{3.1}
\]
where $\lambda_l$ is a non-zero covariant vector. Such a manifold will be called a pseudo-concircular symmetric manifold and denoted by $(PZS)_n$. Permutating $j, k, l$ by cyclic in (3.1), we obtain the following equations
\[
\nabla_j Z_{hikl} = 2\lambda_j Z_{hikl} + \lambda_h Z_{jikl} + \lambda_i Z_{hjkl} + \lambda_k Z_{hijl} + \lambda_l Z_{hikj} \tag{3.2}
\]
and
\[ \nabla_k Z_{hilj} = 2\lambda_k Z_{hilj} + \lambda_h Z_{kilj} + \lambda_i Z_{hkij} + \lambda_l Z_{hilk} + \lambda_j Z_{hikl} \quad (3.3) \]

Adding the equations (3.1), (3.2) and (3.3) and by using the first and the second Bianchi identities, it is obtained that
\[ G_{hijk} \nabla_l R + G_{hikl} \nabla_j R + G_{hilj} \nabla_k R = 0 \quad (3.4) \]

Transvecting (3.4) with \( g^{hk} g^{ij} \), we get
\[ (1 - n)(2 - n) \nabla_l R = 0. \]

Since \( n > 2 \), we find that the scalar curvature of the hypersurface is constant.

Now, we can state the following theorem:

**Theorem 3.1** The scalar curvature of a pseudo concircular symmetric manifold is constant.

**Theorem 3.2** Let us suppose that a hypersurface \((PZS)_n\) of a pseudo concircular symmetric manifold \((PZS)_{n+1}\) be totally umbilical. Then the scalar curvature of \((PZS)_{n+1}\) is constant.

**Proof** Taking the relation \( \frac{A_s}{2} = B_s = D_s = \lambda_s \) in (2.3), (2.4) and (2.5) and using the equation (3.1), we get
\[ \left( \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) \right) \left( 2\lambda_s G_{hijk} + \lambda_i G_{hsjk} + \lambda_j G_{hisk} + \lambda_k G_{hjsi} + \lambda_l G_{sijk} \right) \]
\[ - \frac{1}{n^2} G_{hijk} \nabla_s M^2 - \frac{1}{n} G_{hijk} \nabla_s \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) \]
\[ - \frac{M}{n^2} (G_{hisk} \nabla_j M + G_{ihsj} \nabla_k M + G_{sijk} \nabla_h M + G_{kjsh} \nabla_i M) = 0 \quad (3.5) \]

Multiplying (3.5) by \( g^{hk} g^{ij} \) and \( g^{ik} g^{hs} \), respectively, we obtain
\[ \left( \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) \right) 2\lambda_s (2 + n) - \frac{(n + 2)}{n^2} \nabla_s M^2 \]
\[ - \nabla_s \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) = 0 \quad (3.6) \]

and
\[ \left( \frac{M^2}{n^2} + \frac{1}{n} \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) \right) \lambda_s (2 + n) - \frac{(n + 2)}{2n^2} \nabla_s M^2 \]
\[ - \frac{1}{n} \nabla_s \left( \frac{R}{n - 1} - \frac{\bar{R}}{n + 1} \right) = 0 \quad (3.7) \]

From (3.6) and (3.7), we obtain
\[ - \frac{\bar{R}}{n + 1} + \frac{R}{n - 1} = c \quad (3.8) \]
where \( c \) is a positive constant. By using Theorem 3.1, we can say that
\[
\bar{R} \equiv \text{const.} \tag{3.9}
\]

**Theorem 3.3** If a totally geodesic hypersurface of \((PZS)_{n+1}\) satisfies the condition \( R = (1 - \frac{2}{n+1})\bar{R} \) then the hypersurface is \((PZS)_n\).

**Proof** Let us suppose that a hypersurface of \((PZS)_{n+1}\) be totally geodesic. From the expressions (1.12) and (2.4) and the condition \( \frac{4\alpha}{2} = B_s = D_s = \lambda_s \), the proof is clear. \( \square \)

### 4 An example of a \((WZS)_n\)

In this section, we want to construct a \((WZS)_n\) spaces. On the coordinate space \( R^n \) (with coordinates \( x^1, x^2, \ldots, x^n \)), we define a Riemannian space \( V^n \) and calculate the components of the curvature tensor and its covariant derivative.

Let each Latin index run over 1, 2, \ldots, \( n \) and each Greek index over 2, 3, \ldots, \( n - 1 \). We define a Riemannian metric on \( R^n \) \((n > 3)\) by the formula
\[
d s^2 = \phi (dx^1)^2 + k_{\alpha\beta} dx^\alpha dx^\beta + 2 dx^1 dx^n \tag{4.1}
\]
where \([k_{\alpha\beta}]\) is a symmetric and non-singular matrix consisting of constants and \( \phi \) is a function of \((x^1, x^2, \ldots, x^{n-1})\) and independent of \( x^n \). In the metric considered, the only non-vanishing components of the curvature tensor, \([9]\)
\[
R_{1\alpha\beta 1} = \frac{1}{2} \phi_{,\alpha\beta} \tag{4.2}
\]
where \( \cdot, \cdot \) denotes the partial differentiation with respect to the coordinates and \( k^{\alpha\beta} \) are the elements of the matrix inverse to \([k_{\alpha\beta}]\).

We consider \( V_n \) and
\[
\phi = f(x^1)(V_{\alpha\beta} x^\alpha x^\beta \cos g(x^1) + w_{\alpha\beta} x^\alpha x^\beta \sin g(x^1) + k_{\alpha\beta} x^\alpha x^\beta h(x^1))
\]
where \( f, g, h \) are functions of \( x^1 \) only and the matrices \([w_{\alpha\beta}], [V_{\alpha\beta}]\) and \([k_{\alpha\beta}]\) are the form
\[
w_{\alpha\beta} = -1 \text{ for } \alpha = \beta \quad \text{and} \quad w_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta \tag{4.3}
\]
\[
V_{\alpha\beta} = 1 \text{ for } \alpha = \beta \quad \text{and} \quad V_{\alpha\beta} = 0 \text{ for } \alpha \neq \beta \tag{4.4}
\]
and
\[
k_{\alpha\beta} = \begin{cases} 1 & \text{for } \alpha = \beta \\ 0 & \text{otherwise} \end{cases} \tag{4.5}
\]

From (4.2), the only non-vanishing components of the concircular curvature tensor \( Z_{hijk} \) are
\[
Z_{1\alpha\beta 1} = \begin{cases} f(\cos g - \sin g + h) & \text{for } \alpha = \beta \\ 0 & \text{for } \alpha \neq \beta \end{cases} \tag{4.6}
\]
Here, we consider

$$A_i = B_i = D_i = 0 \quad \text{for } i \neq 1 \text{ and } A_1 + B_1 + D_1 = c_1, \ c_1 \neq 0 \text{ and const.} \quad (4.7)$$

Thus, from (1.4), $V_n$ will be $(WZS)_n$ if and only if the following relations

$$\nabla_1 Z_{1a1} = A_1 Z_{1a1} + B_1 Z_{1a1} + B_a Z_{11a1} + D_a Z_{10a1} + D_1 Z_{1a1} \quad (4.8)$$

$$\nabla_1 Z_{11a1} = A_a Z_{11a1} + B_1 Z_{11a1} + B_1 Z_{1a1} + D_a Z_{11a1} + D_1 Z_{11a1} \quad (4.9)$$

$$\nabla_1 Z_{1a11} = A_a Z_{1a11} + B_1 Z_{a11} + B_a Z_{1a11} + D_1 Z_{1a11} + D_1 Z_{1a1} \quad (4.10)$$

Thus, using (4.8), (4.9) and (4.10), we find

$$f'(x^1)(\cos g - \sin g + h) + f(x^1)(-g' \sin g - g' \cos g + h') = (A_1 + B_1 + D_1) f(x^1)(\cos g - \sin g + h). \quad (4.11)$$

By the aid of (4.11), we get

$$f(\cos g - \sin g + h) = c_2 e^{(A_1 + B_1 + D_1)x^1}, \quad c_2 > 0. \quad (4.12)$$

So, the $n$-dimensional weakly concircular recurrent Riemannian manifold has the metric of the form

$$ds^2 = \phi(dx^1)^2 + k_{a\beta}dx^a dx^\beta + 2dx^1 dx^n,$$

$$\phi = c_2 e^{c_1x} \sum_{k=2}^{n-1} (x^k)^2.$$ 

References