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Integral Presentations of Deviations of de la Vallee Poussin Right-Angled Sums

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Abstract

We investigate approximation properties of de la Vallee Poussin right-angled sums on the classes of periodic functions of several variables with a high smoothness. We obtain integral presentations of deviations of de la Vallee Poussin sums on the classes $C^{m\alpha}_{\beta,\infty}$.

Key words: Right-angled sums of Vallee Poussin, integral presentations, Fourier series.

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1 Introduction

Considering [1] we define $\overline{\psi}$-integral classes of periodic functions of several variables in the following way.

Let R^m be an Euclidean space with elements $\vec{x} = (x_1, x_2, \dots, x_m)$, and let $T^m = \prod_{i=1}^m [-\pi;\pi]$ be an m-dimensional cube with the side 2π,

$$\begin{aligned} N^m &= \{\vec{x} \in R^m \mid x_i \in N,\ i = 1, 2, \dots, m\}, \\ N^m_* &= \{\vec{x} \in R^m \mid x_i \in N_* = N \cup \{0\},\ i = 1, 2, \dots, m\}, \\ N^m_i &= \{\vec{x} \in R^m \mid x_i \in N, x_j \in N_*,\ i \neq j\}, \\ E^m &= \{\vec{x} \in R^m \mid x_i \in \{0; 1\},\ i = 1, 2\}. \end{aligned}$$

We denote by $L(T^m)$ the set of summable on a cube T^m functions $f(\vec{x}) = f(x_1, x_2, \dots, x_m)$ which are 2π-periodic on every variable.

Let $f \in L(T^m)$. Then for every pair of points $\vec{s} \in E^m$, $\vec{k} \in N_*^m$ we have a corresponding value

$$a_{\vec{k}}^{\vec{s}}(f) = \frac{1}{\pi^m} \int\limits_{T^m} f(\vec{x}) \prod_{i=1}^{m} \cos\left(k_i x_i - \frac{s_i \pi}{2}\right) dx_i. \tag{1}$$

Values $a_{\vec{k}}^{\vec{s}}(f)$, $\vec{s} \in E^m$, $\vec{k} \in N_*^m$ are the Fourier coefficients of the function $f(\vec{x})$ [1, p. 546].

For every vector $\vec{k} \in N_*^m$ we have the major harmonic of the function $f(\vec{x})$

$$A_{\vec{k}}(f;\vec{x}) = \sum_{\vec{s} \in E^m} a_{\vec{k}}^{\vec{s}}(f) \prod_{i=1}^{m} \cos\left(k_i x_i - \frac{s_i \pi}{2}\right) \tag{2}$$

and on the variable x_i conjugated harmonic

$$A_{\vec{k}}^{\vec{e_i}}(f;\vec{x}) = \sum_{\vec{s} \in E^m} a_{\vec{k}}^{\vec{s}}(f) \prod_{j \in \overline{m} \setminus \{i\}} \cos\left(k_j x_j - \frac{s_j \pi}{2}\right) \cos\left(k_i x_i - \frac{(s_i+1)\pi}{2}\right).$$

Using [1, p. 545] we define Fourier series of the function $f(\vec{x})$ by the following relation

$$S[f] = \sum_{\vec{k} \in N_*^m} \frac{1}{2^{q(\vec{k})}} A_{\vec{k}}(f,\vec{x}), \tag{3}$$

where $q(\vec{k})$ is a number of zero coordinates of the vector $\vec{k}$.

Let $f \in L(T^m)$ and systems of numbers $\psi_{ij}(k)$, $\Psi_{ij}(k)$, $i = 1, 2, \ldots, m$; $j = 1, 2$, $k \in N_*$ be given.

Let us put

$$\overline{\psi}_i(k) = \sqrt{\psi_{i1}^2(k) + \psi_{i2}^2(k)}, \overline{\Psi}_i(k) = \sqrt{\Psi_{i1}^2(k) + \Psi_{i2}^2(k)}$$

and consider the following conditions be fulfilled: $\overline{\psi}_i(k) \neq 0$, $\overline{\Psi}_i(k) \neq 0$, $k \in N_*$, $\psi_{i1}(0) = 1$, $\Psi_{i1}(0) = 1$, $\psi_{i2}(0) = 0$, $\Psi_{i2}(0) = 0$, $i = 1, 2, \ldots, m$.

Furthermore, let

$$\sum_{\vec{k} \in N_i^m} \frac{1}{2^{q(\vec{k})} \overline{\psi}_i^2(k_i)} [\psi_{i1}(k_i) A_{\vec{k}}(f,\vec{x}) - \psi_{i2}(k_i) A_{\vec{k}}^{\vec{e_i}}(f,\vec{x})] \tag{4}$$

be the Fourier series of some function of $L(T^m)$. It will be denoted by

$$f^{\overline{\psi}_i}(\vec{x}) = \frac{\partial^{\overline{\psi}_i} f(\vec{x})}{\partial x_i}$$

and called $\overline{\psi}_i$-derivative of the function f with respect to the x_i, $i \in \overline{m}$.

Let $\overline{m} = \{1, 2, \ldots, m\}$. For a fixed r-elemental set $\mu(r) \subset \overline{m}$, $\mu(r) = \{i_1, i_2, \ldots, i_r\}$, we define a function $f^{\overline{\Psi}_\mu}(\vec{x})$ by

$$f^{\overline{\Psi}_\mu}(\vec{x}) = \frac{\partial^{\overline{\Psi}_{i_r}} \partial^{\overline{\Psi}_{i_{r-1}}} \ldots \partial^{\overline{\Psi}_{i_1}} f(\vec{x})}{\partial x_{i_r} \partial x_{i_{r-1}} \ldots \partial x_{i_1}}$$

and call it mixed $\overline{\Psi}_\mu$-derivative with respect to variables x_i, $i \in \mu(r)$.

Let a set of functions ψ_{ij}, Ψ_{ij}, $i = 1, 2, \ldots, m$; $j = 1, 2$ be given. The set of continuous functions $f \in L(T^m)$ having the essentially bounded $\overline{\Psi}_\mu$- and $\overline{\psi}_i$-derivatives, i.e.

$$\operatorname{ess\,sup} \left| f^{\overline{\Psi}_\mu}(\vec{x}) \right| \le 1, \quad \operatorname{ess\,sup} \left| f^{\overline{\psi}_i}(\vec{x}) \right| \le 1, \quad i = 1, 2, \ldots, m; \ \mu \subset \overline{m}; \ \vec{x} \in T^m \tag{5}$$

will be denoted by the symbol $C_\infty^{m\overline{\psi}}$.

If for the sets of functions $\psi_{ij}(k)$ and $\Psi_{ij}(k)$, $i = 1, 2, \ldots, m$; $j = 1, 2$, the functions $\psi_i(k)$, $\Psi_i(k)$ and numbers β_i, β_i^*, $i = 1, 2, \ldots, m$, fulfil

$$\psi_{i1}(k) = \psi_i(k) \cos \frac{\beta_i \pi}{2}; \quad \psi_{i2}(k) = \psi_i(k) \sin \frac{\beta_i \pi}{2};$$

$$\Psi_{i1}(k) = \Psi_i(k) \cos \frac{\beta_i^* \pi}{2}, \quad \Psi_{i2}(k) = \Psi_i(k) \sin \frac{\beta_i^* \pi}{2}, \quad i = 1, 2, \ldots, m,$$

then the class $C_\infty^{m\overline{\psi}}$ is the class of (ψ, β)-differentiable periodic functions of m variables (see [2]) and it is denoted by $C_{\beta,\infty}^{m\psi}$. For $m = 2$ these classes are the classes of (ψ, β)-differentiable periodic functions of two variables which are defined in [3] (see also [1]). In the case when the conditions $\Psi_1(k) = k^{-r}$, $\Psi_2(k) = k^{-s}$, $\psi_1(k) = k^{-r_1}$, $\psi_2(k) = k^{-s_1}$, $\beta_1 = r$, $\beta_1^* = s$, $\beta_2 = r_1$, $\beta_2^* = s_1$ for the $r > 0$, $s > 0$, $r_1 \ge r$, $s_1 \ge s$ are also fulfilled the classes $C_{\beta,\infty}^{2\psi}$ and $W_{r_1,s_1}^{r,s}$ are equal(see for example [4]). In [4] (see [5], too) there is proved the asymptotic equality of upper bounds of deviations of Fourier right-angled sums $S_{\vec{n}}(f, \vec{x})$ (taking at the classes $W_{r_1,s_1}^{r,s}$) for $n_i \to \infty$, $i = 1, 2$:

$$\mathcal{E}(W_{r_1,s_1}^{r,s}; S_{\vec{n}}) = \frac{4 \ln n_1}{\pi^2 n_1^{r_1}} + \frac{4 \ln n_2}{\pi^2 n_2^{s_1}} + O(1) \left(\frac{\ln n_1 \ln n_2}{n_1^r n_2^s} + \frac{1}{n_1^{r_1}} + \frac{1}{n_2^{s_1}} \right).$$

Let us put $G_{\vec{n},\vec{p}} = \prod_{i=1}^m [n_i - p_i; n_i - 1]$ for $\vec{n} \in N^m$, $\vec{p} \in N^m$, $p_i < n_i$, $i = 1, 2, \ldots, m,$. Then trigonometric polynomials of the type

$$V_{\vec{n},\vec{p}}(f; \vec{x}) = \frac{1}{\prod_{i=1}^m p_i} \sum_{\vec{k} \in G_{\vec{n},\vec{p}}} S_{\vec{k}}(f; \vec{x}), \tag{6}$$

(where $S_{\vec{k}}(f; \vec{x})$ are partial sums of Fourier series defined (2), $\vec{n} \in N^m$, $p_i \in N$, $p_i < n_i$, $i = 1, 2, \ldots, m$) are called Vallee Poussin right-angled sums.

In this work the problems of approximation of classes $C_{\beta,\infty}^{m\psi}$ by polynomials $V_{\vec{n},\vec{p}}(f; \vec{x})$ are investigated. The functions which determine these classes are defined in the following way:

$$\psi_i(x) = e^{-\alpha_i x}, \quad \Psi_i(x) = e^{-\alpha_i^* x}, \quad \alpha_i > 0, \ \alpha_i^* > 0, \ i = 1, 2, \ldots, m.$$

We denote such classes by $C^{m\alpha}_{\beta,\infty}$ (analagously to the classes of functions of a single variable).

It is proved by S. M. Nikol'skii in [6] (see also [7], [8]) that for upper bounds of the deviations of Fourier sums on the corresponding classes $C^{\alpha}_{\beta,\infty}$ functions of one variabl we obtain the following asymptotic equality for $n \to \infty$:

$$\mathcal{E}\left(C^{\alpha}_{\beta,\infty}; S_n\right) = \frac{8q^n}{\pi^2} K(q) + O(1)\frac{q^n}{n}, \quad q = e^{-\alpha}, \tag{7}$$

where

$$K(q) = \int\limits_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1 - q^2 \sin^2 u}}$$

is the total elliptic integral of the first kind.

Asymptotic equalities for upper bounds of the deviations of de la Vallee Poussin sums on the classes $C^{\alpha}_{\beta,\infty}$ may be found in the [9], [10] (see also [11], [12, p. 217]):

$$\mathcal{E}\left(C^{\alpha}_{\beta,\infty}; V_{n,p}\right) = \frac{4q^{n-p+1}}{\pi p(1-q^2)} + O(1)\left(\frac{q^{n-p+1}}{p(n-p)(1-q)^3} + \frac{q^n}{p(1-q^2)}\right), \ 1 < p < n. \tag{8}$$

The 2-dimensional and m-dimensional analogies of equality (7) for the classes $C^{m\alpha}_{\beta,\infty}$ are in the works [13], [14].

2 Main Results

Let $\Lambda = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_m\}$ be a fixed set of infinite triangle numeric matrices, $\Lambda_i = \{\lambda^{(n_i)}_{k_i}\}$, $i = 1, 2, \ldots, m$, $\lambda^{(n_i)}_0 = 1$, $\lambda^{(n_i)}_{k_i} = 0$ for $k_i \geq n_i$.

Further let $\lambda^{(\vec{n})}_{\vec{k}} = \prod_{i=1}^m \lambda^{(n_i)}_{k_i}$ and let $G_{\vec{n}} = \prod_{i=1}^m [0; n_i - 1]$ be an right-angled parallelepiped corresponding to the vector $\vec{n} \in N^m$.

For every function with Fourier series (1) we have trigonometric polynomial

$$U_{\vec{n}}(f; \vec{x}; \Lambda) = \sum_{\vec{k} \in G_{\vec{n}}} 2^{-q(\vec{k})} \lambda^{(\vec{n})}_{\vec{k}} A_{\vec{k}}(f; \vec{x}).$$

Values $\delta_{\vec{n}}(f; \vec{x}; \Lambda) = f(\vec{x}) - U_{\vec{n}}(f; \vec{x}; \Lambda)$ are the deviations of such polynomials of the function $f(\vec{x})$.

In this work there are found the integral presentations of the deviations

$$\delta_{\vec{n},\vec{p}}(f, \vec{x}) = f(\vec{x}) - V_{\vec{n},\vec{p}}(f, \vec{x})$$

of sums $V_{\vec{n},\vec{p}}(f, \vec{x})$ from function $f(\vec{x})$ out of classes $C^{m\alpha}_{\beta,\infty}$.

The following theorem is the main result of this work.

Theorem 1 *If $\alpha_i > 0$, $\alpha_i^* > 0$, $q_i = e^{-\alpha_i}$, $Q_i = e^{-\alpha_i^*}$, $\beta_i \in R$, $\beta_i^* \in R$, $p_i \in N$, $1 < p_i < n_i$; $i = 1, 2, \ldots, m$,*

then for every function $f \in C^{m\alpha}_{\beta,\infty}$ the following equality is fulfilled

$$\delta_{\vec{n},\vec{p}}(f,\vec{x}) = \sum_{i=1}^{m} \frac{q_i^{n_i-p_i+1}}{p_i\pi} \int\limits_{-\pi}^{\pi} f_{\beta_i}^{\psi_i}(\vec{x}+t_i\vec{e}_i) b_{n_i-p_i}^{\beta_i}(t_i)dt_i$$

$$- \sum_{i=1}^{m} \frac{q_i^{n_i+1}}{p_i\pi} \int\limits_{-\pi}^{\pi} f_{\beta_i}^{\psi_i}(\vec{x}+t_i\vec{e}_i) b_{n_i}^{\beta_i}(t_i)dt_i$$

$$+ O(1)\sum_{r=2}^{m} \sum_{\mu(r)\in\overline{m}} \prod_{j\in\mu(r)} Q_j^{n_j-p_j+1} \int\limits_{T^r} \left| B_{n_j-p_j}^{\beta_j^\star}(t_j) \right| dt_j, \tag{9}$$

where

$$b_{n_i}^{\beta_i}(t_i) = \frac{(q_i^2\cos t_i - 2q_i + \cos t_i)}{(1-2q_i\cos t_i + q_i^2)^2} \cos\left(n_i t_i + \frac{\beta_i\pi}{2}\right)$$

$$+ \frac{(q_i^2 \sin t_i - \sin t_i)}{(1-2q_i\cos t_i + q_i^2)^2} \sin\left(n_i t_i + \frac{\beta_i\pi}{2}\right),$$

$$B_{n_i}^{\beta_i^\star}(t_i) = \frac{(Q_i^2\cos t_i - 2Q_i + \cos t_i)}{(1-2Q_i\cos t_i + Q_i^2)^2} \cos\left(n_i t_i + \frac{\beta_i^\star\pi}{2}\right)$$

$$+ \frac{(Q_i^2 \sin t_i - \sin t_i)}{(1-2Q_i\cos t_i + Q_i^2)^2} \sin\left(n_i t_i + \frac{\beta_i^\star\pi}{2}\right).$$

Proof It is clear that

$$\delta_{\vec{n},\vec{p}}(f;\vec{x}) = \frac{1}{\prod_{i=1}^m p_i} \sum_{\vec{k}\in G_{\vec{n},\vec{p}}} \rho_{\vec{k}}(f;\vec{x}) = \frac{1}{\prod_{i=1}^m p_i} \sum_{i=1}^{m} \sum_{k_i=n_i-p_i}^{n_i-1} \rho_{\vec{k}}(f;\vec{x}), \tag{10}$$

where

$$\rho_{\vec{k}}(f;\vec{x}) = f(\vec{x}) - S_{\vec{k}}(f;\vec{x}), \quad \vec{k} = (k_1;k_2;\ldots;k_m).$$

Let us investigate $\rho_{\vec{k}}(f;\vec{x})$. Using theorem 1 in [13] for $f \in C^{m\alpha}_{\beta,\infty}$ we have

$$\rho_{\vec{n}}(f,\vec{x}) = \sum_{i=1}^{m} \frac{1}{\pi} \int_{-\pi}^{\pi} f_{\beta_i}^{\psi_i}(\vec{x}+t_i\vec{e}_i) \sum_{k=n_i+1}^{\infty} \exp(-\alpha_i k)\cos\left(kt_i + \frac{\beta_i\pi}{2}\right) dt_i$$

$$+ \sum_{r=2}^{m} (-1)^{r+1} \sum_{\mu(r)\in\overline{m}} \frac{1}{\pi^r} \int\limits_{T^r} f_{\beta_\mu^\star}^{\Psi_\mu}\left(\vec{x} + \sum_{i\in\mu(r)} t_i\vec{e}_i\right)$$

$$\times \prod_{j\in\mu(r)} \sum_{k_j=n_j+1}^{\infty} \exp(-\alpha_i^\star k_j)\cos\left(k_j t_j + \frac{\beta_j^\star\pi}{2}\right) dt_j.$$

Denote $q_i = \exp(-\alpha_i)$, $Q_i = \exp(-\alpha_i^\star)$. Using [15, p. 123–124] we obtain

$$\sum_{k=n_i+1}^{\infty} \exp(-\alpha_i k)\cos\left(kt_i + \frac{\beta_i\pi}{2}\right)$$

$$= q_i^{n_i}\left[\frac{q_i\cos t_i - q_i^2}{1-2q_i\cos t_i + q_i^2}\cos\left(n_it_i+\frac{\beta_i\pi}{2}\right) - \frac{q_i\sin t_i}{1-2q_i\cos t_i+q_i^2}\sin\left(n_it_i+\frac{\beta_i\pi}{2}\right)\right].$$

If

$$h_{n_i}^{\beta_i}(t_i) = \frac{(q_i\cos t_i - q_i^2)\cos\left(n_it_i+\frac{\beta_i\pi}{2}\right) - q_i\sin t_i\sin\left(n_it_i+\frac{\beta_i\pi}{2}\right)}{1-2q_i\cos t_i+q_i^2},$$

$$H_{n_i}^{\beta_i^\star}(t_i) = \frac{(Q_i\cos t_i - Q_i^2)\cos\left(n_it_i+\frac{\beta_i\pi}{2}\right) - Q_i\sin t_i\sin\left(n_it_i+\frac{\beta_i\pi}{2}\right)}{1-2Q_i\cos t_i+Q_i^2}$$

then

$$\rho_{\vec n}(f,\vec x) = \sum_{i=1}^{m}\frac{1}{\pi}\int_{-\pi}^{\pi} f_{\beta_i}^{\psi_i}(\vec x + t_i\vec e_i)q_i^{n_i}h_{n_i}^{\beta_i}(t_i)dt_i$$

$$+\sum_{r=2}^{m}(-1)^{r+1}\sum_{\mu(r)\in\overline{m}}\frac{1}{\pi^r}\int_{T^r} f_{\beta_\mu^\star}^{\Psi_\mu}\left(\vec x + \sum_{i\in\mu(r)}t_i\vec e_i\right)\prod_{j\in\mu(r)}Q_j^{n_j}H_{n_j}^{\beta_j^\star}(t_j)\,dt_j.$$

According to (10) we obtain

$$\delta_{\vec n,\vec p}(f,\vec x) = \sum_{i=1}^{m}\frac{1}{p_i\pi}\sum_{k_i=n_i-p_i}^{n_i-1} q_i^{k_i}\int_{-\pi}^{\pi} f_{\beta_i}^{\psi_i}(\vec x + t_i\vec e_i)h_{k_i}^{\beta_i}(t_i)dt_i$$

$$+\sum_{r=2}^{m}(-1)^{r+1}\sum_{\mu(r)\in\overline{m}}\frac{1}{\pi^r}\int_{T^r} f_{\beta_\mu^\star}^{\Psi_\mu}\left(\vec x + \sum_{i\in\mu(r)}t_i\vec e_i\right)$$

$$\times\prod_{j\in\mu(r)}\frac{1}{p_j}\sum_{\nu_j=n_j-p_j}^{n_j-1}Q_j^{\nu_j}H_{\nu_j}^{\beta_j^\star}(t_j)\,dt_j. \tag{11}$$

Let us use [11, p. 232–234]. Applying elementary transformations we obtain

$$\sum_{k_i=n_i-p_i}^{n_i-1} q_i^{k_i}h_{k_i}^{\beta_i}(t) = \sum_{k_i=n_i-p_i}^{n_i-1} q_i^{k_i+1}\Bigg[(\cos(k_i+1)t - q_i\cos k_it)\cos\frac{\beta_i\pi}{2}$$

$$-(\sin(k_i+1)t - q_i\sin k_it)\sin\frac{\beta_i\pi}{2}\Bigg](1-2q_i\cos t+q_i^2)^{-1}$$

$$\overset{\mathrm{df}}{=} \frac{\Sigma_{i,1}(t)\cos\frac{\beta_i\pi}{2} - \Sigma_{i,2}(t)\sin\frac{\beta_i\pi}{2}}{1-2q_i\cos t+q_i^2}. \tag{12}$$

Let us investigate $\Sigma_{i,1}(t)$ and $\Sigma_{i,2}(t)$. We may write

$$\Sigma_1(t) = \sum_{k=n-p}^{n-1} q^{k+1}(\cos(k+1)t - q\cos kt) = \frac{1}{2}\Big[\sum_{k=0}^{n}(qe^{it})^k - \sum_{k=0}^{n-p}(qe^{it})^k\Big]$$

$$+\frac{1}{2}\Big[\sum_{k=0}^{n}(qe^{-it})^k - \sum_{k=0}^{n-p}(qe^{-it})^k\Big] - \frac{q^2}{2}\Big[\sum_{k=0}^{n-1}(qe^{it})^k - \sum_{k=0}^{n-p-1}(qe^{it})^k\Big]$$

$$-\frac{q^2}{2}\Big[\sum_{k=0}^{n-1}(qe^{-it})^k - \sum_{k=0}^{n-p-1}(qe^{-it})^k\Big]$$

$$= \frac{1}{2}\Big[\frac{(qe^{it})^{n+1}-1}{qe^{it}-1} - \frac{(qe^{it})^{n-p+1}-1}{qe^{it}-1}\Big] + \frac{1}{2}\Big[\frac{(qe^{-it})^{n+1}-1}{qe^{-it}-1} - \frac{(qe^{-it})^{n-p+1}-1}{qe^{-it}-1}\Big]$$

$$-\frac{q^2}{2}\Big[\frac{(qe^{it})^{n}-1}{qe^{it}-1} - \frac{(qe^{it})^{n-p}-1}{qe^{it}-1}\Big] - \frac{q^2}{2}\Big[\frac{(qe^{-it})^{n}-1}{qe^{-it}-1} - \frac{(qe^{-it})^{n-p}-1}{qe^{-it}-1}\Big].$$

According to [15, p. 124] we denote

$$\Gamma(t) = (1 - 2q\cos t + q^2)^{-1}. \tag{13}$$

Now we have

$$\Sigma_1(t) = \big(q^{n+2}\cos nt - q^{n+1}\cos(n+1)t - q^{n-p+2}\cos(n-p)t + q^{n-p+1}\cos(n-p+1)t$$
$$-q^2(q^{n+1}\cos(n-1)t - q^n\cos nt - q^{n-p+1}\cos(n-p-1)t + q^{n-p}\cos(n-p)t)\big)\Gamma(t)$$
$$= \big(2q^{n+2}\cos nt - 2q^{n-p+2}\cos(n-p)t - q^{n+1}\cos(n+1)t + q^{n-p+1}\cos(n-p+1)t$$
$$- q^{n+3}\cos(n-1)t + q^{n-p+3}\cos(n-p-1)t\big)\Gamma(t)$$
$$= \big((2q^{n+2}\cos nt - q^{n+1}\cos(n+1)t - q^{n+3}\cos(n-1)t) - (2q^{n-p+2}\cos(n-p)t$$
$$- q^{n-p+1}\cos(n-p+1)t - q^{n-p+3}\cos(n-p-1)t)\big)\Gamma(t). \tag{14}$$

Doing elementary transformation of the term in brackets on the right part of equality (14) we have

$$2q^{n+2}\cos nt - q^{n+1}\cos(n+1)t - q^{n+3}\cos(n-1)t$$
$$= q^{n+1}\big((2q - \cos t - q^2\cos t)\cos nt + (\sin t - q^2\sin t)\sin nt\big), \tag{15}$$

$$2q^{n-p+2}\cos(n-p)t - q^{n-p+1}\cos(n-p+1)t - q^{n-p+3}\cos(n-p-1)t$$
$$= q^{n-p+1}\big((2q - \cos t - q^2\cos t)\cos(n-p)t + (\sin t - q^2\sin t)\sin(n-p)t\big). \tag{16}$$

Comparing (13)–(16) we obtain

$$\Sigma_1(t) = \Big[q^{n+1}\big((2q - \cos t - q^2\cos t)\cos nt + (\sin t - q^2 \sin t)\sin nt\big) - q^{n-p+1}\big((2q - \cos t - q^2\cos t)\cos(n-p)t + (\sin t - q^2\sin t)\sin(n-p)t\big)\Big](1 - 2q\cos t + q^2)^{-1}. \tag{17}$$

Analogously, we may find

$$\Sigma_2(t) = \Big[q^{n+1}\big((q^2\sin t - \sin t)\cos nt + (2q - \cos t - q^2\cos t)\sin nt\big) - q^{n-p+1}\big((q^2\sin t - \sin t)\cos(n-p)t + (2q - \cos t - q^2\cos t)\sin(n-p)t\big)\Big](1 - 2q\cos t + q^2)^{-1}. \tag{18}$$

Respecting the last relation we may the equality (12) write in the following way

$$\frac{1}{p_i}\sum_{k_i=n_i-p_i}^{n_i-1} q_i^{k_i} h_{k_i}^{\beta_i}(t_i) = \frac{q_i^{n_i-p_i+1}}{p_i}\bigg[(q_i^2\cos t_i - 2q_i + \cos t_i)\cos\Big((n_i-p_i)t_i + \frac{\beta_i\pi}{2}\Big) + (q_i^2\sin t_i - \sin t_i)\sin\Big((n_i-p_i)t_i + \frac{\beta_i\pi}{2}\Big)\bigg](1 - 2q_i\cos t_i + q_i^2)^{-2} - \frac{q_i^{n_i+1}}{p_i}\bigg[(q_i^2\cos t_i - 2q_i + \cos t_i)\cos\Big(n_i t_i + \frac{\beta_i\pi}{2}\Big) + (q_i^2\sin t_i - \sin t_i)\sin\Big(n_i t_i + \frac{\beta_i\pi}{2}\Big)\bigg](1 - 2q_i\cos t_i + q_i^2)^{-2}. \tag{19}$$

Analogously,

$$\frac{1}{p_i}\sum_{k_i=n_i-p_i}^{n_i-1} Q_i^{k_i} H_{k_i}^{\beta_i^\star}(t_i) = \frac{Q_i^{n_i-p_i+1}}{p_i}\bigg[(Q_i^2\cos t_i - 2Q_i + \cos t_i)\cos\Big((n_i-p_i)t_i + \frac{\beta_i^\star\pi}{2}\Big) + (Q_i^2\sin t_i - \sin t_i)\sin\Big((n_i-p_i)t_i + \frac{\beta_i^\star\pi}{2}\Big)\bigg](1 - 2Q_i\cos t_i + Q_i^2)^{-2} - \frac{Q_i^{n_i+1}}{p_i}\bigg[(Q_i^2\cos t_i - 2Q_i + \cos t_i)\cos\Big(n_i t_i + \frac{\beta_i^\star\pi}{2}\Big) + (Q_i^2\sin t_i - \sin t_i)\sin\Big(n_i t_i + \frac{\beta_i^\star\pi}{2}\Big)\bigg](1 - 2Q_i\cos t_i + Q_i^2)^{-2}. \tag{20}$$

Considering the condition

$$\operatorname*{ess\,sup}_{\vec{x}\in T^m} |f^{\overline{\Psi}_\mu}(\vec{x})| \le 1, \quad \mu \subset \overline{m}, \quad f \in C^{m\alpha}_{\beta,\infty}$$

and equalities (11), (19), (20) we have the coretness the theorem. □

3 Conclusion

Using the relation (9) we can obtain an asymptotic equality for upper bounds of the deviations of the de la Vallee Poussin right-angled sums taken over classes of periodic functions of several variables with a high smoothness.

References

[1] Stepanec, A. I., Pachulia, N. L.: *Multiple Fourier sums on the sets of* (ψ, β)*-differentiable functions.* Ukrainian Math. J. **43**, 4 (1991), 545–555 (in Russian).

[2] Lassuria, R. A.: *Multiple Fourier sums on the sets of* $\overline{\psi}$*-differentiable functions.* Ukrainian Math. J. **55**, 7 (2003), 911–918 (in Russian).

[3] Zaderey, P. V.: *Integral presentations of deviations of linear means of Fourier series on the classes of differentiable periodic functions of two variables.* Some problems of the theory of functions: collection of scientific works, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev, 1985, 16–28 (in Russian).

[4] Stepanec, A. I.: Uniform approximation by trigonometric polynomials. *Nauk. Dumka, Kiev,* 1981 (in Russian).

[5] Stepanec, A. I.: *Approximation of some classes of periodic functions two variables by Fourier sums.* Ukrainian Math. J. **25**, 5 (1973), 599–609 (in Russian).

[6] Nikol'skii, S. M.: *Approximation of the functions by trigonometric polynomials in the mean.* News of Acad. of Sc. USSR **10**, 3 (1946), 207–256 (in Russian).

[7] Stechkin, S. B.: *Estimation of the remainder of Fourier series for the differentiable functions.* Works of Math. Inst. Acad. of Sc. USSR **145** (1980), 126–151 (in Russian).

[8] Stepanec, A. I.: *Approximation by Fourier sums of de la Poussin integrals of continuous functions.* Lect. of Rus. Acad. of Sc. **373**, 2 (2000), 171–173 (in Russian).

[9] Rukasov, V. I., Chaichenko, S. O.: *Approximation of the classes of analytical functions by de la Vallee-Poussin sums.* Ukrainian Math. J. **55**, 6 (2003), 575–590.

[10] Rukasov, V. I., Chaichenko, S. O.: *Approximation of continuous functions by de la Vallee-Poussin operators.* Ukrainian Math. J. **55**, 3 (2003), 498–511.

[11] Rukasov, V. I., Novikov, O. A.: *Approximation of analytical functions by de la Vallee Poussin sums. Fourier series: Theory and Applications.* Works of the Institute of Mathematics, Ukrainian Academy of Sciences, Kiev, 1998, 228–241 (in Russian).

[12] Stepanec A. I., Rukasov V. I., Chaichehko S. O.: *Approximation by de la Vallee Poussin sums.* Works of the Institute of Mathematics, Ukrainian Academy of Sciences **68**, 2007, 368 pp. (in Russian).

[13] Rukasov, V. R., Novikov, O. A., Velichko, V. E., Rovenska, O. G., Bodraya, V. I.: *Approximation of the periodic functions of many variables with a high smoothness by Fourier right-angled sums.* Works of the Institute of Mathematics and Mechanics, Ukrainian Academy of Sciences, 2008, 163–170 (in Russian).

[14] Rukasov, V. I., Novikov, O. A., Bodraya, V. I.: *Approximation of the classes of functions of two variables with a high smoothness by the right-angled linear means of Fourier series. Problems of the aproximation of the functions theory and closely related concepts.* Works of the Institute of Mathematics, Ukrainian Academy of Sciences **4**, 1 (2007), 270–283 (in Russian).

[15] Stepanec, A. I.: Classification and approximation of periodic functions. *Nauk. Dumka, Kiev,* 1987 (in Russian).