A Result on Segmenting Jungck–Mann Iterates

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Abstract

In this paper, following the concepts in [5, 7], we shall establish a convergence result in a uniformly convex Banach space using the Jungck–Mann iteration process introduced by Singh et al [13] and a certain general contractive condition. The authors of [13] established various stability results for a pair of nonself-mappings for both Jungck and Jungck–Mann iteration processes. Our result is a generalization and extension of that of [7] and its corollaries. It is also an improvement on the result of [7].

Key words: Jungck–Mann iteration process; uniformly convex Banach space.

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1 Introduction

Suppose that $A = (a_{nk})$ is an infinite, lower triangular, regular row-stochastic matrix, $E$ a closed convex subset of a Banach space and $T$ a continuous mapping of $E$ into itself and $x_1 \in E$. Then, the general Mann iteration process $M(x_1, A, T)$ which was introduced in Mann [9] is defined by

$$v_n = \sum_{k=1}^{n} a_{nk} x_k, \quad x_{n+1} = Tv_n, \quad n = 1, 2, \ldots ,$$

(1)
If \( A \) is the identity matrix, then each sequence of \( M(x_1, A, T) \) becomes the sequence of Picard iterates of \( T \) at \( x_1 \). It was established in [9] that if either of the sequences \( \{x_n\} \) and \( \{v_n\} \) converges, then the other also converges to the same point, and their common limit is a fixed point of \( T \).

In [5, 7], it is said that the matrix \( A \) is segmenting for the Mann process if \( a_{n+1,k} = (1 - a_{n+1,n+1})a_{nk} \) for \( k \leq n \). In this case, \( v_{n+1} \) lies on the segment joining \( v_n \) and \(Tv_n\):

\[
v_{n+1} = (1 - d_n)v_n + d_nTv_n, \quad n = 1, 2, \ldots, \tag{2}
\]

where \( d_n = a_{n+1,n+1} \). A segmenting matrix is determined by its sequence of diagonal elements. Some authors including [3, 11, 12] have investigated the case \( d_n = \lambda \), \( 0 < \lambda < 1 \), while Mann [9] approximated the fixed points of continuous functions on a closed interval of the real line using the segmenting matrix determined by \( d_n = \frac{1}{n} \) \( \forall n \). Dotson [6] considered the case when \( d_n \) is bounded away from 0 and 1. Groetsch [7] generalized the results of [3, 6, 9, 11, 12] in a uniformly convex Banach space by employing (2) and assuming that \( A \) is a segmenting matrix for which \( \sum_{n=1}^{\infty} d_n(1 - d_n) = \infty \).

We shall give another definition of a segmenting matrix in the next section with a view to generalizing and extending Groetsch [7] and others mentioned earlier in this paper.

2 Preliminaries

Singh et al [13] introduced the following iteration process: Let \((E, \|\cdot\|)\) be a normed linear space, \(S, T: Y \to E\) and \(T(Y) \subseteq S(Y)\). Then, for \(x_0 \in Y\), consider the iteration process:

\[
Sx_{n+1} = (1 - \alpha_n)Sx_n + \alpha_nTx_n, \quad n = 0, 1, 2, \ldots, \tag{3}
\]

where \(\{\alpha_n\}_{n=0}^{\infty}\) satisfies

(i) \(\alpha_0 = 1\),

(ii) \(0 \leq \alpha_n \leq 1\) for \(n > 0\),

(iii) \(\sum \alpha_n = \infty\), and

(iv) \(\sum_{j=0}^{n} \alpha_j \Pi_{i=j+1}^{n} (1 - \alpha_i + a\alpha_i)\) converges.

The iteration process (3) is called the Jungck–Mann iteration.

For \(Y = E, S = I\) (identity operator) in (3) with \(\{\alpha_n\}_{n=0}^{\infty}\) satisfying (i)–(iv), then we have the Mann iteration process introduced by Mann [9]. Also, if in (3), \(Y = E, S = I\) (identity operator) and \(\alpha_n = 1\), then we obtain the Jungck iteration introduced by Jungck [8].

Following (3), we shall generalize and extend Groetsch [7] and others mentioned earlier in this paper by assuming that \(A\) is a segmenting matrix for which

\[
Sv_{n+1} = (1 - d_n)Sv_n + d_nTv_n, \quad n = 1, 2, \ldots, \tag{*}
\]
such that $\sum_{n=1}^\infty d_n(1 - d_n) = \infty$ and $S,T: C \to C$ are selfmappings on a nonempty convex subset $C$ of a uniformly convex Banach space $E$. The operators $S$ and $T$ are assumed to have a common fixed point and satisfy in addition the contractive condition

$$\|Tx - Ty\| \leq \|Sx - Sy\|, \quad \forall x, y \in C. \quad (**)$$

If $S = I$ (identity operator) in $(\star)$, then we obtain (2) and if $S = I$ in $(**)$ then we have $\|Tx - Ty\| \leq \|x - y\|, \forall x, y \in C$ (that is, $T$ becomes a nonexpansive mapping).

We shall establish our main result in the next section. However, the following lemma is required in the sequel.

**Lemma 2.1** (Groetsch [7]) Let $X$ be a uniformly convex Banach space and let $x, y \in X$. If $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| \geq \epsilon > 0$, then

$$\|\lambda x + (1 - \lambda)y\| \leq 1 - 2\lambda(1 - \lambda)\delta(\epsilon)$$

for $0 \leq \lambda < 1$ and $\delta(\epsilon) > 0$.

The proof of this Lemma is contained in [4, 7].

### 3 The Main Result

**Theorem 3.1** Let $C$ be a convex subset of a uniformly convex Banach space $E$ and $S,T: C \to C$ selfmappings satisfying condition $(**)$ and $T(C) \subseteq S(C)$. Suppose that $S$ and $T$ have at least a common fixed point. Let $\{Sv_n\}_{n=1}^\infty$ be the sequence defined by $(\star)$. Then, the sequence $\{(S - T)v_n\}_{n=1}^\infty$ converges strongly to 0 for each $x_1 \in C$ such that $\sum_{n=1}^\infty d_n(1 - d_n) = \infty$.

**Proof** If $p$ is a common fixed point of $S$ and $T$ (i.e. $Sp = Tp = p$), then

\[
\|Sv_{n+1} - p\| = \|(1 - d_n)Sv_n + d_nTv_n - (1 - d_n + d_n)p\|
\leq (1 - d_n)\|Sv_n - p\| + d_n\|Tv_n - p\|
\leq (1 - d_n)\|Sv_n - p\| + d_n\|Sp - Sv_n - Sp\|
= (1 - d_n)\|Sv_n - p\| + d_n\|Sv_n - p\|
\leq \|Sv_n - p\| \leq \|Sv_{n-1} - p\| \leq \cdots \leq \|Sv_1 - p\|,
\]

from which we have that the sequence $\{Sv_n - p\}_{n=1}^\infty$ is decreasing.

Now,

\[
\|(S - T)v_n\| = \|Sv_n - Tv_n\| \leq \|Sv_n - p\| + \|p - Tv_n\|
= \|Sv_n - p\| + \|Tp - Tv_n\| \leq \|Sv_n - p\| + \|Sp - Sv_n\| = 2\|Sv_n - p\|.
\]
Suppose on the contrary that \(\{(S - T)v_n\}_{n=1}^\infty\) does not converge to 0. Since \(\|Sv_n - Tv_n\| \leq 2\|Sv_n - p\|\), we may assume that there is an \(a > 0\), \(a \in (0, 1)\) such that \(\|Sv_n - p\| \geq a\) for any \(n\). If \(\{(S - T)v_n\}_{n=1}^\infty\) does not converge to 0, then there is an \(\epsilon > 0\) such that \(\|Sv_n - Tv_n\| \geq \epsilon\) for any \(n\).

Let 
\[
    b = 2\delta \left( \frac{\epsilon}{\|Sv_1 - p\|} \right), \quad x_n = \frac{Sv_n - p}{\|Sv_n - p\|} \quad \text{and} \quad y_n = \frac{Tv_n - p}{\|Sv_n - p\|}.
\]

Then, we have 
\[
    \|x_n\| = \left\| \left( \frac{Sv_n - p}{\|Sv_n - p\|} \right) \right\| \leq \frac{\|Sv_n - p\|}{\|Sv_n - p\|} = 1
\]
and 
\[
    \|y_n\| = \left\| \left( \frac{Tv_n - p}{\|Sv_n - p\|} \right) \right\| \leq \frac{\|Tv_n - Tp\|}{\|Sv_n - p\|} \leq \frac{\|Sv_n - Sp\|}{\|Sv_n - p\|} = \frac{\|Sv_n - p\|}{\|Sv_n - p\|} = 1.
\]

Hence, we have by (*) that 
\[
    \|Sv_{n+1} - p\| = \|(1 - d_n)Sv_n + d_nTv_n - (1 - d_n + d_n)p\|
    = \|(1 - d_n)(Sv_n - p) + d_n(Tv_n - p)\|
    = \left\| \left( \|Sv_n - p\| \right) \left( 1 - d_n \right) \frac{(Sv_n - p)}{\|Sv_n - p\|} + d_n \frac{(Tv_n - p)}{\|Sv_n - p\|} \right\|
    \leq \|Sv_n - p\| (1 - d_n) \|x_n + d_n y_n\|
    \leq \|Sv_n - p\| (1 - d_n) \|x_n + d_n y_n\|.
\]

Using (4) and Lemma 2.1 in (5) yield 
\[
    \|Sv_{n+1} - p\| \leq \|Sv_1 - p\| = \|Sv_{n+1} - p\| \leq \|Sv_1 - p\|
    \leq 1 - d_n (1 - d_n) b \|Sv_n - p\|
    = \|Sv_n - p\| - bd_n (1 - d_n) \|Sv_n - p\|
    \leq \|Sv_{n-1} - p\| - bd_{n-1} (1 - d_{n-1}) \|Sv_{n-1} - p\| - bd_n (1 - d_n) \|Sv_n - p\|
    \leq \|Sv_{n-1} - p\| - bd_{n-1} (1 - d_{n-1}) \|Sv_{n-1} - p\| - bd_n (1 - d_n) \|Sv_n - p\|
    = \|Sv_{n-1} - p\| - b [d_{n-1} (1 - d_{n-1}) + d_n (1 - d_n)] \|Sv_n - p\|.
\]

Repeating this process inductively leads to 
\[
    a \leq \|Sv_{n+1} - p\| \leq \|Sv_1 - p\|
    - b \left[ d_1 (1 - d_1) \|Sv_1 - p\| + d_2 (1 - d_2) \|Sv_1 - p\| + \cdots + d_n (1 - d_n) \|Sv_n - p\| \right]
    = \|Sv_1 - p\| - b \sum_{j=1}^n d_j (1 - d_j) \|Sv_1 - p\| \leq \|Sv_1 - p\| - ab \sum_{j=1}^n d_j (1 - d_j).
\]
Therefore, we obtain
\[
a \left[ 1 + b \sum_{j=1}^{n} d_j (1 - d_j) \right] \leq \|Sv_1 - p\|,
\]
from which it follows that
\[
a \leq \frac{\|Sv_1 - p\|}{1 + b \sum_{j=1}^{n} d_j (1 - d_j)} \to 0 \quad \text{as } n \to \infty,
\]
leading to a contradiction. Therefore, we have \(a = 0\). Hence,
\[
\lim_{n \to \infty} \|Sv_n - Tv_n\| = 0.
\]

**Remark 3.1** Theorem 3.1 is also a generalization of the results of [3, 6, 7, 9, 11, 12].

**References**


