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Some Stability Results in Complete Metric Space

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Abstract

In this paper, we obtain some stability results for the Picard iteration process for one and two metrics in complete metric space by using different contractive definitions which are more general than those of Berinde [1], Imoru and Olatinwo [5] some others listed in the reference section. The results generalize and unify some of the results of Harder and Hicks [4], Rhoades [10, 12], Osilike [8], Berinde [1], Imoru and Olatinwo [5] as well as Imoru et al [6].

Key words: Stability results, Picard and Mann iteration processes.

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1 Preliminaries and Introduction

Let (E, d) be a complete metric space, $T : E \to E$ a selfmap of E.

Definition 1.1 [Harder and Hicks [4]]: Suppose that $F_T = \{p \in E \mid Tp = p\}$ is the set of fixed points of T. Let $\{x_n\}_{n=0}^{\infty} \subset E$ be the sequence generated by an iteration procedure involving T which is defined by

$$x_{n+1} = f(T, x_n), \quad n = 0, 1, \dots, \tag{1.1}$$

where $x_0 \in E$ is the initial approximation and f is some function. Suppose $\{x_n\}_{n=0}^{\infty}$ converges to a fixed point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset E$ and set $\epsilon_n = d(y_{n+1}, f(T, y_n))$, $n = 0, 1, 2, \dots$ Then, the iteration procedure (1.1) is said to be $T-$stable or stable with respect to T if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies $\lim_{n\to\infty} y_n = p$.

Definition 1.2 [Singh et al [13]]: Let $S, T\colon Y \to E$, $T(Y) \subseteq S(Y)$ and z a coincidence point of S and T, that is, $Sz = Tz = p$ (say). For any $x_0 \in Y$, let the sequence $\{Sx_n\}_{n=0}^{\infty}$, generated by the iteration procedure

$$Sx_{n+1} = f(T, x_n), \quad n = 0, 1, \ldots \tag{1.2}$$

converge to p. Let $\{Sy_n\}_{n=0}^{\infty} \subset E$ be an arbitrary sequence, and set $\epsilon_n = d(Sy_{n+1}, f(T, y_n))$, $n = 0, 1, \ldots$ Then, the iteration procedure (1.2) will be called (S, T)-stable if and only if $\lim_{n\to\infty} \epsilon_n = 0$ implies that $\lim_{n\to\infty} Sy_n = p$.

This definition reduces to that of the stability of iteration procedure due to Harder and Hicks [4] when $Y = E$ and $S = I$ (identity operator).

If in (1.1),

$$f(T, x_n) = Tx_n, \quad n = 0, 1, \ldots,$$

then we have the Picard iteration process, while we obtain the Jungck-type iteration if in (1.2)

$$f(T, x_n) = Tx_n, \quad n = 0, 1, \ldots$$

Definition 1.3 [Berinde [2]]: A function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if:

(i) ψ is monotone increasing;

(ii) $\lim\limits_{n\to\infty} \psi^n(t) = 0, \ \forall t \geq 0$.

We remark here that every comparison function satisfies the condition $\psi(0) = 0$.

Several stability results have been obtained by various authors using different contractive definitions. Harder and Hicks [4] obtained interesting stability results for some iteration procedures using various contractive definitions. Rhoades [10, 12] generalized the results of Harder and Hicks [4] to a more general contractive mapping. In Osilike [8], a generalization of some of the results of Harder and Hicks [4] and Rhoades [12] was obtained by employing the following contractive definition: there exist a constant $L \geq 0$ and $a \in [0, 1)$ such $\forall x, y \in E$,

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y). \tag{1.3}$$

Condition (1.3) is more general than those of Rhoades [12] and Harder and Hicks [4]. As in Harder and Hicks [4], Berinde [1] obtained the same stability results for the same iteration procedures using the same contractive definitions, but applied a different method. The method of Berinde [1] is similar to that employed in Osilike and Udomene [9].

Recently, Imoru and Olatinwo [5] obtained some stability results for Picard and Mann iteration procedures by using a more general contractive condition than those of Harder and Hicks [4], Rhoades [12], Osilike [8], Osilike and Udomene [9] and Berinde [1]. In the paper [5], the following contractive definition was employed: there exist $a \in [0, 1)$ and a monotone increasing function $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, with $\varphi(0) = 0$, such that $\forall x, y \in E$,

$$d(Tx, Ty) \leq \varphi(d(x, Tx)) + ad(x, y). \tag{1.4}$$

It is our purpose in this paper to obtain several stability results in metric space by applying different contractive definitions. However, we shall employ the following lemmas in the sequel.

Lemma 1.4 [Imoru et al [6]]: *If* $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a subadditive comparison function and* $\{\epsilon_n\}_{n=0}^{\infty}$ *is a sequence of positive numbers such that* $\lim_{n\to\infty} \epsilon_n = 0$, *then for any sequence of positive numbers* $\{u_n\}_{n=0}^{\infty}$ *satisfying*

$$u_{n+1} \leq \sum_{m=0}^{s} \delta_m \psi^m(u_n) + \epsilon_n, \quad n = 0, 1, 2, \ldots,$$

where $\sum_{m=0}^{s} \delta_m = 1$, $\delta_0, \delta_1, \cdots, \delta_s \in [0,1]$, *we have* $\lim_{n\to\infty} u_n = 0$.

Lemma 1.5 [Imoru et al [6]]: *Let* $\{\psi^k(t)\}_{k=0}^{n}$ *be a sequence of comparison functions. Then, any convex linear combination* $\sum_{j=0}^{n} c_j \psi^j(t)$ *of the comparison functions is also a comparison function, where* $\sum_{j=0}^{n} c_j = 1$ *and* $c_o, c_1, \ldots, c_n$ *are positive constants.*

Lemma 1.6 [Imoru et al [6]]: *Let* $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *be a comparison function and* $\{v_n\}_{n=0}^{\infty}$ *a sequence of positive numbers such that* $\lim_{n\to\infty} v_n = 0$. *Then, we have*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \psi^{n-k}(v_k) = 0, \quad \text{for each } k.$$

Lemma 1.7 *If* $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ *is a subadditive comparison function and* $\{\epsilon_n\}_{n=0}^{\infty}$ *is a sequence of positive numbers such that* $\lim_{n\to\infty} \epsilon_n = 0$. *Suppose that* $\epsilon > 0$ *is an arbirarily small given number. Then, for any sequence of positive numbers* $\{u_n\}_{n=0}^{\infty}$ *satisfying*

$$u_{n+1} \leq \sum_{k=0}^{m} \delta_k \psi^k(u_n) + \epsilon_n + \epsilon, \quad n = 0, 1, \ldots, \tag{1.5}$$

where $\delta_k \in [0,1]$, $k = 0, 1, \ldots, m$, $0 \leq \sum_{k=0}^{m} \delta_k \leq 1$, *we have*

$$\lim_{n\to\infty} u_n = 0$$

Proof By putting $\bar{\psi}(u_n) = \sum_{k=0}^{m} \delta_k \psi^k(u_n)$ in (1.5), then we have

$$u_{n+1} \leq \bar{\psi}(u_n) + \epsilon_n + \epsilon, \quad n = 0, 1, \ldots, \tag{1.6}$$

and also by Lemma 1.5, we have that $\bar{\psi}(u_n)$ is a comparison function. It follows from (1.6) that

$$\begin{aligned}
u_1 &\leq \bar{\psi}(u_0) + \epsilon_0 + \epsilon, \\
u_2 &\leq \bar{\psi}(u_1) + \epsilon_1 + \epsilon \leq \bar{\psi}(\bar{\psi}(u_0) + \epsilon_0 + \epsilon) + \epsilon_1 + \epsilon \\
&\leq [\bar{\psi}^2(u_0) + \bar{\psi}(\epsilon_0) + \epsilon_1] + [\bar{\psi}(\epsilon) + \epsilon], \\
u_3 &\leq \bar{\psi}(u_2) + \epsilon_2 + \epsilon \leq \bar{\psi}^3(u_0) + \bar{\psi}^2(\epsilon_0) + \bar{\psi}(\epsilon_1) + \bar{\psi}^2(\epsilon) + \bar{\psi}(\epsilon) + \epsilon_2 + \epsilon \\
&= [\bar{\psi}^3(u_0) + \bar{\psi}^2(\epsilon_0) + \bar{\psi}(\epsilon_1) + \epsilon_2] + [\bar{\psi}^2(\epsilon) + \bar{\psi}(\epsilon) + \epsilon]
\end{aligned}$$

In general,

$$u_{n+1} \leq \bar{\psi}^{n+1}(u_0) + \sum_{k=0}^{n} \bar{\psi}^{n-k}(\epsilon_k) + \sum_{k=0}^{n} \bar{\psi}^{k}(\epsilon). \tag{1.7}$$

Since $\bar{\psi}$ is a comparison function, then $\lim_{n\to\infty} \bar{\psi}^{n+1}(u_0) = 0$. □

Using Lemma 1.6, we obtain that

$$\lim_{n\to\infty} \sum_{k=0}^{n} \bar{\psi}^{n-k}(\epsilon_k) = 0 \quad \text{and} \quad \lim_{n\to\infty} \sum_{k=0}^{n} \bar{\psi}^{k}(\epsilon) = 0$$

since $\epsilon > 0$ is arbitrary. Hence, (1.7) leads to $\lim_{n\to\infty} u_n = 0$.

We shall establish our main results in the next two sections. Section 2 deals with some stability results involving one metric, while stability results involving two metrics are proved in section 3.

2 Stability results involving one metric in complete metric space

Theorem 2.1 *Let (E, d) be a complete metric space and $T\colon E \to E$ a selfmap of E satisfying*

$$d(Tx, Ty) \leq \frac{\varphi_1(d(x, Tx)) + \psi(d(x, y))}{\varphi_2(d(x, Tx))}, \quad \forall x, y \in E, \tag{2.1}$$

where $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous comparison function and $\varphi_1, \varphi_2\colon \mathbb{R}_+ \to \mathbb{R}_+$ are monotone increasing functions such that $\varphi_1(0) = 0$ and $\varphi_2(0) = 1$. Suppose T has a fixed point p. Let $x_0 \in E$ and let $x_{n+1} = Tx_n$, $n = 0, 1, \ldots,$ be the Picard iteration associated to T. Then, the Picard iteration process is T-stable.

Proof Let $\{y_n\}_{n=0}^{\infty} \subset E$ and $\epsilon_n = d(y_{n+1}, Ty_n)$. Assume $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} y_n = p$ by using the contractive condition and the triangle inequality:

$$d(y_{n+1}, p) \leq d(Tp, Ty_n) + \epsilon_n \leq \psi(d(y_n, p)) + \epsilon_n. \tag{2.2}$$

Using Lemma 1.4 in (2.2) yields $\lim_{n\to\infty} d(y_n, p) = 0$, that is, $\lim_{n\to\infty} y_n = p$. Conversely, let $\lim_{n\to\infty} y_n = p$. Then, by the contractive condition and the triangle inequality, we have

$$\epsilon_n = d(y_{n+1}, Ty_n) \leq d(y_{n+1}, p) + \psi(d(y_n, p)) \to 0 \quad \text{as } n \to \infty. \qquad \square$$

Corollary 2.2 *Let (E, d) be a complete metric space and $T\colon E \to E$ a selfmap of E satisfying*

$$d(Tx, Ty) \leq \frac{\varphi(d(x, Tx)) + ad(x, y)}{1 + Ld(x, Tx)}, \quad \forall x, y \in E,$$

where $a \in [0, 1)$, $L \geq 0$ and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone increasing function such that $\varphi(0) = 0$. Suppose T has a fixed point p. Let $x_0 \in E$ and let $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$, be the Picard iteration associated to T. Then, the Picard iteration process is T-stable.

Corollary 2.3 *Let (E, d) be a complete metric space and $T\colon E \to E$ a selfmap of E satisfying*

$$d(Tx, Ty) \leq \varphi_1(d(x, Tx)) + \frac{\psi(d(x, y))}{\varphi_2(d(x, Tx))}, \quad \forall x, y \in E,$$

where $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous comparison function and $\varphi_1, \varphi_2\colon \mathbb{R}_+ \to \mathbb{R}_+$ are monotone increasing functions such that $\varphi_1(0) = 0$ and $\varphi_2(0) = 1$. Suppose T has a fixed point p. Let $x_0 \in E$ and let $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$, be the Picard iteration associated to T. Then, the Picard iteration process is T-stable.

Remark 2.4 Theorem 2.1 and its corollaries generalize and unify Theorem 3.1 of Imoru and Olatinwo [5] and several others in the literature. In particular, see Berinde [1], Imoru and Olatinwo [5], Rhoades [10, 11, 12] and some other references in the reference section of this paper for detail.

We now establish the following stability results for uniform convergence of sequences of operators:

Theorem 2.5 *Let (E, d) be a complete metric space and $\{T_n\}_{n=0}^{\infty}$ a sequence of operators $T_n\colon E \to E$. Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration process. If the sequence $\{T_n\}_{n=0}^{\infty}$ converges uniformly to an operator $T\colon E \to E$ satisfying*

$$d(Tx, Ty) \leq \varphi(d(x, Tx)) + \psi(d(x, y)), \quad \forall x, y \in E, \tag{2.3}$$

where $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone increasing function such that $\varphi(0) = 0$ and $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, subadditive comparison function. Suppose also that T has the fixed point p. Then, the Picard iteration process is T-stable.

Proof Let $\{y_n\}_{n=0}^{\infty} \subset E$ and let $\epsilon_n = d(y_{n+1}, T_n y_n)$, $d(T_n x, Tx) < \epsilon$, $\forall x \in E$, $\forall n \geq N$. Assume $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} y_n = p$ by using the contraction condition (2.3) for T and the triangle inequality:

$$\begin{aligned} d(y_{n+1}, p) &\leq d(y_{n+1}, T_n y_n) + d(T_n y_n, p) \leq d(Tp, Ty_n) + d(Ty_n, T_n y_n) + \epsilon_n \\ &\leq \psi(d(p, y_n)) + \epsilon_n + \epsilon. \end{aligned} \tag{2.4}$$

Using Lemma 1.7 in (2.4) yields

$$d(y_{n+1}, p) \to 0 \quad \text{as } n \to \infty$$

That is, since $\epsilon > 0$ is arbitrary, then $\lim_{n\to\infty} y_n = p$.

Conversely, let $\lim_{n\to\infty} y_n = p$. Then, we have

$$\epsilon_n = d(y_{n+1}, T_n y_n) \leq d(y_{n+1}, p) + \psi(d(p, y_n)) + \epsilon \to 0 \quad \text{as } n \to \infty,$$

since $\epsilon > 0$ is arbitrary. □

Corollary 2.6 *Let (E,d) be a complete metric space and $\{T_n\}_{n=0}^{\infty}$ a sequence of operators $T_n\colon E \to E$. Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration process. If the sequence $\{T_n\}_{n=0}^{\infty}$ converges uniformly to an operator $T\colon E \to E$ satisfying*

$$d(Tx,Ty) \le \varphi(d(x,Tx)) + ad(x,y), \quad \forall x,y \in E,\ a \in [0,1),$$

where $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone increasing function such that $\varphi(0) = 0$. Suppose also that T has the fixed point p. Then, the Picard iteration process is T-stable.

Remark 2.7 We remark that this theorem holds if $\{T_n\}$ converges pointwise to T since uniform convergence is more general than pointwise convergence.

Corollary 2.8 *Let (E,d) be a complete metric space and $\{T_n\}_{n=0}^{\infty}$ a sequence of operators $T_n\colon E \to E$. Let $\{x_n\}_{n=0}^{\infty}$ be the Picard iteration process. If the sequence $\{T_n\}_{n=0}^{\infty}$ converges pointwise to an operator $T\colon E \to E$ satisfying*

$$d(Tx,Ty) \le \varphi(d(x,Tx)) + \psi(d(x,y)), \quad \forall x,y \in E,$$

where $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone increasing function such that $\varphi(0) = 0$ and $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, subadditive comparison function. Suppose also that T has the fixed point p. Then, the Picard iteration process is T-stable.

Remark 2.9 To the best of our knowledge, this is the first time that stability results are being considered using the concepts of uniform and pointwise convergence of sequences of operators.

Theorem 2.10 *Let (E,d) be a complete metric space and Y an arbitrary set. Suppose that $S,T\colon Y \to E$ are nonselfoperators such that $T(Y) \subseteq S(Y)$, $S(Y)$ a complete subspace of E. Let z be a coincidence point of S and T (that is, $Sz = Tz = p$). Suppose that S and T satisfy the contractive condition*

$$d(Tx,Ty) \le \frac{\psi(d(Sx,Sy))}{1 + Md(Sx,Tx)}, \quad M \ge 0,\ \forall x,y \in Y, \tag{2.5}$$

where $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous subadditive comparison function. For $x_0 \in Y$, let $\{Sx_n\}_{n=0}^{\infty}$ be the Jungck-type iteration process defined by $Sx_{n+1} = Tx_n$, $n = 0,1,\ldots$, converging to p. Then, the Jungck-type iteration process is (S,T)-stable.

Proof We now assume that $\lim_{n\to\infty} \epsilon_n = 0$ and establish that $\lim_{n\to\infty} Sy_n = p$, using the contractive condition and triangle inequality. Therefore, we have

$$d(Sy_{n+1},p) \le d(Sy_{n+1},Ty_n) + d(Ty_n,p) \le \psi(d(p,Sy_n)) + \epsilon_n \tag{2.6}$$

By using Lemma 1.4 in (2.6), we get $\lim_{n\to\infty} d(Sy_n,p) = 0$, that is,

$$\lim_{n\to\infty} Sy_n = p.$$

Conversely, let $\lim_{n\to\infty} Sy_n = p$. Then, by the contractive condition on S and T as well as the triangle inequality, we have

$$\begin{aligned}\epsilon_n &= d(Sy_{n+1}, Ty_n) \leq d(Sy_{n+1}, p) + d(p, Ty_n) \\ &\leq d(Sy_{n+1}, p) + \psi(d(p, Sy_n)) \to 0 \quad \text{as } n \to \infty. \qquad \square\end{aligned}$$

Theorem 2.11 *Let S and T be operators on an arbitrary set Y with values in E such that $T(Y) \subseteq S(Y)$ and $S(Y)$ or $T(Y)$ is a complete subspace of E. Let z be a coincidence point of S and T (i.e. $S(z) = T(z) = p$ (say)). Let $x_0 \in Y$ and let $\{Sx_n\}_{n=0}^{\infty} \subset E$ defined by $Sx_{n+1} = Tx_n$, $n = 0, 1, \cdots$, be the Jungck iteration process converging to p. Suppose that $\{Sy_n\}_{n=0}^{\infty} \subset E$ and $\epsilon_n = d(Sy_{n+1}, Ty_n)$, $n = 0, 1, \cdots$ Suppose that S and T satisfy the contractive condition*

$$d(Tx, Ty) \leq \frac{\psi(d(Sx, Sy)) + \varphi(d(Sx, Tx))}{1 + Md(Sx, Tx)}, \quad M \geq 0,\ \forall x, y \in Y, \tag{2.7}$$

where $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous subadditive comparison function and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a monotone increasing function such that $\varphi(0) = 0$. Then, the Jungck iteration process is (S, T)-stable.

Proof The proof of this theorem follows a similar argument as in that of Theorem 2.10. □

Remark 2.12 Theorem 2.10 and others extend some celebrated results of [1, 4, 8, 9, 12] and some results due to the author [5, 6]. Infact, Theorem 2.10 is also a generalization and extension of Theorem 3.1 of Singh et al [13].

3 Stability results involving two metrics d and ρ on a nonempty set E

Theorem 3.1 *Let E be a nonempty set, d and ρ two metrics on E and $T\colon E \to E$ a mapping. Suppose that:*

(i) T has a fixed point p;

(ii) there exist $c > 0$, and a monotone increasing function $\varphi_1\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi_1(0) = 0$ such that

$$d(Tx, Ty) \leq \varphi_1(\rho(x, Tx)) + c\rho(x, y), \quad \forall x, y \in E;$$

(iii) (E, d) is a complete metric space;

(iv) $T\colon (E, \rho) \to (E, \rho)$ satisfies the contractive condition

$$\rho(Tx, Ty) \leq \varphi_2(\rho(x, Tx)) + \psi(\rho(x, y)), \quad \forall x, y \in E,$$

where $\psi^k\colon \mathbb{R}_+ \to \mathbb{R}_+$, $k = 1, 2, \ldots$, are continuous comparison functions (ψ^k is the k-th iterate of ψ) and $\varphi_2\colon \mathbb{R}_+ \to \mathbb{R}_+$, $k = 1, 2, \ldots$, is a monotone increasing function such that $\varphi_2(0) = 0$.

Let $x_0 \in E$ and $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$, be the Picard iteration associated to T. Then, the Picard iteration process with $T\colon (E, d) \to (E, d)$ is T-stable.

Proof Let $\{y_n\}_{n=0}^{\infty} \subset E$, $\epsilon_n = d(y_{n+1}, Ty_n)$, $n = 0, 1, \ldots$, and suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} y_n = p$, using conditions (i)-(iv) and the triangle inequality: Therefore, using (i), (ii) and triangle inequality lead to

$$\begin{aligned} d(y_{n+1}, p) &\leq d(Ty_n, Tp) + \epsilon_n \leq \varphi_1(\rho(p, Tp)) + c\rho(p, y_n) + \epsilon_n \\ &= c\rho(y_n, p) + \epsilon_n. \end{aligned} \tag{3.1}$$

Using (iii), we have that $p \in E$. Condition (iv) shows that T has a unique fixed point. Also by condition (iv), we get

$$\begin{aligned} \rho(y_n, p) &= \rho(Ty_{n-1}, Tp) = \rho(Tp, Ty_{n-1}) \leq \psi(\rho(y_{n-1}, p)) \\ &\leq \psi^2(\rho(y_{n-2}, p)) \leq \cdots \leq \psi^n(\rho(y_0, p)) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.2}$$

Using (3.2) in (3.1), we have

$$d(y_{n+1}, p) \leq c\psi^n(\rho(y_0, p)) + \epsilon_n. \tag{3.3}$$

Taking limits of both sides in (3.3) yields

$$\lim_{n\to\infty} d(y_{n+1}, p) \leq c \lim_{n\to\infty} \psi^n(\rho(y_0, p)) + \lim_{n\to\infty} \epsilon_n \to 0 \quad \text{as } n \to \infty$$

That is, $\lim_{n\to\infty} y_n = p$.

Conversely, let $\lim_{n\to\infty} y_n = p$. Then, by condition (ii) and (3.2) we have

$$\begin{aligned} \epsilon_n &= d(y_{n+1}, Ty_n) \leq d(y_{n+1}, p) + d(Tp, Ty_n) \\ &\leq d(y_{n+1}, p) + c\psi^n(\rho(p, y_0)) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

□

Corollary 3.2 *Let E be a nonempty set, d and ρ two metrics on E and $T\colon E \to E$ a mapping. Suppose that:*

(i) T has a fixed point p;

(ii) there exist $c > 0$, $M \geq 0$ such that

$$d(Tx, Ty) \leq M\rho(x, Tx) + c\rho(x, y), \quad \forall x, y \in E;$$

(iii) (E, d) is a complete metric space;

(iv) $T\colon (E, \rho) \to (E, \rho)$ satisfies the contractive condition

$$\rho(Tx, Ty) \leq \varphi(\rho(x, Tx)) + \psi(\rho(x, y)), \quad \forall x, y \in E,$$

where $\psi^k\colon \mathbb{R}_+ \to \mathbb{R}_+$, $k = 1, 2, \ldots$, are continuous comparison functions (ψ^k is the k-th iterate of ψ) and $\varphi\colon \mathbb{R}_+ \to \mathbb{R}_+$, $k = 1, 2, \ldots$, monotone increasing functions such that $\varphi(0) = 0$.

Let $x_0 \in E$ and $x_{n+1} = Tx_n$, $n = 0, 1, \ldots$, be the Picard iteration associated to T. Then, the Picard iteration process with $T\colon (E, d) \to (E, d)$ is T-stable.

Theorem 3.3 *Let E be a nonempty set and Y an arbitrary set. Let d and ρ two metrics on Y and $S, T\colon Y \to E$ nonselfmappings such that $T(Y) \subseteq S(Y)$ and $S(Y)$ is a complete subspace of E. Suppose that:*

(i) S and T have a coincidence point z (that is $Tz = Sz = p$);

(ii) there exist $c > 0$, and a monotone increasing function $\varphi_1\colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\varphi_1(0) = 0$ such that

$$d(Tx, Ty) \leq \varphi_1(\rho(Sx, Tx)) + c\rho(Sx, Sy), \quad \forall x, y \in Y;$$

(iii) (E, d) is a complete metric space;

(iv) $T\colon (Y, \rho) \to (E, \rho)$ satisfies the contractive condition

$$\rho(Tx, Ty) \leq \varphi_2(\rho(Sx, Tx)) + \psi(\rho(Sx, Sy)), \quad \forall x, y \in Y,$$

where $\psi^k\colon \mathbb{R}_+ \to \mathbb{R}_+$, $k = 1, 2, \dots$, are continuous comparison functions (ψ^k is the k-th iterate of ψ) and $\varphi_2\colon \mathbb{R}_+ \to \mathbb{R}_+$, $k = 1, 2, \dots$, is a monotone increasing function such that $\varphi_2(0) = 0$.

Let $x_0 \in E$ and $x_{n+1} = Tx_n$, $n = 0, 1, \dots$, be the Jungck-type iteration associated to S and T. Then, the Jungck-type iteration process with $T\colon (Y, d) \to (E, d)$ is (S, T)-stable.

Proof Let $\{Sy_n\}_{n=0}^{\infty} \subset E$, $\epsilon_n = d(Sy_{n+1}, Ty_n)$, $n = 0, 1, \dots$, and suppose that $\lim_{n\to\infty} \epsilon_n = 0$. Then, we shall establish that $\lim_{n\to\infty} Sy_n = p$, using conditions (i)–(iv) and the triangle inequality: Therefore, using (i), (ii) and triangle inequality lead to

$$\begin{aligned} d(Sy_{n+1}, p) &\leq d(Sy_{n+1}, Ty_n) + d(Ty_n, p) = d(Tz, Ty_n) + \epsilon_n \\ &\leq \varphi_1(\rho(Sz, Tz)) + c\rho(Sz, Sy_n) + \epsilon_n = c\rho(p, Sy_n) + \epsilon_n. \end{aligned} \tag{3.4}$$

Using (iii), we have that $p \in E$. Condition (iv) shows that T has a unique fixed point. Also by condition (iv), we get

$$\begin{aligned} \rho(p, Sy_n) &= \rho(Tz, Ty_{n-1}) \leq \psi(\rho(Sy_{n-1}, p)) \\ &\leq \psi^2(\rho(Sy_{n-2}, p)) \leq \cdots \leq \psi^n(\rho(Sy_0, p)) \to 0 \quad \text{as } n \to \infty. \end{aligned} \tag{3.5}$$

Using (3.5) in (3.4), we have

$$d(Sy_{n+1}, p) \leq c\psi^n(\rho(Sy_0, p)) + \epsilon_n. \tag{3.6}$$

Taking limits of both sides in (3.6) yields

$$\lim_{n\to\infty} d(Sy_{n+1}, p) \leq c \lim_{n\to\infty} \psi^n(\rho(Sy_0, p)) + \lim_{n\to\infty} \epsilon_n = 0$$

That is, $\lim_{n\to\infty} Sy_n = p$.

Conversely, let $\lim_{n\to\infty} Sy_n = p$. Then, by condition (ii) and (3.5) we have

$$\begin{aligned} \epsilon_n &= d(Sy_{n+1}, Ty_n) \leq d(Sy_{n+1}, p) + d(Tz, Ty_n) \\ &\leq d(Sy_{n+1}, p) + c\psi^n(\rho(p, Sy_0)) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

□

Remark 3.4 Theorem 3.1 and Theorem 3.3 as well as the corollary generalize and extend the well-known stability results in the literature. In particular, see Singh et al [13], Berinde [1], Imoru and Olatinwo [5], Rhoades [10, 11, 12] and some other references in the reference section of this paper for detail. Indeed, Theorem 3.1 and Theorem 3.3 are generalizations and extensions of Theorem 3.1 and Theorem 3.4 of Singh et al [13].

Remark 3.5 To the best of our knowledge, this is the first time the stability of the Picard and Jungck-type iteration processes is being investigated for the case of two metrics.

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