Linearization Regions for a Confidence Ellipsoid in Singular Nonlinear Regression Models

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Abstract

A construction of confidence regions in nonlinear regression models is difficult mainly in the case that the dimension of an estimated vector parameter is large. A singularity is also a problem. Therefore some simple approximation of an exact confidence region is welcome. The aim of the paper is to give a small modification of a confidence ellipsoid constructed in a linearized model which is sufficient under some conditions for an approximation of the exact confidence region.

Key words: Nonlinear regression model, confidence region, singularity.

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1 Introduction

A construction of a confidence region for unbiasedly estimable functions of nonlinear singular regression model parameters can be a difficult numerical problem (for more detail on nonlinear models cf. [6]). Mainly the case of a large dimension of a vector parameter is unwelcome. If a confidence region can be

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approximated by a confidence ellipsoid (in the case of normally distributed observation vector), then a numerical calculation and an interpretation of results are much more easier and simpler.

Therefore an attempt to find a simple measure of nonlinearity which enable us to decide whether an approximate confidence ellipsoid can be used instead of exact confidence region, is the aim of the paper.

2 Notation and some useful statements

The following notation is used.

\[
Y \sim N_n(f(\beta), \Sigma) \tag{1}
\]

means that \(Y\) is an \(n\)-dimensional normally distributed random vector with the mean value \(E(Y) = f(\beta)\) and with the covariance matrix \(\text{Var}(Y) = \Sigma\). Let the function \(f(\cdot): \mathbb{R}^k \to \mathbb{R}^n\) (\(\mathbb{R}^n\) is the \(n\)-dimensional real linear vector space) can be expressed by the Taylor series of the second order, i.e.

\[
f(\beta) = f_0 + F\delta\beta + \frac{1}{2}\kappa(\delta\beta), \quad \delta\beta = \beta - \beta_0,
\]

\[
f_0 = f(\beta_0), \quad \beta_0 \text{ is an approximate value of } \beta,
\]

\[
F = \left. \frac{\partial f(u)}{\partial u} \right|_{u=\beta_0}, \quad \kappa(\delta\beta) = \left[ \kappa_1(\delta\beta), \ldots, \kappa_n(\delta\beta) \right]',
\]

\[
\kappa_i(\delta\beta) = \left. \delta\beta^T \frac{\partial^2 f_i(u)}{\partial u \partial u^T} \right|_{u=\beta_0} \delta\beta, \quad i = 1, \ldots, n.
\]

The matrix \(F\) need not be of the full rank in columns and \(\Sigma\) need not be positive definite.

The linearized version of the model (1) is

\[
Y - f_0 \sim N_n(F\delta\beta, \Sigma) \tag{2}
\]

and the quadratized version is

\[
Y - f_0 \sim N_n \left( F\delta\beta + \frac{1}{2}\kappa(\delta\beta), \Sigma \right). \tag{3}
\]

In the following text the notations

\(A^-\) ... g-inverse (generalized inverse) of the matrix \(A\),
\(A^+\) ... the Moore–Penrose g-inverse of the matrix \(A\),
\(A_{m(W)}^-\) ... minimum \(W\)-seminorm g-inverse of the matrix \(A\), (\(W\) is positive semidefinite matrix),
\(\mathcal{M}(A_{m,n}) = \{Au: u \in \mathbb{R}^n\}\) (column space of the matrix) \(A\),
\(I\) ... identity matrix,
\( P_{F'} = F'(FF')^{-1}F \) the projection matrix on the space \( \mathcal{M}(F') \) in the Euclidean norm,

\( r(A) \) the rank of the matrix \( A \),

\[ U = \text{Var}(P_{F'}\delta\beta), \]

\[ T = \Sigma + FF', \]

will be used. More on a \( g \)-inverse of a matrix cf. [7].

In the model (2) a representative of all unbiasedly estimable linear functions of the parameter \( \beta \) is the vector

\[ \gamma = P_{F'}\beta = P_{F'}\beta_0 + P_{F'}\delta\beta = \gamma_0 + \delta\gamma. \]

**Lemma 1** In the model (2) the \((1 - \alpha)\)-confidence ellipsoid of the vector \( P_{F'}\delta\beta \) is

\[ E_{P_{F'},\delta\beta} = \left\{ P_{F'}u : P_{F'}u - P_{F'}\delta\beta \in \mathcal{M} \left[ \text{Var}(P_{F'}\delta\beta) \right], \left( P_{F'}u - P_{F'}\delta\beta \right)' \times \left[ \text{Var}(P_{F'}\delta\beta) \right]^{-1} \left( P_{F'}u - P_{F'}\delta\beta \right) \leq \chi^2_n(\Sigma + FF' - \Sigma) \right\}, \]

where

\[ P_{F'}\delta\beta = P_{F'} \left[ (F')_m(\Sigma) \right]' (Y - f_0), \]

\[ \text{Var}(P_{F'}\delta\beta) = P_{F'} \left[ (F'TF)' - I \right] P_{F'}, \quad T = \Sigma + FF'. \]

**Proof** is given in [2]. \( \square \)

In the following text it is necessary to take into account the fact that even \( \beta_0 \) can be considered to be known, only \( P_{F'}(\beta - \beta_0) = P_{F'}\delta\beta \) can be unbiasedly estimated. Let

\[ \beta_0 = \gamma_0 + \omega_0, \quad \gamma_0 = P_{F'}\beta_0, \quad \omega_0 = M_{F'}\beta_0; \]

the parameter \( \delta\gamma = P_{F'}(\beta - \beta_0) \) is unbiasedly estimable in the model (2), however \( \delta\omega = M_{F'}(\beta - \beta_0) \) is not. Therefore the model

\[ Y \sim N_n \left[ f(\beta_0) + F\delta\gamma + \frac{1}{2} \kappa_{\omega_0}(\delta\gamma), \Sigma \right] \]

will be considered instead the model (3). Here

\[ \kappa_{\omega_0} = (\kappa_{\omega_0,1}, \ldots, \kappa_{\omega_0,n})', \]

\[ \kappa_{\omega_{n,i}} = \delta\gamma \frac{\partial^2 f_i(\gamma_0 + \omega_0)}{\partial\gamma' \partial\gamma}, \quad i = 1, \ldots, n, \]

\[ F = \frac{\partial f(\gamma_0 + \omega_0)}{\partial\gamma'}. \]
Lemma 2  The bias $b$ of the estimator

$$\hat{\delta} = P F' \delta = P F'[\hat{(F')_m}')(Y - f_0)$$

in the model (4) is

$$b = E(\hat{\delta}) - \delta = \frac{1}{2} P F' [(F')_m)' \kappa_{\omega}(\delta)$$

$$= \frac{1}{2} P F' (F'T^T F) - F'T^T \kappa_{\omega}(\delta) .$$

Proof is implied by the definition of the bias. \qed

Lemma 3  Let $Y \sim N_k(\mu, \Sigma)$. Then

$$Y' \Sigma^+ Y \sim \chi^2_r(\delta),$$

where $\delta = \mu' \Sigma^+ \mu = \mu' P \Sigma \Sigma^+ P \mu$.

Proof  Let $J$ be a $k \times r(\Sigma)$ matrix such that $JJ' = \Sigma$ and $K$ such a $k \times r(\Sigma)$ matrix that $KK' = \Sigma^+$ (i.e. $J'K = I$). Then $K'Y = K'\mu + \eta, \eta \sim N_r(0, I)$. Thus

$$Y'KK'Y = Y' \Sigma^+ Y = \eta' \eta + 2\eta'K'\mu + \mu' \Sigma^+ \mu \sim \chi^2_r(\mu' \Sigma^+ \mu).$$

However $\Sigma^+ = P \Sigma \Sigma^+ P \Sigma$ since

$$\Sigma P \Sigma \Sigma^+ P \Sigma = \Sigma, \quad \Sigma P \Sigma \Sigma^+ P \Sigma = \Sigma P \Sigma \Sigma^+ P \Sigma,$$

$$(in\ more\ detail\ cf. [7]). \quad \square$$

3  A linearization region for a confidence ellipsoid

Since $r[\text{Var}(P F' \delta)] = r[F' (\Sigma + FF')^{-1} \Sigma]$, it can happen that

$$r[\text{Var}(P F' \delta)] = r(\Sigma) < r(F').$$

Therefore the vector $b$ need not be an element of $\mathcal{M}[\text{Var}(P F' \delta)]$.

The relation

$$\delta \gamma = P F' \delta = E(P F' \delta) = b = E(\hat{\delta}) - b,$$

valid in the model (3) and (4), respectively, implies that in general case the vector $P F' \delta$ need not be an element of $\mathcal{E}_{P F' \delta}$ from Lemma 1. Thus it seems to be reasonable to enlarge the ellipsoid $\mathcal{E}_{P F' \delta}$ by $\hat{b}$ in such a way that $P F' \delta \in \mathcal{E}$.

In the following text the notation $U = \text{Var}(P F' \delta)$ will be used.
Definition 1 Let a set $\mathcal{E}$ be defined as
\[
\mathcal{E} = \left\{ P_F \cdot u : u \in \mathbb{R}^k, (P_F \cdot u - \overline{P_F} \cdot \delta \beta) \right\}^+(U + c^2(P_F - P_U)) + \times (P_F \cdot u - \overline{P_F} \cdot \delta \beta) \leq \chi^2(\mathbb{F}^T - \Sigma)(0; 1 - \alpha) \right\},
\]
where $T = \Sigma + FF'$ and the choice $c^2$ depends on the opinion of the user (cf. the following remark).

Remark 1 The number $c^2$ should be comparable with the spectral numbers of the matrix $U$. The semi-axes of $\mathcal{E}$ in the space $M(P_F - P_U)$ have the same size equal to
\[
a = c \sqrt{\frac{\chi^2(\mathbb{F}^T - \Sigma)(0; 1 - \alpha)}{\lambda_{max}(P_F - P_U)}},
\]
The smaller is $c$, the smaller is the probability $P \left\{ P_F \cdot \delta \beta \notin \mathcal{E} \right\}$. Thus $c$ cannot be smaller than some reasonable bound. If $b \in M(U)$, then it can be tolerated in the case $b'U - b \leq \varepsilon$. Let
\[
U = \sum_{i=1}^f \lambda_i f'_i f_i, \quad f = r(F'T^T \Sigma),
\]
be spectral decomposition of the matrix $U$ and
\[
\lambda_{max} = \max\{ \lambda_i : i = 1, \ldots, r(F'T^T \Sigma) \}.
\]
If $h = s f_{max}$ (the vector $f_{max}$ corresponds to $\lambda_{max}$), then, regarding the Scheffé theorem [8] ($b'U - b \leq \varepsilon \Leftrightarrow \forall \{ h \in M(U) \} |h'b| \leq \varepsilon \sqrt{|h'Uf|}$),
\[
|h'b| = s|f'_{max} b| \leq s \varepsilon \sqrt{\lambda_{max}}.
\]
In the worst case (i.e. $b = t f_{max}$) $|b| = t < \varepsilon \sqrt{\lambda_{max}}$. It implies that the bias $b$ with the norm smaller than $\varepsilon \sqrt{\lambda_{max}}$ can be tolerated and thus the choice $c^2 = \lambda_{max}$ is reasonable.

Definition 2 Let the measure of nonlinearity for the confidence ellipsoid be
\[
C^{(ell)} = \sup \left\{ \frac{2 \sqrt{b'(\delta \gamma)[U + \lambda_{max}(P_F - P_U)]^b(\delta \gamma)}}{\delta \gamma'[U + \lambda_{max}(P_F - P_U)]^b \delta \gamma} : \delta \gamma \in \mathbb{R}^n(F) \right\},
\]
where
\[
b(\delta \gamma) = \frac{1}{2} P_F (F'T^T F)' - F'T^T k(\delta \gamma).
\]

Theorem 1 If $\delta \beta \in \mathcal{L}^{(ell)}_{\delta \gamma}$, where
\[
\mathcal{L}^{(ell)}_{\delta \gamma} = \left\{ \delta \gamma : \delta \gamma \in M(F'), \delta \gamma' [U + \lambda_{max}(P_F - P_U)]^{-1} \delta \gamma \leq \frac{2 \sqrt{\lambda_{max}}}{C^{(ell)}} \right\},
\]
then
\[ P\left\{ \delta \gamma \in \mathfrak{F} \right\} \geq 1 - \alpha - \varepsilon. \]

Here \( \delta_{\text{max}} \) is a solution of the equation
\[ P\left\{ \chi_{\gamma}^2(\delta_{\text{max}}) \leq \chi_{\gamma}^2(0; 1 - \alpha) \right\} = 1 - \alpha - \varepsilon \]
and \( f = r(\mathbf{F}^T\mathbf{T}^{-1}) \).

**Proof** Regarding Definition 6
\[ 2\sqrt{b'(\delta \gamma)[\mathbf{U} + \lambda_{\text{max}}(\mathbf{P}_F' - \mathbf{P}_U)]^+ b(\delta \gamma)} \leq \delta \gamma'[\mathbf{U} + \lambda_{\text{max}}(\mathbf{P}_F' - \mathbf{P}_U)]^{-} \delta \gamma C^{(\ell\ell)}. \]

Let
\[ \delta \gamma'[\mathbf{U} + \lambda_{\text{max}}(\mathbf{P}_F' - \mathbf{P}_U)]^{-} \delta \gamma \leq \frac{2\sqrt{\delta_{\text{max}}}}{C^{(\ell\ell)}}. \]

Further
\[ (\hat{\delta \gamma} - \delta \gamma)'[\mathbf{U} + \lambda_{\text{max}}(\mathbf{P}_F' - \mathbf{P}_U)]^+ (\hat{\delta \gamma} - \delta \gamma) = \]
\[ = \left[ \hat{\delta \gamma} - E(\hat{\delta \gamma}) + E(\hat{\delta \gamma}) - \delta \gamma \right]'[\mathbf{U} + \lambda_{\text{max}}(\mathbf{P}_F' - \mathbf{P}_U)]^+ \]
\[ \times \left[ \hat{\delta \gamma} - E(\hat{\delta \gamma}) + E(\hat{\delta \gamma}) - \delta \gamma \right] \]
\[ = (\hat{\delta \gamma} - E(\hat{\delta \gamma}))'[\mathbf{U} + \lambda_{\text{max}}(\mathbf{P}_F' - \mathbf{P}_U)]^+ (\hat{\delta \gamma} - E(\hat{\delta \gamma})) \]
\[ + 2b'(\delta \gamma)[\mathbf{U} + \lambda_{\text{max}}(\mathbf{P}_F' - \mathbf{P}_U)]^+ b(\delta \gamma) = \chi_{\gamma}^2(\delta), \]
where
\[ \delta = b'(\delta \gamma)[\mathbf{U} + \lambda_{\text{max}}(\mathbf{P}_F' - \mathbf{P}_U)]^+ b(\delta \gamma), \]
what is implied by Lemma 3. The relation
\[ [(\mathbf{Y} - \mu) + \mu']\Sigma^+ [(\mathbf{Y} - \mu) + \mu] = \]
\[ = (\mathbf{Y} - \mu)\Sigma^-(\mathbf{Y} - \mu) + 2\mu'\Sigma^+ (\mathbf{Y} - \mu) + \mu'\Sigma^+ \mu = \chi_{r(\Sigma)}^2(\mu'\Sigma^+ \mu), \]
based on Lemma 3 is used as well.

Thus
\[ (\hat{\delta \gamma} - \delta \gamma)'[\mathbf{U} + \lambda_{\text{max}}(\mathbf{P}_F' - \mathbf{P}_U)]^+ (\hat{\delta \gamma} - \delta \gamma) = \chi_{\gamma}^2(\delta), \]
where
\[ \delta = b'(\delta \gamma)[\mathbf{U} + \lambda_{\text{max}}(\mathbf{P}_F' - \mathbf{P}_U)]^+ b(\delta \gamma). \]

If \( \delta \leq \delta_{\text{max}} \), then
\[ P\left\{ \chi_{\gamma}(\delta) \leq \chi_{\gamma}^2(0; 1 - \alpha) \right\} \geq P\left\{ \chi_{\gamma}(\delta_{\text{max}}) \leq \chi_{\gamma}^2(0; 1 - \alpha) \right\} = 1 - \alpha - \varepsilon, \]
what means \( P\left\{ \delta \gamma \in \mathfrak{F} \right\} \geq 1 - \alpha - \varepsilon. \) \( \square \)
Remark 2 Let us apply Theorem 1 on the regular linearized model. Then $P_{F'} = P_U = I$, $E = E_{\delta \gamma}$ and $C^{(ell)} = K^{(par)}$, where $K^{(par)}$ is the Bates and Watts parametric curvature

$$K^{(par)} = \sup \left\{ \sqrt{\kappa'(\delta \beta) \Sigma^{-1} P_{F'}^{-1} \kappa(\delta \beta)} \cdot \delta \beta \in R^k \right\}$$

(in more detail cf. [1]). In this case the statement

$$\delta \beta \in \left\{ u: u^T \Sigma^{-1} Fu \leq \frac{2 \sqrt{\delta_{max}}}{K^{(par)}} \right\} \Rightarrow P \left\{ P_{F'} \delta \beta \in E_{\delta \beta} \right\} = P \left\{ \delta \beta \in E_{\delta \beta} \right\} \geq 1 - \alpha - \varepsilon$$

is true (cf. also [4]). Thus Theorem 7 is a reasonable generalization suitable for the singular model.

Remark 3 In the case that only one function of the parameter $\beta$, i.e. $h(\gamma) = h'\gamma + h'\delta \gamma$, $\delta \gamma \in \mathcal{M}(F')$, is important, a very simple procedure can be used. Let in the first case $h'P_{F'}[(F'T^-F)^{-1} - I]P_{F'}h > 0$.

Since

$$b_h = E(h'\delta \gamma) - h'\delta \gamma = \frac{1}{2} h'P_{F'}[(F')_{m(\Sigma)}]' \kappa_{\omega_0}(\delta \gamma) = \delta \gamma' A_h \delta \gamma,$$

where

$$A_h = \sum_{i=1}^n \left\{ \frac{1}{2} h'P_{F'}[(F')_{m(\Sigma)}]' \right\} \left\{ \frac{\partial f_i(u + \omega_0)}{\partial u} \right\}_{u=\gamma_0},$$

we obtain

$$\delta \gamma \in L_{h',\delta \gamma} = \left\{ u: u \in \mathcal{M}(F'), |u'A_h \delta \gamma| \leq \sqrt{\delta_{1,max}} \right\}$$

$$\Rightarrow P \left\{ |h'\delta \gamma - \delta \gamma| \leq \sqrt{\chi^2(0;1 - \alpha)} \sqrt{h'P_{F'}UP_{F'}h} \right\} \geq 1 - \alpha - \varepsilon.$$

Here $\delta_{1,max}$ is a solution of the equation

$$P \left\{ \chi^2(\delta_{1,max}) \leq \chi^2(0;1 - \alpha) \right\} = 1 - \alpha - \varepsilon.$$

If $h'Uh = 0$, then

$$P \left\{ h'\delta \gamma - E(h'\delta \gamma) = 0 \right\} = 1$$

and thus

$$P \left\{ h'\delta \gamma = h'\delta \gamma + h'b(\delta \gamma) \right\} = 1.$$

Thus

$$\delta \gamma \in L_{h',\delta \gamma} = \left\{ u: u \in \mathcal{M}(F'), |u'A_h u| \leq \Delta \right\}$$

$$\Rightarrow P \left\{ h'\delta \gamma \in \left\{ u: u \in R^1, |u - h'\delta \gamma| \leq \Delta \right\} \right\} = 1.$$
4 Numerical example

Let us consider the regression model

\[
\begin{pmatrix}
Y_1 \\
Y_2 \\
Y_3 \\
Y_4 \\
Y_5 \\
Y_6 \\
\end{pmatrix}
\sim N_6
\begin{pmatrix}
\beta_1 \exp(-\beta_3) \\
\beta_1 \exp(-\beta_3) \\
\beta_1 \exp(-\beta_3) \\
\beta_2 \exp(-\beta_3) \\
\beta_2 \exp(-\beta_3) \\
\beta_2 \exp(-\beta_3) \\
\end{pmatrix}, \Sigma_{6,6}, \beta = \begin{pmatrix}
\beta_1 \\
\beta_2 \\
\beta_3 \\
\end{pmatrix} \in \mathbb{R}^3, \\
\Sigma_{6,6} = \sigma^2 I_{6,6}, \sigma^2 = (0.5)^2.
\]

Then

\[
F = \frac{\partial f(u + \omega_0)}{\partial u'} \bigg|_{u=\gamma_0} = \begin{pmatrix}
1_3, 0, -1_3 \\
0, 1_3, -1_3 \\
\end{pmatrix}, \quad 1_3 = \begin{pmatrix}
1 \\
1 \\
1 \\
\end{pmatrix},
\]

\[
F_1 = F_2 = F_3 = \begin{pmatrix}
0, 0, -1 \\
0, 0, 0 \\
-1, 0, 1 \\
\end{pmatrix}, \quad F_4 = F_5 = F_6 = \begin{pmatrix}
0, 0, 0 \\
0, 0, -1 \\
0, -1, 1 \\
\end{pmatrix}.
\]

Here

\[
F_i = \frac{\partial^2 f_i(u + \omega_0)}{\partial u \partial u'} \bigg|_{u=\gamma_0}, \quad i = 1, \ldots, 6,
\]

\[
P_{F'} = F'(FF')^{-1}F = \frac{1}{3} \begin{pmatrix}
2, -1, -1 \\
-1, 2, -1 \\
-1, -1, 2 \\
\end{pmatrix},
\]

\[
\text{Var}(P_{F'} \delta \beta) = U = P_{F'} \{ [F'(\Sigma + FF')^{-1}F]^{-1} - I \} P_{F'} = \frac{\sigma^2}{54} \begin{pmatrix}
10, -8, -2 \\
-8, 10, -2 \\
-2, -2, 4 \\
\end{pmatrix},
\]

\[
P_U = U(U^2)^{-1}U = \frac{1}{3} \begin{pmatrix}
2, -1, -1 \\
-1, 2, -1 \\
-1, -1, 2 \\
\end{pmatrix},
\]

\[
U = \sum_{i=1}^{2} \lambda_i f_i f_i' = \sum_{i=1}^{2} \lambda_i f_i f_i', \quad \lambda_1 = \frac{1}{3} \sigma^2, \quad \lambda_2 = \frac{1}{9} \sigma^2, \quad \lambda_{\max} = \frac{1}{3} \sigma^2,
\]

\[
\delta_{\max} = 0.48 \text{ is a solution of the equation}
\]

\[
P \{ \chi^2_f(0; 1 - \alpha) \} = 1 - \alpha - \varepsilon,
\]

and \( f = r[F'(\Sigma + FF')^{-1} \Sigma] = 2, \alpha = 0.05, \varepsilon = 0.04. \)
Further

\[ C^{\text{ell}} = \sup \left\{ \frac{2\sqrt{\Delta(\delta \gamma)[U + \lambda_{\max}(P_{F'} - P_U)]^+\Delta(\delta \gamma)}}{\delta \gamma'[U + \lambda_{\max}(P_{F'} - P_U)]^{-\delta \gamma}} : \delta \gamma \in \mathbb{R}^2 \right\} = \sigma \cdot 0.191273, \]

where

\[ \Delta = \frac{1}{2} P_{F'} (F' T^{-1} F)^{-1} F' T^{-1} \kappa_{\omega}(\delta \gamma). \]

The linearization region for \( \delta \gamma = P_{F'} \delta \beta \) is

\[ \mathcal{L}_{\delta \gamma} = \left\{ \mathbf{u} : \mathbf{u} \in \mathcal{M}(F'), \mathbf{u}'[U + \lambda_{\max}(P_{F'} - P_U)]^+ \mathbf{u} \leq \frac{2\sqrt{\sigma_{\max}}}{C^{\text{ell}}} \right\} \]

and the set \( \overline{\mathcal{E}_{\delta \gamma}} \) is

\[ \overline{\mathcal{E}_{\delta \gamma}} = \left\{ \mathbf{u} : \mathbf{u} \in \mathcal{M}(F'), (\mathbf{u} - \overline{\delta \gamma})'[U + \lambda_{\max}(P_{F'} - P_U)]^+ \times (\mathbf{u} - \overline{\delta \gamma}) \leq \chi^2_{\nu(F'T^{-1}F)}(0; 1 - \alpha) \right\} \]

The linearization region \( \mathcal{L}_{\delta \gamma} \) is the ellipse in the subspace \( \mathcal{M}(F') \) with the semi-axes

\[ a_{\mathcal{L}, 1} = 1.5539 \sqrt{\sigma}, \quad a_{\mathcal{L}, 2} = 0.8972 \sqrt{\sigma} \]

and \( \overline{\mathcal{E}_{\delta \gamma}} \) is the ellipse in \( \mathcal{M}(F') \) with the semi-axes

\[ a_{\mathcal{E}, 1} = 0.2359 \sigma, \quad a_{\mathcal{E}, 2} = 0.1362 \sigma. \]

For \( \sigma = 0.5 \) it means

\[ a_{\mathcal{L}, 1} = 1.099, \quad a_{\mathcal{L}, 2} = 0.634 \]

and

\[ a_{\mathcal{E}, 1} = 0.118, \quad a_{\mathcal{E}, 2} = 0.068. \]

Thus the linearization is possible.

As far as the single function of \( \beta \) is concerned let us consider \( h = (1, 0, 0)' \).

\[ A_h = \sum_{s=1}^{6} \left\{ \frac{1}{2} h' P_{F'} \left[ F' (\Sigma + FF')^{-1} F (\Sigma + FF')^{-1} \right] s \right\} F_s \]

\[ = \frac{1}{18} \begin{pmatrix} 0, 0, -6 \\ 0, 0, 3 \\ -6, 3, 3 \end{pmatrix} \]

and

\[ \mathcal{L}_{h'; \delta \gamma} = \left\{ \mathbf{u} : \mathbf{u} \in \mathcal{M}(F'), \mathbf{u}' A_h \mathbf{u} \leq \delta_{1, \max} \right\} \]
where \( \delta_{1,\text{max}} = 0.339 \) is a solution of the equation

\[
P\left\{ \chi_1^2(\delta_{1,\text{max}}) \leq \chi_1^2(0; 0.95) \right\} = 1 - 0.05 - 0.04.
\]

The linearization region \( L_{h,\delta\gamma} \) is the hyperbola in \( \mathcal{M}(\mathbf{F}') \) with the real semi-axis \( a = 1.1768 \) and the imaginary \( b_i, b = 1.714 \). Thus the linearization region for the confidence interval for \( \delta\gamma_1 \) is essentially larger (in the case \( \sigma = 0.5 \)) than the linearization region for the whole vector \( \delta\gamma \).

References