

# Eliminating Transformations for Nuisance Parameters in Linear Regression Models with Type I Constraints

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## Abstract

The linear regression model in which the vector of the first order parameter is divided into two parts: to the vector of the useful parameters and to the vector of the nuisance parameters is considered. The type I constraints are given on the useful parameters. We examine eliminating transformations which eliminate the nuisance parameters without loss of information on the useful parameters.

**Key words:** Regular linear regression model; nuisance parameters; BLUE; constraints.

**2000 Mathematics Subject Classification:** 62J05

## 1 Introduction, notations

Transformations for nuisance parameters in linear regression models with nuisance parameters are studied for instance in [3], [4], [6]. This paper deals with similar problems in models to which type I constraints are added.

The following notation will be used throughout the paper:

$R^n$	the space of all $n$ -dimensional real vectors;
$u_p, A_{m,n}$	the real column $p$ -dimensional vector, the real $m \times n$ matrix;
$A', r(A)$	the transpose, the rank of the matrix $A$ ;
$\mathcal{M}(A), \text{Ker}(A)$	the range, the null space of the matrix $A$ ;
$A^-$	a generalized inverse of a matrix $A$ (satisfying $AA^-A = A$ );
$A^+$	the Moore–Penrose generalized inverse of a matrix $A$ (satisfying $AA^+A = A$ , $A^+AA^+ = A^+$ , $(AA^+)' = AA^+$ , $(A^+A)' = A^+A$ );
$P_A$	the orthogonal projector in the Euclidean norm onto $\mathcal{M}(A)$ ;
$M_A = I - P_A$	the orthogonal projector in the Euclidean norm onto $\mathcal{M}^\perp(A)$ ;
$I_k$	the $k \times k$ identity matrix;
$O_{m,n}$	the $m \times n$ null matrix;
$o$	the null vector.

If  $\mathcal{M}(A) \subset \mathcal{M}(U)$ ,  $U$  p.s.d., then the symbol  $P_A^{U^-}$  denotes the projector projecting vectors in  $\mathcal{M}(U)$  onto  $\mathcal{M}(A)$  along  $\mathcal{M}(UA^\perp)$ . A general representation of all such projectors  $P_A^{U^-}$  is given by  $A(A'U^-A)^-A'U^- + B(I - UU^-)$ , where  $B$  is arbitrary, (see [7], (2.14)).  $M_A^{U^-} = I - P_A^{U^-}$ .

Let  $N_{n,n}$  is p.d. (p.s.d.) matrix and  $A_{m,n}$  an arbitrary matrix, then the symbol  $A_{m(N)}^-$  denotes the matrix satisfying  $AA_{m(N)}^-A = A$  and  $NA_{m(N)}^-A = (NA_{m(N)}^-A)'$ . [ $A_{m(N)}^-y$  is any solution of the consistent system  $Ax = y$  whose  $N$ -seminorm is minimal]. In general  $A_{m(N)}^- = (N + A'A)^-A'[A(N + AA^-)^-A']^-$ . If the condition  $\mathcal{M}(A') \subset \mathcal{M}(N)$  is fulfilled, then  $A_{m(N)}^- = N^-A'(AN^-A')^-$ , (see [2], pp. 14–15).

**Assertion 1** (see [3], Lemma 10.1.35) *Let  $X$  be any  $n \times k$  matrix and  $\Sigma$  an  $n \times n$  p.s.d. matrix.*

(i) *If  $\Sigma$  is p.d., then*

$$(M_X \Sigma M_X)^+ = \Sigma^{-1} - \Sigma^{-1} X (X' \Sigma^{-1} X)^- X' \Sigma^{-1} = \Sigma^{-1} M_X^{\Sigma^{-1}}.$$

(ii) *If  $\Sigma$  is not p.d. however  $\mathcal{M}(X) \subset \mathcal{M}(\Sigma)$ , then*

$$(M_X \Sigma M_X)^+ = \Sigma^+ - \Sigma^+ X (X' \Sigma^- X)^- X' \Sigma^+.$$

(iii) *In general case*

$$(M_X \Sigma M_X)^+ = (\Sigma + XX')^+ - (\Sigma + XX')^+ X [X'(\Sigma + XX')^- X]^- X'(\Sigma + XX')^+.$$

(iv)

$$(M_X \Sigma M_X)^+ = M_X (M_X \Sigma M_X)^+ = (M_X \Sigma M_X)^+ M_X = M_X (M_X \Sigma M_X)^+ M_X.$$

**Assertion 2** *Let  $D = \begin{pmatrix} A, & B \\ B', & C \end{pmatrix}$  be symmetric and positive semidefinite matrix. If  $\mathcal{M}(B') \subset \mathcal{M}(C - B'A^+B)$ , then*

$$D^+ = \begin{pmatrix} A, & B \\ B', & C \end{pmatrix}^+ = \begin{pmatrix} A^+ + A^+B(C - B'A^+B)^+B'A^+, & -A^+B(C - B'A^+B)^+ \\ -(C - B'A^+B)^+B'A^+, & (C - B'A^+B)^+ \end{pmatrix}.$$

If  $\mathcal{M}(\mathbf{B}) \subset \mathcal{M}(\mathbf{A} - \mathbf{B}\mathbf{C}^+\mathbf{B}')$ , then

$$\mathbf{D}^+ = \begin{pmatrix} \mathbf{A}, & \mathbf{B} \\ \mathbf{B}', & \mathbf{C} \end{pmatrix}^+ = \begin{pmatrix} (\mathbf{A} - \mathbf{B}\mathbf{C}^+\mathbf{B}')^+, & -(\mathbf{A} - \mathbf{B}\mathbf{C}^+\mathbf{B}')^+\mathbf{B}\mathbf{C}^+ \\ -\mathbf{C}^+\mathbf{B}'(\mathbf{A} - \mathbf{B}\mathbf{C}^+\mathbf{B}')^+, & \mathbf{C}^+ + \mathbf{C}^+\mathbf{B}'(\mathbf{A} - \mathbf{B}\mathbf{C}^+\mathbf{B}')^+\mathbf{B}\mathbf{C}^+ \end{pmatrix}.$$

**Proof** Assertions can be proved directly. As  $\mathbf{D}$  is p.s.d. matrix, there exists block matrix  $\begin{pmatrix} \mathbf{J} \\ \mathbf{K} \end{pmatrix}$  such that

$$\begin{pmatrix} \mathbf{A}, & \mathbf{B} \\ \mathbf{B}', & \mathbf{C} \end{pmatrix} = \begin{pmatrix} \mathbf{J} \\ \mathbf{K} \end{pmatrix} (\mathbf{J}', \mathbf{K}') = \begin{pmatrix} \mathbf{J}\mathbf{J}', & \mathbf{J}\mathbf{K}' \\ \mathbf{K}\mathbf{J}', & \mathbf{K}\mathbf{K}' \end{pmatrix} \Rightarrow \mathcal{M}(\mathbf{B}) = \mathcal{M}(\mathbf{J}\mathbf{K}') \subset \mathcal{M}(\mathbf{J}) = \mathcal{M}(\mathbf{A}),$$

analogously  $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C})$ . It implies that  $\mathbf{A}\mathbf{A}^+\mathbf{B} = \mathbf{B}$ ,  $\mathbf{B}'\mathbf{A}^+\mathbf{A} = \mathbf{B}'$ ,  $\mathbf{C}\mathbf{C}^+\mathbf{B}' = \mathbf{B}'$ ,  $\mathbf{B}\mathbf{C}^+\mathbf{C} = \mathbf{B}$ . These matrices don't depend on the choice of g-inverses. We can easily prove, that relations  $\mathbf{D}\mathbf{D}^+\mathbf{D} = \mathbf{D}$ ,  $\mathbf{D}^+\mathbf{D}\mathbf{D}^+ = \mathbf{D}^+$  are valid for both formulas. Matrices  $\mathbf{D}^+\mathbf{D}$ ,  $\mathbf{D}\mathbf{D}^+$  are symmetric, if conditions  $\mathcal{M}(\mathbf{B}') \subset \mathcal{M}(\mathbf{C} - \mathbf{B}'\mathbf{A}^+\mathbf{B})$  and  $\mathcal{M}(\mathbf{B}) \subset \mathcal{M}(\mathbf{A} - \mathbf{B}\mathbf{C}^+\mathbf{B}')$  are satisfied. It is to be remarked that these conditions are valid if  $r(\mathbf{D}) = r(\mathbf{A}) + r(\mathbf{C})$ .  $\square$

Let us consider following linear model with nuisance parameters

$$\mathbf{Y} \sim [(\mathbf{X}, \mathbf{S}) \begin{pmatrix} \beta \\ \kappa \end{pmatrix}, \Sigma_{\vartheta}], \quad \Sigma_{\vartheta} \text{ known matrix}, \quad (1)$$

where  $\mathbf{Y} = (\mathbf{Y}_1, \dots, \mathbf{Y}_n)'$  is a random observation vector;  $\beta \in R^k$  is a vector of the useful parameters;  $\kappa \in R^l$  is a vector of the nuisance parameters;  $\mathbf{X}_{n,k}$  is a design matrix belonging to the vector  $\beta$ ;  $\mathbf{S}_{n,l}$  is a design matrix belonging to the vector  $\kappa$ .

We suppose that

1.  $E(\mathbf{Y}) = \mathbf{X}\beta + \mathbf{S}\kappa$ ,  $\forall \beta \in R^k$ ,  $\forall \kappa \in R^l$ ,

2.  $\text{var}(\mathbf{Y}) = \Sigma_{\vartheta} = \sum_{i=1}^p \vartheta_i \mathbf{V}_i$ ,  $\vartheta = (\vartheta_1, \dots, \vartheta_p)' \in \underline{\vartheta} \subset R^p$ ,  $\underline{\vartheta}$  is supposed to be with nonempty topological interior.

In this paper we consider that the given matrices  $\mathbf{V}_1, \dots, \mathbf{V}_p$  are p.s.d. and that the variance components  $\vartheta_1, \dots, \vartheta_p$  are positive (mixed linear model, see [1], Chapter 4).

3.  $\Sigma_{\vartheta}$  is not a function of the vector  $(\beta', \kappa)'$ .

If matrix  $\Sigma_{\vartheta}$  is positive definite and  $r(\mathbf{X}, \mathbf{S}) = k + l < n$ , the model is said to be *regular*, (see [3], p.13).

Parametric function  $\mathbf{f}'\beta$  is unbiasedly estimable in model (1) iff  $\mathbf{f} \in \mathcal{M}(\mathbf{X}'\mathbf{M}_{\mathbf{S}})$ , see [6], Remark 2.

There are situations in the practice that auxiliary information on the vector of useful regression coefficients  $\beta$  is known, it means that the parametric space for  $\beta$  is not  $R^k$  but its subset only,

$$\beta \in \{\mathbf{u} \in R^k : \mathbf{b} + \mathbf{B}\mathbf{u} = \mathbf{o}\}, \quad (2)$$

where  $\mathbf{B}$  is a  $q \times k$  known matrix. Since no assumption on the  $r(\mathbf{B})$  is considered, it must be assumed that a given  $q$ -dimensional vector  $\mathbf{b}$  satisfies  $\mathbf{b} \in \mathcal{M}(\mathbf{B})$ . This constraints on the useful parameters will be called *type I constraints*.

**Lemma 1** *The class of unbiasedly estimable functions of the useful parameters in model (1) with constraints (2) is created by all functions  $\mathbf{h}'\beta$  possessing*

$$\mathbf{h} \in \mathcal{M}(\mathbf{X}'\mathbf{M}_S, \mathbf{B}').$$

**Proof** Function  $\mathbf{h}'\beta + a$ ,  $\mathbf{h} \in R^k$ ,  $a \in R$  is in model (1) with constraints (2) unbiasedly estimable iff there exists statistic  $\mathbf{g}'\mathbf{Y} + c$ ,  $\mathbf{g} \in R^n$ ,  $c \in R$  such that

$$E(\mathbf{g}'\mathbf{Y} + c) = \mathbf{g}'[\mathbf{X}\beta + \mathbf{S}\kappa] + c = \mathbf{h}'\beta + a, \quad \forall \beta, \forall \kappa$$

$$\Leftrightarrow (\mathbf{g}'\mathbf{X} - \mathbf{h}')\beta + c - a = 0 \wedge \mathbf{g}'\mathbf{S} = \mathbf{o}', \quad \forall \beta$$

$$\Leftrightarrow (\mathbf{u}'\mathbf{M}_S\mathbf{X} - \mathbf{h}')\beta + c - a = 0, \quad \forall \beta, \mathbf{u} \in R^n$$

$$\Leftrightarrow \text{there exists vector } \mathbf{k} \in R^q \text{ such that } \mathbf{k}'\mathbf{B} = \mathbf{u}'\mathbf{M}_S\mathbf{X} - \mathbf{h}' \wedge \mathbf{k}'\mathbf{b} = c - a.$$

Because  $c$  can be chosen arbitrarily, the necessary and sufficient condition for unbiasedly estimable function is

$$\mathbf{u}'\mathbf{M}_S\mathbf{X} - \mathbf{k}'\mathbf{B} = \mathbf{h}' \Leftrightarrow \mathbf{h} = \mathbf{X}'\mathbf{M}_S\mathbf{u} - \mathbf{B}'\mathbf{k} \Leftrightarrow \mathbf{h} \in \mathcal{M}(\mathbf{X}'\mathbf{M}_S, \mathbf{B}'). \quad \square$$

**Remark 1** The BLUE (best linear unbiased estimator) of the vector function  $\mathbf{M}_S\mathbf{X}\beta$  in the singular model (1) with constraints (2) is

$$\widehat{\mathbf{M}_S\mathbf{X}\beta} = \mathbf{M}_S\mathbf{P}_{(\mathbf{X}\mathbf{M}_{B'}, \mathbf{S})}^{[\Sigma_\vartheta + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}' + \mathbf{S}\mathbf{S}']^+} \mathbf{Y} - \mathbf{M}_S\mathbf{M}_{(\mathbf{X}\mathbf{M}_{B'}, \mathbf{S})}^{[\Sigma_\vartheta + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}' + \mathbf{S}\mathbf{S}']^+} \mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^{-1}\mathbf{b}.$$

It is proved in [1], 2.10.2. and enables us to get BLUE of the unbiasedly estimable functions  $\mathbf{h}'\beta$ ,  $\mathbf{h} \in \mathcal{M}(\mathbf{X}'\mathbf{M}_S)$  in singular model (1) with constraints (2).

In the regular model (1) with constraints (2) the BLUE of the parameter  $\beta$  is given by

$$\hat{\beta} = [\mathbf{I} - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{B}]\beta^* - \mathbf{C}^{-1}\mathbf{B}'(\mathbf{B}\mathbf{C}^{-1}\mathbf{B}')^{-1}\mathbf{b},$$

where

$$\mathbf{C} = \mathbf{X}'(\mathbf{M}_S\Sigma_\vartheta\mathbf{M}_S)^+\mathbf{X},$$

and where

$$\beta^* = [\mathbf{X}'(\mathbf{M}_S\Sigma_\vartheta\mathbf{M}_S)^+\mathbf{X}]^{-1}\mathbf{X}'(\mathbf{M}_S\Sigma_\vartheta\mathbf{M}_S)^+\mathbf{Y},$$

(estimator in the regular model (1) without constraints).

The variance matrix of the estimator  $\hat{\beta}$  in regular model (1) with constraints (2) is given by

$$\text{var}(\hat{\beta}) = (\mathbf{M}_{B'}\mathbf{C}\mathbf{M}_{B'})^+.$$

These assertions are proved in [5], Theorem 1, Theorem 2.

In the literature there are investigated properties of estimators of the parameters  $\beta, \kappa$  in model (1) under constraints (2), see for example [1], [5]. In cases when we are interested on useful parameters only it is possible to simplify model (1) by the propriate eliminating transformation, see [3], [4], [6].

In this paper we join both of the procedures mentioned. Firstly we use eliminating transformation and then we add constraints to the transformed model.

## 2 Type I constraints in the transformed model

Our task will be to eliminate the matrix  $\mathbf{S}$  belonging to the vector of nuisance parameters, i.e. we consider the following class of eliminating matrices

$$\mathcal{T} = \{\mathbf{T} : \mathbf{T}\mathbf{S} = \mathbf{O}\},$$

where  $\mathbf{T}$  is matrix of the proper dimension, say of the type  $r \times n$ .

That leads us to linear models

$$\mathbf{T}\mathbf{Y} \sim [\mathbf{T}\mathbf{X}\beta, \mathbf{T}\Sigma_{\vartheta}\mathbf{T}']. \quad (3)$$

If we now add constraints (2) to the model (3), we get model

$$\begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix} \sim \left[ \begin{pmatrix} \mathbf{T}\mathbf{X} \\ \mathbf{B} \end{pmatrix} \beta, \begin{pmatrix} \mathbf{T}\Sigma_{\vartheta}\mathbf{T}', & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} \right]. \quad (4)$$

**Lemma 2** *Linear function  $\mathbf{f}'\beta + a$ ,  $\mathbf{f} \in R^k$ ,  $a \in R$  is unbiasedly estimable in model (4), iff*

$$\mathbf{f} \in \mathcal{M}(\mathbf{X}'\mathbf{T}', \mathbf{B}').$$

**Proof** The assertion can be proved in the same way as in Lemma 1.  $\square$

In the following text we consider only transformation matrices  $\mathbf{T}$  with the property

$$\mathcal{M}(\mathbf{X}'\mathbf{T}') = \mathcal{M}(\mathbf{X}'\mathbf{M}_S),$$

it means that transformations do not cause a loss of information on the parameter  $\beta$ .

**Theorem 1** *For the BLUE of the function of the parameter  $\beta$  in the model (4) holds*

$$\widehat{\mathbf{T}\mathbf{X}}\beta = \mathbf{P}_{\mathbf{T}\mathbf{X}\mathbf{M}_{B'}}^{[\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^+} \mathbf{T}\mathbf{Y} - \mathbf{M}_{\mathbf{T}\mathbf{X}\mathbf{M}_{B'}}^{[\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^+} \mathbf{T}\mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^{-1}\mathbf{b}.$$

**Proof** According to Theorem 3.1.3. in [3]

$$\begin{aligned} \widehat{\begin{pmatrix} \mathbf{T}\mathbf{X} \\ \mathbf{B} \end{pmatrix}}\beta &= \begin{pmatrix} \mathbf{T}\mathbf{X} \\ \mathbf{B} \end{pmatrix} \left[ (\mathbf{X}'\mathbf{T}', \mathbf{B}')_{m(\mathbf{T}\Sigma_{\vartheta}\mathbf{T}', \mathbf{O})} \right]' \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{T}\mathbf{X} \\ \mathbf{B} \end{pmatrix} \left[ (\mathbf{X}'\mathbf{T}', \mathbf{B}') \left\{ \begin{pmatrix} \mathbf{T}\Sigma_{\vartheta}\mathbf{T}', & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} + \begin{pmatrix} \mathbf{T}\mathbf{X} \\ \mathbf{B} \end{pmatrix} (\mathbf{X}'\mathbf{T}', \mathbf{B}') \right\}^{-1} \begin{pmatrix} \mathbf{T}\mathbf{X} \\ \mathbf{B} \end{pmatrix} \right]^{-1} \\ &\quad \times (\mathbf{X}'\mathbf{T}', \mathbf{B}') \left\{ \begin{pmatrix} \mathbf{T}\Sigma_{\vartheta}\mathbf{T}', & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} + \begin{pmatrix} \mathbf{T}\mathbf{X}\mathbf{X}'\mathbf{T}', & \mathbf{T}\mathbf{X}\mathbf{B}' \\ \mathbf{B}\mathbf{X}'\mathbf{T}', & \mathbf{B}\mathbf{B}' \end{pmatrix} \right\}^{-1} \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix}. \end{aligned}$$

By the help of the Rohde's formula for g-inverse of the p.s.d. partitioned matrix (see [3], Theorem 10.1.40) we can write

$$\begin{pmatrix} \mathbf{T}[\Sigma_{\vartheta} + \mathbf{X}\mathbf{X}']\mathbf{T}', & \mathbf{T}\mathbf{X}\mathbf{B}' \\ \mathbf{B}\mathbf{X}'\mathbf{T}', & \mathbf{B}\mathbf{B}' \end{pmatrix}^{-1} = \begin{pmatrix} \boxed{11}, & \boxed{12} \\ \boxed{21}, & \boxed{22} \end{pmatrix},$$

where

$$\begin{aligned}\boxed{11} &= [\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^{-}, \\ \boxed{12} &= -[\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^{-}\mathbf{T}\mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^{-}, \\ \boxed{21} &= -(\mathbf{B}\mathbf{B}')^{-}\mathbf{B}\mathbf{X}'\mathbf{T}'[\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^{-}, \\ \boxed{22} &= (\mathbf{B}\mathbf{B}')^{-} + (\mathbf{B}\mathbf{B}')^{-}\mathbf{B}\mathbf{X}'\mathbf{T}'[\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^{-}\mathbf{T}\mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^{-}.\end{aligned}$$

Then (we use Moore–Penrose g-inverse matrix for the sake of simplicity)

$$\begin{aligned}& \left[ (\mathbf{X}'\mathbf{T}', \mathbf{B}') \begin{pmatrix} \boxed{11} & \boxed{12} \\ \boxed{21} & \boxed{22} \end{pmatrix} \begin{pmatrix} \mathbf{T}\mathbf{X} \\ \mathbf{B} \end{pmatrix} \right]^{+} \\ &= [\mathbf{M}_{B'}\mathbf{X}'\mathbf{T}'(\mathbf{T}[\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}']\mathbf{T}')^{-}\mathbf{T}\mathbf{X}\mathbf{M}_{B'} + \mathbf{P}_{B'}]^{+},\end{aligned}$$

thus

$$\begin{aligned}& \widehat{\begin{pmatrix} \mathbf{T}\mathbf{X} \\ \mathbf{B} \end{pmatrix}} \beta = \\ &= \begin{pmatrix} \mathbf{T}\mathbf{X} \\ \mathbf{B} \end{pmatrix} \{ [\mathbf{M}_{B'}\mathbf{X}'\mathbf{T}'(\mathbf{T}[\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}']\mathbf{T}')^{-}\mathbf{T}\mathbf{X}\mathbf{M}_{B'}]^{+} + \mathbf{P}_{B'} \} (\mathbf{X}'\mathbf{T}', \mathbf{B}') \\ & \quad \times \begin{pmatrix} \boxed{11} & \boxed{12} \\ \boxed{21} & \boxed{22} \end{pmatrix} \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix}.\end{aligned}$$

After some calculations we get

$$\widehat{\begin{pmatrix} \mathbf{T}\mathbf{X} \\ \mathbf{B} \end{pmatrix}} \beta = \begin{pmatrix} \mathbf{P}_{\mathbf{T}\mathbf{X}\mathbf{M}_{B'}}^{[\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^{+}} \mathbf{T}\mathbf{Y} - \mathbf{M}_{\mathbf{T}\mathbf{X}\mathbf{M}_{B'}}^{[\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^{+}} \mathbf{T}\mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^{-}\mathbf{b} \\ -\mathbf{b} \end{pmatrix}.$$

In the course of the proof following assertion has been used

$$\mathbf{A}'\mathbf{B} = \mathbf{O} \wedge \mathbf{B}\mathbf{A}' = \mathbf{O} \Rightarrow (\mathbf{A} + \mathbf{B})^{+} = \mathbf{A}^{+} + \mathbf{B}^{+}. \quad \square$$

**Theorem 2** *The covariance matrix of the estimator  $\widehat{\mathbf{T}\mathbf{X}}\beta$  in model (4) is*

$$\text{var}[\widehat{\mathbf{T}\mathbf{X}}\beta] = \mathbf{T}\mathbf{X}\{[\mathbf{M}_{B'}\mathbf{X}'\mathbf{T}'(\mathbf{T}[\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}']\mathbf{T}')^{-}\mathbf{T}\mathbf{X}\mathbf{M}_{B'}]^{+} - \mathbf{M}_{B'}\}\mathbf{X}'\mathbf{T}'.$$

**Proof**

$$\begin{aligned}\text{var}[\widehat{\mathbf{T}\mathbf{X}}\beta] &= \\ &= \mathbf{P}_{\mathbf{T}\mathbf{X}\mathbf{M}_{B'}}^{[\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^{+}} [\mathbf{T}\Sigma_{\vartheta}\mathbf{T}' + \mathbf{T}\mathbf{X}\mathbf{M}_{B'}\mathbf{X}'\mathbf{T}' - \mathbf{T}\mathbf{X}\mathbf{M}_{B'}\mathbf{X}'\mathbf{T}'] (\mathbf{P}_{\mathbf{T}\mathbf{X}\mathbf{M}_{B'}}^{[\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^{+}})' \\ & \quad = \mathbf{T}\mathbf{X}\mathbf{M}_{B'} (\mathbf{M}_{B'}\mathbf{X}'\mathbf{T}'[\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^{+}\mathbf{T}\mathbf{X}\mathbf{M}_{B'})^{+} \mathbf{M}_{B'}\mathbf{X}'\mathbf{T}' \\ & \quad \times [\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^{+} [\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}'] [\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^{+} \\ & \quad \times \mathbf{T}\mathbf{X}\mathbf{M}_{B'} (\mathbf{M}_{B'}\mathbf{X}'\mathbf{T}'[\mathbf{T}(\Sigma_{\vartheta} + \mathbf{X}\mathbf{M}_{B'}\mathbf{X}')\mathbf{T}']^{+}\mathbf{X}'\mathbf{T}'\mathbf{M}_{B'})^{+} \mathbf{M}_{B'}\mathbf{X}'\mathbf{T}'\end{aligned}$$

$$\begin{aligned}
& - \text{TXM}_{B'} (\text{M}_{B'} \text{X}' \text{T}' [\text{T}(\Sigma_\vartheta + \text{XM}_{B'} \text{X}') \text{T}']^+ \text{TXM}_{B'})^+ \text{M}_{B'} \text{X}' \text{T}' \\
& \times [\text{T}(\Sigma_\vartheta + \text{XM}_{B'} \text{X}') \text{T}']^+ \text{TXM}_{B'} \text{M}_{B'} \text{X}' \text{T}' [\text{T}(\Sigma_\vartheta + \text{XM}_{B'} \text{X}') \text{T}']^+ \\
& \times \text{TXM}_{B'} (\text{M}_{B'} \text{X}' \text{T}' [\text{T}(\Sigma_\vartheta + \text{XM}_{B'} \text{X}') \text{T}']^+ \text{TXM}_{B'})^+ \text{M}_{B'} \text{X}' \text{T}' \\
& = \text{TXM}_{B'} (\text{M}_{B'} \text{X}' \text{T}' [\text{T}(\Sigma_\vartheta + \text{XM}_{B'} \text{X}') \text{T}']^+ \text{TXM}_{B'})^+ \text{M}_{B'} \text{X}' \text{T}' - \text{TXM}_{B'} \text{X}' \text{T}' \\
& = \text{TX} \left\{ (\text{M}_{B'} \text{X}' \text{T}' [\text{T}(\Sigma_\vartheta + \text{XM}_{B'} \text{X}') \text{T}']^+ \text{TXM}_{B'})^+ - \text{M}_{B'} \right\} \text{X}' \text{T}'.
\end{aligned}$$

In the course of the proof we have used Assertion 1, (ii) and following statement

$$\mathcal{M}(B') \subset \mathcal{M}(A') \Leftrightarrow \text{BA}^{-1} \text{A} = \text{B},$$

for matrices  $\text{A} = \text{T}(\Sigma_\vartheta + \text{XM}_{B'} \text{X}') \text{T}'$  and  $\text{B} = \text{M}_{B'} \text{X}' \text{T}'$ .  $\square$

**Theorem 3** *Let the transformed model (4) where  $\Sigma_\vartheta = \sum_{i=1}^p \vartheta_i \text{V}_i$ ,  $\text{V}_i$  p.s.d.,  $\vartheta_i > 0$ ,  $\forall i = 1, \dots, p$ , (mixed linear model) be under consideration. Let  $\Sigma_0 = \sum_{i=1}^p \vartheta_i^0 \text{V}_i$ , where  $\vartheta^0 = (\vartheta_1^0, \dots, \vartheta_p^0)'$  is as near to the actual value  $\vartheta^*$  of the parameter as possible. The linear function  $\mathbf{g}'\vartheta$ ,  $\vartheta \in \underline{\vartheta}$  can be estimated by MINQUE (minimum norm quadratic unbiased estimator) iff*

$$\mathbf{g} \in \mathcal{M} \left[ \text{S} \left( \text{M}_{\left( \begin{smallmatrix} \text{TX} \\ \text{B} \end{smallmatrix} \right)} \left( \begin{smallmatrix} \text{T}\Sigma_0 \text{T}' & 0 \\ 0 & 0 \end{smallmatrix} \right) \text{M}_{\left( \begin{smallmatrix} \text{TX} \\ \text{B} \end{smallmatrix} \right)} \right)^+ \right], \quad (5)$$

where the  $(i, j)$ -th element of the matrix  $\text{S} \left( \text{M}_{\left( \begin{smallmatrix} \text{TX} \\ \text{B} \end{smallmatrix} \right)} \left( \begin{smallmatrix} \text{T}\Sigma_0 \text{T}' & 0 \\ 0 & 0 \end{smallmatrix} \right) \text{M}_{\left( \begin{smallmatrix} \text{TX} \\ \text{B} \end{smallmatrix} \right)} \right)^+ \text{is}$

$$\begin{aligned}
& \left\{ \text{S} \left( \text{M}_{\left( \begin{smallmatrix} \text{TX} \\ \text{B} \end{smallmatrix} \right)} \left( \begin{smallmatrix} \text{T}\Sigma_0 \text{T}' & 0 \\ 0 & 0 \end{smallmatrix} \right) \text{M}_{\left( \begin{smallmatrix} \text{TX} \\ \text{B} \end{smallmatrix} \right)} \right)^+ \right\}_{i,j} = \\
& = \text{Tr} \left[ \left( \text{M}_{\text{TXM}_{B'}} \text{T}\Sigma_0 \text{T}' \text{M}_{\text{TXM}_{B'}} \right)^+ \text{TV}_i \text{T}' \left( \text{M}_{\text{TXM}_{B'}} \text{T}\Sigma_0 \text{T}' \text{M}_{\text{TXM}_{B'}} \right)^+ \text{TV}_j \text{T}' \right], \\
& \quad \quad \quad i, j = 1, \dots, p.
\end{aligned}$$

If the condition (5) is satisfied, then the  $\vartheta^0$ -MINQUE is

$$\begin{aligned}
& \widehat{\mathbf{g}'\vartheta} = \sum_{i=1}^p \lambda_i \left( \begin{array}{c} \text{TY} \\ -\mathbf{b} \end{array} \right)' \\
& \times \left( \begin{array}{cc} \text{ZTV}_i \text{T}' \text{Z}; & -\text{ZTV}_i \text{T}' \text{ZTXB}' (\text{BB}')^{-} \\ -(\text{BB}')^{-} \text{BX}' \text{T}' \text{Z}' \text{TV}_i \text{T}' \text{Z}'; & (\text{BB}')^{-} \text{BX}' \text{T}' \text{Z}' \text{TV}_i \text{T}' \text{Z}' \text{TXB}' (\text{BB}')^{-} \end{array} \right) \left( \begin{array}{c} \text{TY} \\ -\mathbf{b} \end{array} \right),
\end{aligned}$$

where  $\text{Z} = [\text{M}_{\text{TXM}_{B'}} \text{T}\Sigma_0 \text{T}' \text{M}_{\text{TXM}_{B'}}]^+$ , and where the vector  $\lambda = (\lambda_1, \dots, \lambda_p)'$  is a solution of the equation

$$\text{S} \left( \text{M}_{\left( \begin{smallmatrix} \text{TX} \\ \text{B} \end{smallmatrix} \right)} \left( \begin{smallmatrix} \text{T}\Sigma_0 \text{T}' & 0 \\ 0 & 0 \end{smallmatrix} \right) \text{M}_{\left( \begin{smallmatrix} \text{TX} \\ \text{B} \end{smallmatrix} \right)} \right)^+ \lambda = \mathbf{g}.$$

**Proof** We use following statement (see [4], p. 101) valid for the linear model  $Y \sim [X\beta, \Sigma_\vartheta]$  where  $\beta \in R^k$ ,  $\Sigma_\vartheta = \sum_{i=1}^p \vartheta_i V_i$ ,  $\vartheta = (\vartheta_1, \dots, \vartheta_p)' \in \underline{\vartheta} \subset R^p$ ,  $\vartheta_i > 0$ ,  $\forall i = 1, \dots, p$ ,  $V_1, \dots, V_p$  p.s.d. matrices (mixed linear model):

a) Let  $\Sigma_0 = \sum_{i=1}^p \vartheta_i^0 V_i$ . The function  $g'\vartheta = \sum_{i=1}^p g_i \vartheta_i$ ,  $\vartheta \in \underline{\vartheta}$ , can be unbiasedly quadratically and invariantly estimated [i.e. the estimator has the form  $Y'AY$ , where  $A_{n,n}$  is symmetric matrix, the estimator is invariant with respect to the change of the vector  $\beta$ ] if and only if  $g \in \mathcal{M}(S_{(M_X \Sigma_0 M_X)^+})$ , where

$$\{S_{(M_X \Sigma_0 M_X)^+}\}_{i,j} = \text{Tr}[(M_X \Sigma_0 M_X)^+ V_i (M_X \Sigma_0 M_X)^+ V_j],$$

$i, j = 1, \dots, p$ .

b) If the function  $g'\vartheta$  satisfies the condition from a), then the  $\vartheta^0$ -MINQUE of  $g'\vartheta$  is given as

$$\widehat{g'\vartheta} = \sum_{i=1}^p \lambda_i Y' (M_X \Sigma_0 M_X)^+ V_i (M_X \Sigma_0 M_X)^+ Y,$$

where the vector  $\lambda = (\lambda_1, \dots, \lambda_p)'$  is a solution of the system of equations

$$S_{(M_X \Sigma_0 M_X)^+} \lambda = g.$$

We use this statement for the model (4) by following substitutions

$$Y \rightarrow \begin{pmatrix} TY \\ -b \end{pmatrix} \quad X \rightarrow \begin{pmatrix} TX \\ B \end{pmatrix} \quad \Sigma_0 \rightarrow \begin{pmatrix} T\Sigma_0 T' & O \\ O & O \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^p \vartheta_i^0 TV_i T' & O \\ O & O \end{pmatrix}.$$

Thus

$$\begin{aligned} & \{S_{\left(M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \begin{pmatrix} T\Sigma_0 T' & O \\ O & O \end{pmatrix} M_{\begin{pmatrix} TX \\ B \end{pmatrix}}\right)^+}\}_{i,j} \\ &= \text{Tr} \left\{ \left[ M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \begin{pmatrix} T\Sigma_0 T' & O \\ O & O \end{pmatrix} M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \right]^+ \begin{pmatrix} TV_i T' & O \\ O & O \end{pmatrix} \right. \\ & \quad \left. \times \left[ M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \begin{pmatrix} T\Sigma_0 T' & O \\ O & O \end{pmatrix} M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \right]^+ \begin{pmatrix} TV_j T' & O \\ O & O \end{pmatrix} \right\}. \end{aligned}$$

Let us denote

$$\left[ \begin{pmatrix} T\Sigma_0 T' & O \\ O & O \end{pmatrix} + \begin{pmatrix} TX \\ B \end{pmatrix} (X'T', B') \right]^+ = \begin{pmatrix} \boxed{aa} & \boxed{ab} \\ \boxed{ba} & \boxed{bb} \end{pmatrix},$$

where (see Assertion 2)

$$\begin{aligned} \boxed{aa} &= [T(\Sigma_0 + XM_{B'}X')T']^+ \\ \boxed{ab} &= -[T(\Sigma_0 + XM_{B'}X')T']^+ TXB'(BB')^+, \\ \boxed{ba} &= -(BB')^+ BX'T'[T(\Sigma_0 + XM_{B'}X')T']^+, \\ \boxed{bb} &= (BB')^+ + (BB')^+ BX'T'[T(\Sigma_0 + XM_{B'}X')T']^+ TXB'(BB')^+. \end{aligned}$$



By Assertion 1,(ii) (the Moore–Penrose matrices are used because of uniqueness of matrix expressions)

$$\begin{aligned}
& \left[ M_{\binom{TX}{B}} \begin{pmatrix} T\Sigma_0 T', & \mathbf{O} \\ \mathbf{O} & , \mathbf{O} \end{pmatrix} M_{\binom{TX}{B}} \right]^+ = \begin{pmatrix} \boxed{aa}, & \boxed{ab} \\ \boxed{ba}, & \boxed{bb} \end{pmatrix} \\
& - \begin{pmatrix} \boxed{aa}, & \boxed{ab} \\ \boxed{ba}, & \boxed{bb} \end{pmatrix} \begin{pmatrix} TX \\ B \end{pmatrix} \left\{ (X'T', B') \begin{pmatrix} \boxed{aa}, & \boxed{ab} \\ \boxed{ba}, & \boxed{bb} \end{pmatrix} \begin{pmatrix} TX \\ B \end{pmatrix} \right\}^+ \\
& \quad \times (X'T', B') \begin{pmatrix} \boxed{aa}, & \boxed{ab} \\ \boxed{ba}, & \boxed{bb} \end{pmatrix} = \begin{pmatrix} \boxed{aa}, & \boxed{ab} \\ \boxed{ba}, & \boxed{bb} \end{pmatrix} \\
& - \begin{pmatrix} \boxed{aa}, & \boxed{ab} \\ \boxed{ba}, & \boxed{bb} \end{pmatrix} \begin{pmatrix} TX \\ B \end{pmatrix} \{ P_{B'} + M_{B'} X'T' [T(\Sigma_0 + X M_{B'} X') T']^+ T X M_{B'} \}^+ \\
& \quad \times (X'T', B') \begin{pmatrix} \boxed{aa}, & \boxed{ab} \\ \boxed{ba}, & \boxed{bb} \end{pmatrix} = \begin{pmatrix} \boxed{aa}, & \boxed{ab} \\ \boxed{ba}, & \boxed{bb} \end{pmatrix} - \begin{pmatrix} \boxed{I}, & \boxed{II} \\ \boxed{III}, & \boxed{IV} \end{pmatrix},
\end{aligned}$$

where by notation

$$U = [T(\Sigma_0 + X M_{B'} X') T']^+,$$

$$\boxed{I} = U T X (M_{B'} X'T' U T X M_{B'})^+ X'T' U,$$

$$\boxed{II} = -U T X (M_{B'} X'T' U T X M_{B'})^+ X'T' U T X B' (B B')^+ = \boxed{III}' ,$$

$$\boxed{IV} = (B B')^+ B X'T' U T X (M_{B'} X'T' U T X M_{B'})^+ X'T' U T X B' (B B')^+ + (B B')^+.$$

After some calculations using notation

$$Z = (M_{T X M_{B'}} T \Sigma_0 T' M_{T X M_{B'}})^+,$$

we get

$$\begin{aligned}
& \left[ M_{\binom{TX}{B}} \begin{pmatrix} T\Sigma_0 T', & \mathbf{O} \\ \mathbf{O} & , \mathbf{O} \end{pmatrix} M_{\binom{TX}{B}} \right]^+ \\
& = \begin{pmatrix} Z, & -Z T X B' (B B')^+ \\ -(B B')^+ B X'T' Z', & (B B')^+ B X'T' Z T X B' (B B')^+ \end{pmatrix} = \begin{pmatrix} E, & F \\ F', & G \end{pmatrix}.
\end{aligned}$$

Thus

$$\begin{aligned}
& \{ S \left( M_{\binom{TX}{B}} \begin{pmatrix} T\Sigma_0 T', & \mathbf{0} \\ \mathbf{0} & , \mathbf{0} \end{pmatrix} M_{\binom{TX}{B}} \right)^+ \}_{i,j} \\
& = \text{Tr} \left[ \begin{pmatrix} E, & F \\ F', & G \end{pmatrix} \begin{pmatrix} T V_i T', & \mathbf{O} \\ \mathbf{O} & , \mathbf{O} \end{pmatrix} \begin{pmatrix} E, & F \\ F', & G \end{pmatrix} \begin{pmatrix} T V_j T', & \mathbf{O} \\ \mathbf{O} & , \mathbf{O} \end{pmatrix} \right] = \text{Tr} [E T V_i T' E T V_j T'] \\
& = \text{Tr} [(M_{T X M_{B'}} T \Sigma_0 T' M_{T X M_{B'}})^+ T V_i T' (M_{T X M_{B'}} T \Sigma_0 T' M_{T X M_{B'}})^+ T V_j T'], \\
& i, j = 1, \dots, p.
\end{aligned}$$

If

$$\mathbf{g} \in \mathcal{M}\left(\mathbf{S} \left( M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \left( \begin{pmatrix} T\Sigma_0 T' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \right)^+ \right),$$

then under the model (4)

$$\begin{aligned} \widehat{\mathbf{g}}^{\vartheta} &= \sum_{i=1}^p \lambda_i \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix}' \left[ M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \left( \begin{pmatrix} T\Sigma_0 T' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \right)^+ \begin{pmatrix} \mathbf{T}\mathbf{V}_i T' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right. \\ &\quad \times \left. \left[ M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \left( \begin{pmatrix} T\Sigma_0 T' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \right)^+ \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix} \right] \\ &= \sum_{i=1}^p \lambda_i \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix}' \begin{pmatrix} \mathbf{E}\mathbf{T}\mathbf{V}_i T' \mathbf{E} & \mathbf{E}\mathbf{T}\mathbf{V}_i T' \mathbf{F} \\ \mathbf{F}' \mathbf{T}\mathbf{V}_i T' \mathbf{E} & \mathbf{F}' \mathbf{T}\mathbf{V}_i T' \mathbf{F} \end{pmatrix} \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix} = \sum_{i=1}^p \lambda_i \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix}' \\ &\quad \times \begin{pmatrix} \mathbf{Z}\mathbf{T}\mathbf{V}_i T' \mathbf{Z} & -\mathbf{Z}\mathbf{T}\mathbf{V}_i T' \mathbf{Z}\mathbf{T}\mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^+ \\ -(\mathbf{B}\mathbf{B}')^+ \mathbf{B}\mathbf{X}' T' \mathbf{Z}' \mathbf{T}\mathbf{V}_i T' \mathbf{Z}' & (\mathbf{B}\mathbf{B}')^+ \mathbf{B}\mathbf{X}' T' \mathbf{Z}' \mathbf{T}\mathbf{V}_i T' \mathbf{Z}\mathbf{T}\mathbf{X}\mathbf{B}'(\mathbf{B}\mathbf{B}')^+ \end{pmatrix} \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix}, \end{aligned}$$

where  $\mathbf{Z} = (\mathbf{M}_{TXM_B}, T\Sigma_0 T' \mathbf{M}_{TXM_B'})^+$ .  $\square$

**Theorem 4** *Function  $\mathbf{g}'_1 \mathbf{T}\mathbf{Y}$  is the best unbiased estimator of its mean value in the model (4) iff*

$$\mathbf{g}_1 \in \mathcal{M}[\mathbf{M}_{(T\Sigma_\vartheta T' [I - TX(X' T' TX + BB')^{-1} X' T], T\Sigma_\vartheta T' TX(X' T' TX + BB')^{-1} B')}]$$

**Proof** Function  $\mathbf{g}' \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix}$ ,  $\mathbf{g} = \begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \end{pmatrix}$ ,  $\mathbf{g}_1 \in R^r$ ,  $\mathbf{g}_2 \in R^q$ , is in the model (4) the best unbiased estimator of its mean value iff

$$\text{cov} \left\{ \mathbf{g}' \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix}, \tau_0 \left[ \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix} \right] \right\} = 0,$$

where  $\tau_0 \left[ \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix} \right]$  is arbitrary unbiased estimator of the null function  $\mathbf{g}_0(\beta, \vartheta) = 0$ , (see [4], p. 84). Any unbiased estimator of this function is of the form

$$\tau_0 \left[ \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix} \right] = \mathbf{f}' \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}, \quad \mathbf{f} \in \mathcal{M}(\mathbf{M}_{\begin{pmatrix} TX \\ B \end{pmatrix}}),$$

as

$$E[\mathbf{f}'_1 \mathbf{T}\mathbf{Y} + \mathbf{f}'_2 (-\mathbf{b})] = \mathbf{f}'_1 \mathbf{T}\mathbf{X}\beta + \mathbf{f}'_2 (-\mathbf{b}) = (\mathbf{f}'_1 \mathbf{T}\mathbf{X} + \mathbf{f}'_2 \mathbf{B})\beta = 0, \quad \forall \beta,$$

$$\Leftrightarrow (\mathbf{f}'_1, \mathbf{f}'_2) \begin{pmatrix} \mathbf{T}\mathbf{X} \\ \mathbf{B} \end{pmatrix} = \mathbf{o}', \quad \Leftrightarrow \mathbf{f} \in \mathcal{M}(\mathbf{M}_{\begin{pmatrix} TX \\ B \end{pmatrix}}).$$

Let  $\mathbf{u} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}$ ,  $\mathbf{u}_1 \in R^r$ ,  $\mathbf{u}_2 \in R^q$ , be arbitrary. Then the covariance

$$\text{cov} \left( \mathbf{g}' \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix}, \mathbf{u}' M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \begin{pmatrix} \mathbf{T}\mathbf{Y} \\ -\mathbf{b} \end{pmatrix} \right) = \mathbf{g}' \begin{pmatrix} T\Sigma_\vartheta T' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} M_{\begin{pmatrix} TX \\ B \end{pmatrix}} \mathbf{u}$$

$$\begin{aligned}
&= \mathbf{g}' \begin{pmatrix} \mathbf{T}\Sigma_\vartheta\mathbf{T}', & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{I}_r - \mathbf{TX}(\mathbf{X}'\mathbf{T}'\mathbf{TX} + \mathbf{B}'\mathbf{B})^{-1}\mathbf{X}'\mathbf{T}', & -\mathbf{TX}(\mathbf{X}'\mathbf{T}'\mathbf{TX} + \mathbf{B}'\mathbf{B})^{-1}\mathbf{B}' \\ -\mathbf{B}(\mathbf{X}'\mathbf{T}'\mathbf{TX} + \mathbf{B}'\mathbf{B})^{-1}\mathbf{X}'\mathbf{T}', & \mathbf{I}_q - \mathbf{B}(\mathbf{X}'\mathbf{T}'\mathbf{TX} + \mathbf{B}'\mathbf{B})^{-1}\mathbf{B}' \end{pmatrix} \mathbf{u} \\
&= (\mathbf{g}'_1, \mathbf{g}'_2) \begin{pmatrix} \mathbf{T}\Sigma_\vartheta\mathbf{T}'(\mathbf{I}_r - \mathbf{TX}(\mathbf{X}'\mathbf{T}'\mathbf{TX} + \mathbf{B}'\mathbf{B})^{-1}\mathbf{X}'\mathbf{T}'), & -\mathbf{T}\Sigma_\vartheta\mathbf{T}'\mathbf{TX}(\mathbf{X}'\mathbf{T}'\mathbf{TX} + \mathbf{B}'\mathbf{B})^{-1}\mathbf{B}' \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} \mathbf{u} \\
&= 0.
\end{aligned}$$

$$\Leftrightarrow \mathbf{g}'_1(\mathbf{T}\Sigma_\vartheta\mathbf{T}'[\mathbf{I}_r - \mathbf{TX}(\mathbf{X}'\mathbf{T}'\mathbf{TX} + \mathbf{B}'\mathbf{B})^{-1}\mathbf{X}'\mathbf{T}'], -\mathbf{T}\Sigma_\vartheta\mathbf{T}'\mathbf{TX}(\mathbf{X}'\mathbf{T}'\mathbf{TX} + \mathbf{B}'\mathbf{B})^{-1}\mathbf{B}') = \mathbf{o}'.$$

Thus  $\mathbf{g}'_1\mathbf{T}\mathbf{Y}$  is the best unbiased estimator of its mean value iff

$$\mathbf{g}_1 \in \mathcal{M}[\mathbf{M}_{(\mathbf{T}\Sigma_\vartheta\mathbf{T}'[\mathbf{I}_r - \mathbf{TX}(\mathbf{X}'\mathbf{T}'\mathbf{TX} + \mathbf{B}'\mathbf{B})^{-1}\mathbf{X}'\mathbf{T}'], \mathbf{T}\Sigma_\vartheta\mathbf{T}'\mathbf{TX}(\mathbf{X}'\mathbf{T}'\mathbf{TX} + \mathbf{B}'\mathbf{B})^{-1}\mathbf{B}')}. \quad \square$$

**Remark 2** If we change the ordering of the procedures described at the beginning of this section, we get the same model. Indeed by joining linear model (1) with constraints (2), we can write

$$\begin{pmatrix} \mathbf{Y} \\ -\mathbf{b} \end{pmatrix} \sim \left[ \begin{pmatrix} \mathbf{X}, & \mathbf{S} \\ \mathbf{B}, & \mathbf{O} \end{pmatrix} \begin{pmatrix} \beta \\ \kappa \end{pmatrix}, \begin{pmatrix} \Sigma, & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} \right].$$

The transformation by the matrix  $\begin{pmatrix} \mathbf{T}, & \mathbf{O} \\ \mathbf{O}, & \mathbf{I} \end{pmatrix}$ , such that  $\mathbf{TS} = \mathbf{O}$ , leads to the model (4).

### 3 Examples of the transformation matrices

The general solution of the matrix equation  $\mathbf{TS} = \mathbf{O}$  is of the form

$$\mathbf{T} = \mathbf{A}(\mathbf{I} - \mathbf{SS}^-),$$

where  $\mathbf{A}$  is an arbitrary matrix of the corresponding type,  $\mathbf{S}^-$  is some version of generalized inverse of the matrix  $\mathbf{S}$ .

If we choose  $\mathbf{S}^- = (\mathbf{S}^- \mathbf{W}\mathbf{S})^{-1} \mathbf{S}' \mathbf{W}$ , where  $\mathbf{W}$  is an arbitrary p.s.d. matrix such that

$$\mathcal{M}(\mathbf{S}') = \mathcal{M}(\mathbf{S}'\mathbf{W}\mathbf{S}), \quad (6)$$

then  $\mathbf{T} = \mathbf{A}\mathbf{M}_S^W$ , where  $\mathbf{M}_S^W$  is given uniquely.

First we confine us to the transformation matrix

$$\mathbf{a) \quad T} = \mathbf{M}_S^W,$$

i.e. we consider transformed linear model

$$\mathbf{M}_S^W \mathbf{Y} \sim [\mathbf{M}_S^W \mathbf{X} \beta, \mathbf{M}_S^W \Sigma (\mathbf{M}_S^W)']. \quad (7)$$

Thus model with the type I constraints is following

$$\begin{pmatrix} \mathbf{M}_S^W \mathbf{Y} \\ -\mathbf{b} \end{pmatrix} \sim \left[ \begin{pmatrix} \mathbf{M}_S^W \mathbf{X} \\ \mathbf{B} \end{pmatrix} \beta, \begin{pmatrix} \mathbf{M}_S^W \Sigma_\vartheta (\mathbf{M}_S^W)', & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} \right]. \quad (8)$$

It can be proved (see [6], chapter 3) that

$$\mathcal{M}(\mathbf{M}_S) = \mathcal{M}((\mathbf{M}_S^W)'),$$

thus

$$\mathcal{M}(\mathbf{X}'\mathbf{M}_S, \mathbf{B}') = \mathcal{M}(\mathbf{X}'(\mathbf{M}_S^W)', \mathbf{B}'),$$

i.e. the classes of unbiasedly estimable functions  $\mathbf{g}'\beta$  in model (1) with constraints (2) and in model (8) are identical.

According to Theorem 1 and Theorem 2

$$\begin{aligned} \widehat{\mathbf{M}_S^W \mathbf{X} \beta} &= \mathbf{P}_{\mathbf{M}_S^W \mathbf{X} \mathbf{M}_{B'}}^{[\mathbf{M}_S^W (\Sigma + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}') (\mathbf{M}_S^W)']^+} \mathbf{M}_S^W [\mathbf{Y} + \mathbf{X} \mathbf{B}' (\mathbf{B} \mathbf{B}')^{-1} \mathbf{b}] - \mathbf{M}_S^W \mathbf{X} \mathbf{B}' (\mathbf{B} \mathbf{B}')^{-1} \mathbf{b} \\ &= \mathbf{M}_S^W \mathbf{X} \mathbf{M}_{B'} \left[ \mathbf{M}_{B'} \mathbf{X}' (\mathbf{M}_S^W)' \left( \mathbf{M}_S^W (\Sigma + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}') (\mathbf{M}_S^W)' \right)^+ \mathbf{M}_S^W \mathbf{X} \mathbf{M}_{B'} \right]^+ \mathbf{M}_{B'} \\ &\times \mathbf{X}' (\mathbf{M}_S^W)' \left( \mathbf{M}_S^W (\Sigma + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}') (\mathbf{M}_S^W)' \right)^+ \mathbf{M}_S^W [\mathbf{Y} + \mathbf{X} \mathbf{B}' (\mathbf{B} \mathbf{B}')^{-1} \mathbf{b}] - \mathbf{M}_S^W \mathbf{X} \mathbf{B}' (\mathbf{B} \mathbf{B}')^{-1} \mathbf{b}. \end{aligned}$$

$$\begin{aligned} \text{var}[\widehat{\mathbf{M}_S^W \mathbf{X} \beta}] &= \\ &= \mathbf{M}_S^W \mathbf{X} \left\{ \left[ \mathbf{M}_{B'} \mathbf{X}' (\mathbf{M}_S^W)' \left( \mathbf{M}_S^W [\Sigma + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}] (\mathbf{M}_S^W)' \right)^- \mathbf{M}_S^W \mathbf{X} \mathbf{M}_{B'} \right]^+ - \mathbf{M}_{B'} \right\} \mathbf{X}' (\mathbf{M}_S^W)'. \end{aligned}$$

**Remark 3** If the matrix  $\Sigma + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}'$  is regular or if

$$\mathcal{M}(\mathbf{S}) \subset \mathcal{M}(\Sigma + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}'),$$

it can be proved that (see [6], Lemma 1)

$$(\mathbf{M}_S^W)' \left[ \mathbf{M}_S^W (\Sigma + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}') (\mathbf{M}_S^W)' \right]^+ \mathbf{M}_S^W = [\mathbf{M}_S (\Sigma + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}') \mathbf{M}_S]^+.$$

Then

$$\begin{aligned} \widehat{\mathbf{M}_S^W \mathbf{X} \beta} &= \mathbf{M}_S^W \mathbf{X} \left( \mathbf{M}_{B'} \mathbf{X}' [\mathbf{M}_S (\Sigma + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}') \mathbf{M}_S]^+ \mathbf{X} \mathbf{M}_{B'} \right)^+ \mathbf{X}' [\mathbf{M}_S (\Sigma + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}') \mathbf{M}_S]^+ \\ &\times (\mathbf{Y} + \mathbf{X} \mathbf{B}' (\mathbf{B} \mathbf{B}')^{-1} \mathbf{b}) - \mathbf{M}_S^W \mathbf{X} \mathbf{B}' (\mathbf{B} \mathbf{B}')^{-1} \mathbf{b}. \end{aligned}$$

$$\text{var}[\widehat{\mathbf{M}_S^W \mathbf{X} \beta}] = \mathbf{M}_S^W \mathbf{X} \left\{ \left( \mathbf{M}_{B'} \mathbf{X}' [\mathbf{M}_S (\Sigma + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}') \mathbf{M}_S]^+ \mathbf{X} \mathbf{M}_{B'} \right)^+ - \mathbf{M}_{B'} \right\} \mathbf{X}' (\mathbf{M}_S^W)'.$$

When we choose transformation matrix

$$\mathbf{b) \quad T} = \mathbf{M}_S^{(\mathbf{M}_X \Sigma \mathbf{M}_X)^+},$$

we get the model with type I constraints with the same design matrix belonging to the vector  $\beta$ ,

$$\begin{pmatrix} \mathbf{M}_S^{(\mathbf{M}_X \Sigma \mathbf{M}_X)^+} \mathbf{Y} \\ -\mathbf{b} \end{pmatrix} \sim \left[ \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} \beta, \begin{pmatrix} \mathbf{M}_S^{(\mathbf{M}_X \Sigma \mathbf{M}_X)^+} \Sigma (\mathbf{M}_S^{(\mathbf{M}_X \Sigma \mathbf{M}_X)^+})' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \right],$$

because it is

$$\mathbf{M}_S^{(M_X \Sigma M_X)^+} \mathbf{S} = \mathbf{O}, \quad \mathbf{M}_S^{(M_X \Sigma M_X)^+} \mathbf{X} = \mathbf{X}.$$

According to assumption (6) it should be

$$\mathcal{M}(\mathbf{S}') = \mathcal{M}(\mathbf{S}'[\mathbf{M}_X \Sigma \mathbf{M}_X]^+ \mathbf{S}).$$

It is valid if the model (1) is regular (see [3], page 189).

In this model

$$\begin{aligned} \widehat{\mathbf{X}}\beta &= \mathbf{P}_{\mathbf{X} \mathbf{M}_{B'}}^{[\mathbf{M}_S^{(M_X \Sigma M_X)^+} \Sigma(\mathbf{M}_S^{(M_X \Sigma M_X)^+})' + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}']^+} \mathbf{M}_S^{(M_X \Sigma M_X)^+} \mathbf{Y} \\ &\quad - \mathbf{M}_{\mathbf{X} \mathbf{M}_{B'}}^{[\mathbf{M}_S^{(M_X \Sigma M_X)^+} \Sigma(\mathbf{M}_S^{(M_X \Sigma M_X)^+})' + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}']^+} \mathbf{X} \mathbf{B}' (\mathbf{B} \mathbf{B}')^{-1} \mathbf{b}, \end{aligned}$$

$$\text{var}[\widehat{\mathbf{X}}\beta] =$$

$$= \mathbf{X} \left\{ \left[ \mathbf{M}_{B'} \mathbf{X}' \left( \mathbf{M}_S^{(M_X \Sigma M_X)^+} \Sigma(\mathbf{M}_S^{(M_X \Sigma M_X)^+})' + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}' \right)^{-1} \mathbf{X} \mathbf{M}_{B'} \right]^+ - \mathbf{M}_{B'} \right\} \mathbf{X}'.$$

If we suppose, that

$$\mathcal{M}(\mathbf{X}') \subset \mathcal{M}(\mathbf{X}'[\mathbf{M}_S \Sigma \mathbf{M}_S]^+ \mathbf{X}), \quad (9)$$

we can use transformation matrix

$$\mathbf{c) \quad T} = \mathbf{P}_X^{(M_S \Sigma M_S)^+}$$

that leads to the model

$$\begin{pmatrix} \mathbf{P}_X^{(M_S \Sigma M_S)^+} \mathbf{Y} \\ -\mathbf{b} \end{pmatrix} \sim \begin{bmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{B} \end{pmatrix} \beta, \begin{pmatrix} \mathbf{X}(\mathbf{X}'[\mathbf{M}_S \Sigma \mathbf{M}_S]^+ \mathbf{X})^{-1} \mathbf{X}', & \mathbf{O} \\ \mathbf{O}, & \mathbf{O} \end{pmatrix} \end{bmatrix},$$

because under assumption (9) it is

$$\mathbf{P}_X^{(M_S \Sigma M_S)^+} \mathbf{X} = \mathbf{X}, \quad \mathbf{P}_X^{(M_S \Sigma M_S)^+} \mathbf{S} = \mathbf{O},$$

$$\mathbf{P}_X^{(M_S \Sigma M_S)^+} \Sigma \left( \mathbf{P}_X^{(M_S \Sigma M_S)^+} \right)' = \mathbf{X}(\mathbf{X}'[\mathbf{M}_S \Sigma \mathbf{M}_S]^+ \mathbf{X})^{-1} \mathbf{X}'.$$

$$\begin{aligned} \widehat{\mathbf{X}}\beta &= \mathbf{P}_{\mathbf{X} \mathbf{M}_{B'}}^{[\mathbf{X}(\mathbf{X}'[\mathbf{M}_S \Sigma \mathbf{M}_S]^+ \mathbf{X})^{-1} \mathbf{X}' + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}']^+} \mathbf{P}_X^{(M_S \Sigma M_S)^+} \mathbf{Y} \\ &\quad - \mathbf{M}_{\mathbf{X} \mathbf{M}_{B'}}^{[\mathbf{X}(\mathbf{X}'[\mathbf{M}_S \Sigma \mathbf{M}_S]^+ \mathbf{X})^{-1} \mathbf{X}' + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}']^+} \mathbf{X} \mathbf{B}' (\mathbf{B} \mathbf{B}')^{-1} \mathbf{b} \end{aligned}$$

$$\begin{aligned} &= \left\{ \mathbf{X} \mathbf{M}_{B'} \left[ \mathbf{M}_{B'} \mathbf{X}' \left( \mathbf{X}(\mathbf{X}'[\mathbf{M}_S \Sigma \mathbf{M}_S]^+ \mathbf{X})^{-1} \mathbf{X}' + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}' \right)^{-1} \mathbf{X} \mathbf{M}_{B'} \right]^+ - \mathbf{M}_{B'} \mathbf{X}' - \mathbf{X} \mathbf{M}_{B'} \mathbf{X}' \right\} \\ &\quad \times (\mathbf{M}_S \Sigma \mathbf{M}_S)^+ [\mathbf{Y} + \mathbf{X} \mathbf{B}' (\mathbf{B} \mathbf{B}')^{-1} \mathbf{b}] - \mathbf{X} \mathbf{B}' (\mathbf{B} \mathbf{B}')^{-1} \mathbf{b}. \end{aligned}$$

$$\text{var}[\widehat{\mathbf{X}}\beta] = \mathbf{X} \left\{ \left[ \mathbf{M}_{B'} \mathbf{X}' \left( \mathbf{X}(\mathbf{X}'[\mathbf{M}_S \Sigma \mathbf{M}_S]^+ \mathbf{X})^{-1} \mathbf{X}' + \mathbf{X} \mathbf{M}_{B'} \mathbf{X}' \right)^{-1} \mathbf{X} \mathbf{M}_{B'} \right]^+ - \mathbf{M}_{B'} \right\} \mathbf{X}'.$$

**Remark 4** In the practice we have to decide, whether to use transformation or not. We should compute variance matrices of the estimators in the original model and in the transformed model and decide according to the accuracy of the estimates. We can use following formulas:

a) if the model (1) is regular, then under condition (2) without transformation (see Remark 1)

$$\text{var}(\widehat{X}\beta) = X[M_{B'}X'(M_S\Sigma_\vartheta M_S)^+XM_{B'}]^+X',$$

b) in the singular model (1) with constraints (2) without transformation (see Remark 1)

$$\text{var}[\widehat{X}\beta] = P_{(XM_{B'},S)}^{[\Sigma_\vartheta + XM_{B'}X' + SS']^+} \Sigma_\vartheta \left( P_{(XM_{B'},S)}^{[\Sigma_\vartheta + XM_{B'}X' + SS']^+} \right)',$$

c) in the transformed singular model (4) (see Theorem 2)

$$\text{var}[\widehat{TX}\beta] = TX\{[M_{B'}X'T'(T[\Sigma + XM_{B'}X']T')^{-1}TXM_{B'}]^+ - M_{B'}\}X'T'.$$

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