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Uncertainty of the design and covariance matrices in linear statistical model*

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Abstract

The aim of the paper is to determine an influence of uncertainties in design and covariance matrices on estimators in linear regression model.

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1 Introduction

Uncertainties in entries of design and covariance matrices influence the variance of estimators and cause their bias. A problem occurs mainly in a linearization of nonlinear regression models, where the design matrix is created by derivatives of some functions. The question is how precise must these derivatives be. Uncertainties of covariance matrices must be suppressed under some reasonable bound as well.

The aim of the paper is to give the simple rules which enables us to decide how many ciphers an entry of the mentioned matrices must be consisted of.

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2 Symbols used

In the following text a linear regression model (in more detail cf. [2]) is denoted as

$$\mathbf{Y} \sim_n (\mathbf{F}\boldsymbol{\beta}, \boldsymbol{\Sigma}), \quad \boldsymbol{\beta} \in R^k, \tag{1}$$

where $\mathbf{Y}$ is an n-dimensional random vector with the mean value $E(\mathbf{Y})$ equal to $\mathbf{F}\boldsymbol{\beta}$ and with the covariance matrix $\mathrm{Var}(\mathbf{Y}) = \boldsymbol{\Sigma}$. The symbol R^k means the k-dimensional linear vector space. The $n \times k$ matrix $\mathbf{F}$ is given. It is assumed that the rank $r(\mathbf{F})$ of the matrix $\mathbf{F}$ is $r(\mathbf{F}) = k < n$ and the given matrix $\boldsymbol{\Sigma}$ is positive definite. The k-dimensional unknown vector parameter $\boldsymbol{\beta}$ must be estimated on the basis of the realization $\mathbf{y}$ of the random vector $\mathbf{Y}$. Symbol $\mathbf{e}_i^{(n)}$ means n-dimensional vector with the entry 1 at the i-th position; other entries are zero. The matrix of the normal equation $\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}$ is denoted as $\mathbf{C}$; its (i,j)-th entry is $\{\mathbf{C}\}_{i,j}$ and the (i,j)-th entry of $\mathbf{C}^{-1}$ is $\{\mathbf{C}\}^{i,j}$. $\mathbf{F}'$ means the transpose of the matrix $\mathbf{F}$. The (i,j)-th entry of the matrix $\boldsymbol{\Sigma}$ is $\sigma_{i,j} = \{\boldsymbol{\Sigma}\}_{i,j}$ and the i-th component of the vector $\mathbf{v}$ is $\{\mathbf{v}\}_i$.

The symbol $\partial \mathbf{l}_h'\mathbf{Y}/\partial\mathbf{F}$ means

$$\frac{\partial \mathbf{l}_h'\mathbf{Y}}{\partial \mathbf{F}} = \begin{pmatrix} \frac{\partial \mathbf{l}_h'\mathbf{Y}}{\partial F_{1,1}}, & \dots & \frac{\partial \mathbf{l}_h'\mathbf{Y}}{\partial F_{1,k}} \\ \dots & \dots & \dots \\ \frac{\partial \mathbf{l}_h'\mathbf{Y}}{\partial F_{n,1}}, & \dots & \frac{\partial \mathbf{l}_h'\mathbf{Y}}{\partial F_{n,k}} \end{pmatrix}, \tag{2}$$

where $F_{i,j} = \{\mathbf{F}\}_{i,j}, i = 1,\dots,n, j = 1,\dots,k$, and $\mathbf{l}_h' = \mathbf{h}'\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}$ for an arbitrary $\mathbf{h} \in \mathbb{R}^k$, $\mathbf{h} \neq \mathbf{0}$.

The Kronecker multiplication of matrices $\mathbf{A}$ and $\mathbf{B}$ is denoted as $\mathbf{A} \otimes \mathbf{B}$ (in more detail cf. [3]). If $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_m)$, then $\mathrm{vec}(\mathbf{A}) = (\mathbf{a}_1', \dots, \mathbf{a}_m')'$. The identity matrix is denoted as $\mathbf{I}$.

3 Uncertainty in the design matrix

In the following text a sensitivity approach is used, i.e. the influence of uncertainty in the design matrix is judged according to the linear term of the Taylor series (cf. also in [1], chpt. VI). The Taylor series of the quantity $\mathbf{l}_h'\mathbf{Y} = \mathbf{h}'\widehat{\boldsymbol{\beta}}$ will be considered.

Lemma 3.1 *Let $\mathbf{h}' \in R^k$ be an arbitrary vector. It is valid that*

$$\frac{\partial \mathbf{h}'\widehat{\boldsymbol{\beta}}}{\partial \mathbf{F}} = -\mathbf{l}_h\widehat{\boldsymbol{\beta}}' + \boldsymbol{\Sigma}^{-1}\mathbf{v}\mathbf{h}'\mathbf{C}^{-1}, \quad \mathbf{l}_h = \boldsymbol{\Sigma}^{-1}\mathbf{F}\mathbf{C}^{-1}\mathbf{h}, \tag{3}$$

$$\widehat{\boldsymbol{\beta}} = \mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}, \tag{4}$$

$$\mathbf{v} = \mathbf{Y} - \mathbf{F}\widehat{\boldsymbol{\beta}}. \tag{5}$$

Proof The BLUE (best linear unbiased eastimator) of the linear function $h(\boldsymbol{\beta}) = \mathbf{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in R^k$, is $\mathbf{h}'\widehat{\boldsymbol{\beta}} = \mathbf{l}_h'\mathbf{Y} = \mathbf{h}'\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}$. Thus

$$\frac{\partial \mathbf{h}'\widehat{\boldsymbol{\beta}}}{\partial F_{i,j}} = \mathbf{h}'\frac{\partial \mathbf{C}^{-1}}{\partial F_{i,j}}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} + \mathbf{h}'\mathbf{C}^{-1}\frac{\partial \mathbf{F}'}{\partial F_{i,j}}\boldsymbol{\Sigma}^{-1}\mathbf{Y}$$

and

$$\begin{aligned}\frac{\partial \mathbf{C}^{-1}}{\partial F_{i,j}} &= \frac{\partial(\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F})^{-1}}{\partial F_{i,j}} = -\mathbf{C}^{-1}\left(\frac{\partial \mathbf{F}'}{\partial F_{i,j}}\boldsymbol{\Sigma}^{-1}\mathbf{F} + \mathbf{F}'\boldsymbol{\Sigma}^{-1}\frac{\partial \mathbf{F}}{\partial F_{i,j}}\right)\mathbf{C}^{-1} \\ &= -\mathbf{C}^{-1}\left\{\left[\mathbf{e}_i^{(n)}(\mathbf{e}_j^{(k)})'\right]'\boldsymbol{\Sigma}^{-1}\mathbf{F} + \mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{e}_i^{(n)}(\mathbf{e}_j^{(k)})'\right\}\mathbf{C}^{-1} \\ &= -\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{e}_i^{(n)}(\mathbf{e}_j^{(k)})'\mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{e}_j^{(k)}(\mathbf{e}_i^{(n)})'\boldsymbol{\Sigma}^{-1}\mathbf{F}\mathbf{C}^{-1}.\end{aligned}$$

It implies

$$\begin{aligned}\frac{\partial \mathbf{h}'\widehat{\boldsymbol{\beta}}}{\partial F_{i,j}} &= -\mathbf{l}_h'\mathbf{e}_i^{(n)}(\mathbf{e}_j^{(k)})'\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} - \mathbf{h}'\mathbf{C}^{-1}\mathbf{e}_j^{(k)}(\mathbf{e}_i^{(n)})'\boldsymbol{\Sigma}^{-1}\mathbf{F}\mathbf{C}^{-1}\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\ &\quad + \mathbf{h}'\mathbf{C}^{-1}\mathbf{e}_j^{(k)}(\mathbf{e}_i^{(n)})'\boldsymbol{\Sigma}^{-1}\mathbf{Y} \\ &= -\mathbf{l}_h'\mathbf{e}_i^{(n)}(\mathbf{e}_j^{(k)})'\widehat{\boldsymbol{\beta}} + \mathbf{h}'\mathbf{C}^{-1}\mathbf{e}_j^{(k)}(\mathbf{e}_i^{(n)})'\boldsymbol{\Sigma}^{-1}\mathbf{v} \\ &= \left\{-\mathbf{l}_h\widehat{\boldsymbol{\beta}}' + \boldsymbol{\Sigma}^{-1}\mathbf{v}\mathbf{h}'\mathbf{C}^{-1}\right\}_{i,j}, \quad i = 1,\dots,n,\ j = 1,\dots,k. \qquad \square\end{aligned}$$

Lemma 3.2 *Let in the model from Lemma 3.1 the symbol* $\delta\mathbf{F}$ *denote the matrix of uncertainties in the design matrix* $\mathbf{F}$. *Then*

$$(i) \quad E\left[\mathrm{Tr}\left(\delta\mathbf{F}'\frac{\partial \mathbf{h}'\widehat{\boldsymbol{\beta}}}{\partial \mathbf{F}}\right)\right] = -\,\mathrm{Tr}(\delta\mathbf{F}'\mathbf{l}_h\boldsymbol{\beta}'), \tag{6}$$

$$\begin{aligned}(ii) \quad \mathrm{Var}\left[\mathrm{Tr}\left(\delta\mathbf{F}'\frac{\partial \mathbf{h}'\widehat{\boldsymbol{\beta}}}{\partial \mathbf{F}}\right)\right] &= \mathbf{l}_h'\delta\mathbf{F}\mathbf{C}^{-1}\delta\mathbf{F}'\mathbf{l}_h + \mathbf{h}'\mathbf{C}^{-1}\delta\mathbf{F}'(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^+ \\ &\quad \times \delta\mathbf{F}\mathbf{C}^{-1}\mathbf{h},\end{aligned} \tag{7}$$

where

$$(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^+ = \boldsymbol{\Sigma}^{-1} - \boldsymbol{\Sigma}^{-1}\mathbf{F}\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}$$

is the Moore–Penrose generalized inverse of the matrix $\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F$ *(in more detail cf. [3]).*

Proof The statement (i) is obvious. As far as (ii) is concerned, it is valid that

$$\begin{aligned}\mathrm{Var}\left[\mathrm{Tr}\left(\delta\mathbf{F}'\frac{\partial \mathbf{h}'\widehat{\boldsymbol{\beta}}}{\partial \mathbf{F}}\right)\right] &= \mathrm{Var}\left\{[\mathrm{vec}(\delta\mathbf{F})]'\,\mathrm{vec}\left(\frac{\partial \mathbf{h}'\widehat{\boldsymbol{\beta}}}{\partial \mathbf{F}}\right)\right\} \\ &= \mathrm{Var}\left([\mathrm{vec}(\delta\mathbf{F})]'\left\{-(\mathbf{I}\otimes\mathbf{l}_h)\widehat{\boldsymbol{\beta}} + \left[(\mathbf{C}^{-1}\mathbf{h})\otimes\boldsymbol{\Sigma}^{-1}\right]\mathbf{v}\right\}\right).\end{aligned}$$

Since $\widehat{\boldsymbol\beta}$ and $\mathbf{v}$ are noncorrelated, $\operatorname{Var}(\widehat{\boldsymbol\beta}) = \mathbf{C}^{-1}$ and $\operatorname{Var}(\mathbf{v}) = \boldsymbol\Sigma - \mathbf{F}\mathbf{C}^{-1}\mathbf{F}'$, we have

$$\operatorname{Var}\left[\operatorname{Tr}\left(\delta\mathbf{F}'\frac{\partial \mathbf{h}'\widehat{\boldsymbol\beta}}{\partial \mathbf{F}}\right)\right] = [\operatorname{vec}(\delta\mathbf{F})]'(\mathbf{I}\otimes\mathbf{l}_h)\mathbf{C}^{-1}(\mathbf{I}\otimes\mathbf{l}_h')\operatorname{vec}(\delta\mathbf{F})$$

$$+ [\operatorname{vec}(\delta\mathbf{F})]'[(\mathbf{C}^{-1}\mathbf{h})\otimes\boldsymbol\Sigma^{-1}](\boldsymbol\Sigma - \mathbf{F}\mathbf{C}^{-1}\mathbf{F}')[(\mathbf{h}'\mathbf{C}^{-1})\otimes\boldsymbol\Sigma^{-1}]\operatorname{vec}(\delta\mathbf{F})$$

$$= [\operatorname{vec}(\delta\mathbf{F})]'[\mathbf{C}^{-1}\otimes(\mathbf{l}_h\mathbf{l}_h')]\operatorname{vec}(\delta\mathbf{F}) + [\operatorname{vec}(\delta\mathbf{F})]'[(\mathbf{C}^{-1}\mathbf{h}\mathbf{h}'\mathbf{C}^{-1})\otimes(\mathbf{M}_F\boldsymbol\Sigma\mathbf{M}_F)^+]$$

$$\times \operatorname{vec}(\delta\mathbf{F}) = \operatorname{Tr}[(\delta\mathbf{F})'\mathbf{l}_h\mathbf{l}_h'\delta\mathbf{F}\mathbf{C}^{-1}] + \operatorname{Tr}[(\delta\mathbf{F})'(\mathbf{M}_F\boldsymbol\Sigma\mathbf{M}_F)^+\delta\mathbf{F}\mathbf{C}^{-1}\mathbf{h}\mathbf{h}'\mathbf{C}^{-1}]$$

$$= \mathbf{l}_h'\delta\mathbf{F}\mathbf{C}^{-1}(\delta\mathbf{F})'\mathbf{l}_h + \mathbf{h}'\mathbf{C}^{-1}(\delta\mathbf{F})'(\mathbf{M}_F\boldsymbol\Sigma\mathbf{M}_F)^+\delta\mathbf{F}\mathbf{C}^{-1}\mathbf{h}. \qquad \square$$

Remark 3.1 Regarding Lemma 3.1 the influence of $\delta\mathbf{F}$ on the estimate of the function $\mathbf{h}'\boldsymbol\beta$, $\boldsymbol\beta \in R^k$, can be evaluated. If $\delta\mathbf{F} \neq \mathbf{0}$, then instead of $\mathbf{h}'\widehat{\boldsymbol\beta} = \mathbf{h}'\mathbf{C}^{-1}\mathbf{F}'\boldsymbol\Sigma^{-1}\mathbf{y}$ ($\mathbf{y}$ is a realization of $\mathbf{Y}$) we obtain

$$\mathbf{h}'\tilde{\boldsymbol\beta} \approx \mathbf{h}'\mathbf{C}^{-1}\mathbf{F}'\boldsymbol\Sigma^{-1}\mathbf{y} - \operatorname{Tr}[(\delta\mathbf{F})'\mathbf{l}_h\widehat{\boldsymbol\beta}'] + \operatorname{Tr}[(\delta\mathbf{F})'\boldsymbol\Sigma^{-1}\mathbf{v}\mathbf{h}'\mathbf{C}^{-1}] \tag{8}$$

(for practical purposes the values $\tilde{\boldsymbol\beta}$ and $\mathbf{y} - \mathbf{F}\tilde{\boldsymbol\beta}$ can be used on the right hand side of the last approximate equality instead of $\widehat{\boldsymbol\beta}$ and $\mathbf{v}$).

In an actual case we can judge whether uncertainty $\delta\mathbf{F}$ in the used matrix $\mathbf{F}$ satisfy the inequality

$$|-\operatorname{Tr}[(\delta\mathbf{F})'\mathbf{l}_h\widehat{\boldsymbol\beta}'] + \operatorname{Tr}[(\delta\mathbf{F})'\boldsymbol\Sigma^{-1}\mathbf{v}\mathbf{h}'\mathbf{C}^{-1}]| < \varepsilon\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}},$$

where $\varepsilon > 0$ is sufficiently small (according to an opinion of a statistician) number.

If $\delta\mathbf{F} = \mathbf{e}_i^{(n)}(\mathbf{e}_j^{(k)})'\Delta$, then

$$-\operatorname{Tr}[(\delta\mathbf{F})'\mathbf{l}_h\widehat{\boldsymbol\beta}'] + \operatorname{Tr}[(\delta\mathbf{F})'\boldsymbol\Sigma^{-1}\mathbf{v}\mathbf{h}'\mathbf{C}^{-1}] =$$

$$= -\operatorname{Tr}\left[\mathbf{e}_j^{(k)}(\mathbf{e}_i^{(n)})'\mathbf{l}_h\widehat{\boldsymbol\beta}'\right] + \operatorname{Tr}\left[\mathbf{e}_j^{(k)}(\mathbf{e}_i^{(n)})'\boldsymbol\Sigma^{-1}\mathbf{v}\mathbf{h}'\mathbf{C}^{-1}\right]$$

$$= -\{\mathbf{l}_h\}_i\{\widehat{\boldsymbol\beta}\}_j + \{\boldsymbol\Sigma^{-1}\mathbf{v}\}_i\left\{\mathbf{C}^{-1}\mathbf{h}\right\}_j.$$

Remark 3.2 According to Lemma 3.2 the influnce of $\delta\mathbf{F}$ on the estimator of the function $\mathbf{h}'\boldsymbol\beta$, $\boldsymbol\beta \in R^k$, can be evaluated. As far as the bias of the estimator $\mathbf{h}'\widehat{\boldsymbol\beta}$ is concerned, if

$$\tilde{\boldsymbol\beta} = \left[(\mathbf{F}+\delta\mathbf{F})'\boldsymbol\Sigma^{-1}(\mathbf{F}+\delta\mathbf{F})\right]^{-1}(\mathbf{F}+\delta\mathbf{F})'\boldsymbol\Sigma^{-1}\mathbf{Y},$$

then

$$E(\mathbf{h}'\tilde{\boldsymbol\beta}) \approx \mathbf{h}'\boldsymbol\beta - \operatorname{Tr}\left[(\delta\mathbf{F})'\mathbf{l}_h\boldsymbol\beta'\right],$$

i.e. the bias of the estimator is $-\operatorname{Tr}\left[(\delta\mathbf{F})'\mathbf{l}_h\boldsymbol\beta'\right]$. It must be suppressed under some reasonable bound, i.e. it must be

$$|\operatorname{Tr}\left[(\delta\mathbf{F})'\mathbf{l}_h\boldsymbol\beta'\right]| < \varepsilon\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}.$$

(Instead of $\boldsymbol\beta$ the estimator of it can be used what could be sufficient for practical purposes.)

For the sake of simplicity let $\delta\mathbf{F} = \mathbf{e}_i^{(n)}(\mathbf{e}_j^{(k)})'\Delta$. Then

$$\mathrm{Tr}\left[(\delta\mathbf{F})'\mathbf{l}_h\boldsymbol{\beta}'\right] = \Delta\,\mathrm{Tr}\left[\mathbf{e}_j^{(k)}(\mathbf{e}_i^{(n)})'\mathbf{l}_h\boldsymbol{\beta}'\right] = \Delta\{\mathbf{l}_h\}_i\{\boldsymbol{\beta}\}_j;$$

thus it should be valid

$$\Delta \ll \varepsilon\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}\frac{1}{\{\mathbf{l}_h\}_i\{\boldsymbol{\beta}\}_j}.$$

The value

$$\triangle^{(F)}_{crit,i,j} = \varepsilon\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}\frac{1}{\{\mathbf{l}_h\}_i\{\boldsymbol{\beta}\}_j} \tag{9}$$

is the maximum admissible contamination of the (i,j)-th entry of the design matrix $\mathbf{F}$. It causes a bias of the estimator $\mathbf{h}'\tilde{\boldsymbol{\beta}}$ not larger than $\varepsilon\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}$.

As far as the variance of the estimator $\mathbf{h}'\widehat{\boldsymbol{\beta}}$ is concerned, we have

$$\begin{aligned}\mathbf{h}'\tilde{\boldsymbol{\beta}} &= \mathbf{h}'\widehat{\boldsymbol{\beta}} + \Big\{\,\mathrm{Tr}\left[-(\delta\mathbf{F})'\mathbf{l}_h\widehat{\boldsymbol{\beta}}'\right] + \mathrm{Tr}\left[(\delta\mathbf{F})'\boldsymbol{\Sigma}^{-1}\mathbf{v}\mathbf{h}'\mathbf{C}^{-1}\right]\Big\}\\ &= (\mathbf{h}' - \mathbf{l}_h'\delta\mathbf{F})\widehat{\boldsymbol{\beta}} + \mathbf{h}'\mathbf{C}^{-1}\delta\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{v}\end{aligned}$$

and thus

$$\begin{aligned}\mathrm{Var}(\mathbf{h}'\tilde{\boldsymbol{\beta}}) &= (\mathbf{h}' - \mathbf{l}_h'\delta\mathbf{F})\mathbf{C}^{-1}\left[\mathbf{h} - (\delta\mathbf{F})'\mathbf{l}_h\right] + \mathbf{h}'\mathbf{C}^{-1}(\delta\mathbf{F})'(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\delta\mathbf{F}\mathbf{C}^{-1}\mathbf{h}\\ &= \mathrm{Var}(\mathbf{h}'\widehat{\boldsymbol{\beta}}) - 2\mathbf{l}_h'\delta\mathbf{F}\mathbf{C}^{-1}\mathbf{h} + \mathbf{l}_h'\delta\mathbf{F}\mathbf{C}^{-1}(\delta\mathbf{F})'\mathbf{l}_h + \mathbf{h}'\mathbf{C}^{-1}(\delta\mathbf{F})'\\ &\quad\times(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\delta\mathbf{F}\mathbf{C}^{-1}\mathbf{h}.\end{aligned}$$

The variance of the estimator with an uncertain design matrix differs from the variance of the estimator with the proper design matrix. The difference is

$$-2\mathbf{l}_h'\delta\mathbf{F}\mathbf{C}^{-1}\mathbf{h} + \mathbf{l}_h'\delta\mathbf{F}\mathbf{C}^{-1}(\delta\mathbf{F})'\mathbf{l}_h + \mathbf{h}'\mathbf{C}^{-1}(\delta\mathbf{F})'(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\delta\mathbf{F}\mathbf{C}^{-1}\mathbf{h}.$$

For the sake of simplicity let $\delta\mathbf{F} = \mathbf{e}_i^{(n)}(\mathbf{e}_j^{(k)})'\Delta$. Then the difference is

$$\begin{aligned}&\gamma_{h,(i,j)} =\\ &= -2\Delta\{\mathbf{l}_h\}_i\{\mathbf{C}^{-1}\mathbf{h}\}_j + \Delta^2\Big[\{\mathbf{C}\}^{j,j}(\{\mathbf{l}_h\}_i)^2 + (\{\mathbf{C}^{-1}\mathbf{h}\}_j)^2\{(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\}_{i,i}\Big].\end{aligned}$$

It can be assumed that $\gamma_{h,(i,j)} \ll \mathbf{h}'\mathbf{C}^{-1}\mathbf{h}$ and thus

$$\begin{aligned}\sqrt{\mathrm{Var}(\mathbf{h}'\tilde{\boldsymbol{\beta}})} &= \sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h} + \gamma_{h,(i,j)}} = \sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}\left(1 + \frac{\gamma_{h,(i,j)}}{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}\right)^{1/2}\\ &\approx \sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}\left(1 + \frac{1}{2}\frac{\gamma_{h,(i,j)}}{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}\right).\end{aligned}$$

The solution $\triangle^{(V)}_{crit,i,j}$ of the quadratic equation

$$\begin{aligned}\Delta^2&\Big[\{\mathbf{C}\}^{j,j}(\{\mathbf{l}_h\}_i)^2 + (\{\mathbf{C}^{-1}\mathbf{h}\}_j)^2\{(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\}_{i,i}\Big]\\ &- 2\Delta\{\mathbf{l}_h\}_i\{\mathbf{C}^{-1}\mathbf{h}\}_j - 2\varepsilon\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}} = 0\end{aligned} \tag{10}$$

is the maximum admissible contamination of the (i,j)-th entry of the design matrix $\mathbf{F}$. It causes an enlargement of the standard deviation $\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}$ not larger than $\varepsilon\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}$. The value of the quantity $\gamma_{h,(i,j)}$ is the same for both roots of the quadratic equation.

It is useful to arrange tables of the values $\triangle^{(F)}_{crit,i,j}$ (cf. (9)) and $\triangle^{(V)}_{crit,i,j}$ (cf. (10)) for all $i=1,\dots,n$ and $j=1,\dots,k$, cf. section 5 Numerical examples.

Remark 3.3 The most dangerous shift $\delta\mathbf{F}$ of the matrix $\mathbf{F}$ with respect to the bias of the estimator is in the direction of the gradient, i.e.

$$\delta\mathbf{F}^* = kE\left(\frac{\partial \mathbf{h}'\widehat{\boldsymbol{\beta}}}{\partial \mathbf{F}}\right) = -k\mathbf{l}_h\boldsymbol{\beta}'.$$

(The number k will be determined later.) The bias of the estimator caused by $\delta\mathbf{F}^*$ is

$$-\operatorname{Tr}\left[(\delta\mathbf{F}^*)'\mathbf{l}_h\boldsymbol{\beta}'\right] = k\boldsymbol{\beta}'\boldsymbol{\beta}\mathbf{l}_h'\mathbf{l}_h.$$

The number k now can be bounded according to the condition

$$k\boldsymbol{\beta}'\boldsymbol{\beta}\mathbf{l}_h'\mathbf{l}_h < \varepsilon\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}.$$

The matrix

$$\delta\mathbf{F}^* = \frac{\varepsilon\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}}{\boldsymbol{\beta}'\boldsymbol{\beta}\mathbf{l}_h'\mathbf{l}_h}\mathbf{l}_h\boldsymbol{\beta}' \tag{11}$$

can serve as a good information on the necessary accuracy of the matrix $\mathbf{F}$ in connection with the bias of the estimator $\mathbf{h}'\widehat{\boldsymbol{\beta}}$.

It is to be remarked that in the case $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}$, the number k must satisfy the inequality $k < \sigma\varepsilon/\left(\boldsymbol{\beta}'\boldsymbol{\beta}\sqrt{\mathbf{h}'(\mathbf{F}'\mathbf{F})^{-1}\mathbf{h}}\right)$.

4 Uncertainty in the covariance matrix

Lemma 4.1 *In the regular linear model* $\mathbf{Y} \sim_n (\mathbf{F}\boldsymbol{\beta},\boldsymbol{\Sigma})$, $\boldsymbol{\beta}\in R^k$, *for a given linear function* $\mathbf{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta}\in R^k$, *it is valid that*

$$\frac{\partial \mathbf{h}'\widehat{\boldsymbol{\beta}}}{\partial \sigma_{i,j}} = -\{\mathbf{l}_h\}_i\{\boldsymbol{\Sigma}^{-1}\mathbf{v}\}_j - \{\mathbf{l}_h\}_j\{\boldsymbol{\Sigma}^{-1}\mathbf{v}\}_i,\ i,j=1,\dots,n.$$

Proof Since $\mathbf{h}'\widehat{\boldsymbol{\beta}} = \mathbf{h}'(\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F})^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}$, it is valid that

$$\begin{aligned}\frac{\partial \mathbf{h}'\widehat{\boldsymbol{\beta}}}{\partial \sigma_{i,j}} &= \mathbf{h}'\frac{\partial(\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F})^{-1}}{\partial\sigma_{i,j}}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{Y} + \mathbf{h}'(\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F})^{-1}\mathbf{F}'\frac{\partial\boldsymbol{\Sigma}^{-1}}{\partial\sigma_{i,j}}\mathbf{Y}\\ &= \mathbf{h}'\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\left[\mathbf{e}_i^{(n)}(\mathbf{e}_j^{(n)})' + \mathbf{e}_j^{(n)}(\mathbf{e}_i^{(n)})'\right]\boldsymbol{\Sigma}^{-1}\mathbf{F}\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}\\ &\quad - \mathbf{h}'\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\left[\mathbf{e}_i^{(n)}(\mathbf{e}_j^{(n)})' + \mathbf{e}_j^{(n)}(\mathbf{e}_i^{(n)})'\right]\boldsymbol{\Sigma}^{-1}\mathbf{Y}\\ &= \left\{-\left[\mathbf{l}_h(\boldsymbol{\Sigma}^{-1}\mathbf{v})' + \boldsymbol{\Sigma}^{-1}\mathbf{v}\mathbf{l}_h'\right]\right\}_{i,j},\quad i,j=1,\dots,n.\end{aligned}$$

□

Remark 4.1 Since uncertainty in the covariance matrix does not cause the bias of the estimator, only a change of the variance of the estimator must be taken into account. Since it is valid that

$$\mathbf{h}'\big[\mathbf{F}'(\boldsymbol{\Sigma}+\delta\boldsymbol{\Sigma})^{-1}\mathbf{F}\big]^{-1}\mathbf{F}'(\boldsymbol{\Sigma}+\delta\boldsymbol{\Sigma})^{-1}\mathbf{Y} \approx \mathbf{h}'\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{Y}$$
$$- \mathrm{Tr}\left[\delta\boldsymbol{\Sigma}(\mathbf{l}_h\mathbf{v}'\boldsymbol{\Sigma}^{-1}+\boldsymbol{\Sigma}^{-1}\mathbf{v}\mathbf{l}_h')\right] = \mathbf{h}'\widehat{\boldsymbol{\beta}} - 2\mathbf{l}'\delta\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\mathbf{v},$$

we have

$$\mathrm{Var}_{\Sigma}\left\{\mathbf{h}'\big[\mathbf{F}'(\boldsymbol{\Sigma}+\delta\boldsymbol{\Sigma})^{-1}\mathbf{F}\big]^{-1}\mathbf{F}'(\boldsymbol{\Sigma}+\delta\boldsymbol{\Sigma})^{-1}\mathbf{Y}\right\}$$
$$\approx \mathbf{h}'\mathbf{C}^{-1}\mathbf{h} + 4\mathbf{l}_h'\delta\boldsymbol{\Sigma}(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\delta\boldsymbol{\Sigma}\mathbf{l}_h.$$

If

$$\delta\boldsymbol{\Sigma} = \begin{cases} \big[\mathbf{e}_i^{(n)}(\mathbf{e}_j^{(n)})' + \mathbf{e}_j^{(n)}(\mathbf{e}_i^{(n)})'\big]\Delta, & i \neq j \\ \big[\mathbf{e}_i^{(n)}(\mathbf{e}_i^{(n)})'\big]\Delta, & i = j \end{cases}$$

then if $i \neq j$

$$d_{h,(i,j)} = 4\mathbf{l}_h'\delta\boldsymbol{\Sigma}(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\delta\boldsymbol{\Sigma}\mathbf{l}_h$$
$$= 4\big(\{\mathbf{l}_h\}_j, \{\mathbf{l}_h\}_i\big)\begin{pmatrix} \{(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\}_{i,i}, & \{(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\}_{i,j} \\ \{(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\}_{j,i}, & \{(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\}_{j,j} \end{pmatrix}\begin{pmatrix} \{\mathbf{l}_h\}_j \\ \{\mathbf{l}_h\}_i \end{pmatrix}\Delta^2,$$

if $i = j$

$$d_{h,(i,i)} = 4(\{\mathbf{l}_h\}_i)^2\{(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\}_{i,i}\Delta^2.$$

Since we can assume that $d_{h,(i,j)} \ll \mathbf{h}'\mathbf{C}^{-1}\mathbf{h}$, we can write

$$\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h} + d_{h,(i,j)}} \approx \sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}\left(1 + \frac{1}{2}\frac{d_{h,(i,j)}}{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}\right).$$

The matrix $\mathbf{D}_h$ with the (i,j)-th entry

$$\{\mathbf{D}_h\}_{i,j} = \left(1 + \frac{1}{2}\frac{d_{h,(i,j)}}{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}\right), \quad i,j = 1,\dots,n,$$

can help to analyze the influence of $\delta\boldsymbol{\Sigma}$ on the standard deviation of the estimator $\mathbf{h}'\widehat{\boldsymbol{\beta}}$. The value $\{\mathbf{D}_h\}_{i,j}$ means the ratio of the standard deviation of the estimator calculated with the covariance matrix $\boldsymbol{\Sigma} + \delta\boldsymbol{\Sigma}$ to the standard deviation of the estimator calculated with proper covariance matrix $\boldsymbol{\Sigma}$.

The solution $\triangle^{(\boldsymbol{\Sigma})}_{\boldsymbol{crit,i,j}}$ of the equation (for $i \neq j$)

$$2\Delta^2\big(\{\mathbf{l}_h\}_j, \{\mathbf{l}_h\}_i\big)\begin{pmatrix} \{(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\}_{i,i}, & \{(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\}_{i,j} \\ \{(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\}_{j,i}, & \{(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\}_{j,j} \end{pmatrix}\begin{pmatrix} \{\mathbf{l}_h\}_j \\ \{\mathbf{l}_h\}_i \end{pmatrix}$$
$$= \varepsilon\mathbf{h}'\mathbf{C}^{-1}\mathbf{h} \quad (12)$$

and the equation (for $i = j$)

$$2\Delta^2(\{\mathbf{l}_h\}_i)^2\{(\mathbf{M}_F\boldsymbol{\Sigma}\mathbf{M}_F)^{+}\}_{i,i} = \varepsilon\mathbf{h}'\mathbf{C}^{-1}\mathbf{h} \quad (13)$$

is the maximum admissible contamination of the (i,j)-th entry of the variance matrix Σ. It causes an enlargement of the standard deviation $\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}$ not greater than $\varepsilon\sqrt{\mathbf{h}'\mathbf{C}^{-1}\mathbf{h}}$.

5 Numerical examples

Example 5.1 Let the regression model be

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} \sim_n \left[\begin{pmatrix} 1, & 1 \\ 1, & 2 \\ 1, & 3 \\ 1, & 4 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}, \sigma^2 \mathbf{I} \right], \ \sigma = 0.1$$

and $\mathbf{y} = (1.6, 1.9, 2.6, 3.1)'$.

Then

$$(\mathbf{F}'\mathbf{F})^{-1} = \begin{pmatrix} 1.5, & -0.5 \\ -0.5, & 0.2 \end{pmatrix}, \quad \sigma^2(\mathbf{F}'\mathbf{F})^{-1} = \begin{pmatrix} 0.0150, & -0.0050 \\ -0.0050, & 0.0020 \end{pmatrix},$$

$$(\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}' = \begin{pmatrix} 1.0, & 0.5, & 0.0, & -0.5 \\ -0.3, & -0.1, & 0.1, & 0.3 \end{pmatrix},$$

$$\widehat{\boldsymbol{\beta}} = (\mathbf{F}'\mathbf{F})^{-1}\mathbf{F}'\mathbf{y} = \begin{pmatrix} 1.00 \\ 0.52 \end{pmatrix}, \quad \mathbf{v} = \mathbf{y} - \mathbf{F}\widehat{\boldsymbol{\beta}} = (0.08, -0.14, 0.04, 0.02)'.$$

Let $\mathbf{h}_1 = (1, 0)'$ in situation A, $\mathbf{h}_2 = (0, 1)'$ in situation B and $\varepsilon = 0.2$.

Then in situation A according Remark 3.4 formulas (9) and (10) we will determine:

$$\triangle^{(\boldsymbol{F})}_{\boldsymbol{crit}} = \begin{pmatrix} 0.0245 & 0.0471 \\ 0.0490 & 0.0942 \\ \infty & \infty \\ -0.0490 & -0.0942 \end{pmatrix}, \quad \delta\mathbf{F}^* = \begin{pmatrix} 0.0129 & 0.0067 \\ 0.0064 & 0.0033 \\ 0 & 0 \\ -0.0064 & -0.0033 \end{pmatrix},$$

from (10) two solution ${}_1\triangle^{(\boldsymbol{V})}_{\boldsymbol{crit}}$ and ${}_2\triangle^{(\boldsymbol{V})}_{\boldsymbol{crit}}$ are obtained

$${}_1\triangle^{(\boldsymbol{V})}_{\boldsymbol{crit}} = \begin{pmatrix} -0.9620 & -6.4139 \\ -1.2464 & -5.9078 \\ -1.7637 & -5.2910 \\ -2.9893 & -4.5720 \end{pmatrix}, \quad {}_2\triangle^{(\boldsymbol{V})}_{\boldsymbol{crit}} = \begin{pmatrix} 2.3413 & 2.7775 \\ 2.0156 & 3.6855 \\ 1.7637 & 5.2910 \\ 1.5608 & 8.5720 \end{pmatrix}.$$

These two matrices cause an enlargement of standard deviation not more ε-times. As a criterion the value

$$\min\left\{ |{}_1\triangle^{(\boldsymbol{V})}{}_{crit,i,j}|, |{}_2\triangle^{(\boldsymbol{V})}{}_{crit,i,j}| \right\}$$

must be choosen in practice.

$$\triangle^{(\boldsymbol{\Sigma})}_{\boldsymbol{crit}} = \begin{pmatrix} 0.0071 & 0.0063 & 0.0046 & 0.0093 \\ 0.0063 & 0.0093 & 0.0093 & 0.0071 \\ 0.0046 & 0.0093 & \infty & 0.0093 \\ 0.0093 & 0.0071 & 0.0093 & 0.0141 \end{pmatrix},$$

For example the value $\triangle^{(\boldsymbol{F})}{}_{crit,(3,1)}$ and $\triangle^{(\boldsymbol{F})}{}_{crit,(3,2)}$ for $\mathbf{h} = (1, 0)'$ cannot be determined, since $\{\mathbf{l}_h\}_3\{\boldsymbol{\beta}\}_1$ and $\{\mathbf{l}_h\}_3\{\boldsymbol{\beta}\}_2$, respectively are zero. Ever it seems that the contamination of the design matrix $\mathbf{F}$ in the third row can be any larger number, it is not so. An aproach to determination of the value $\triangle^{(\boldsymbol{F})}_{\boldsymbol{crit,i,j}}$ is infinitesimal and therefore some carefulness it necessary. If, e.g. $\triangle^{(\boldsymbol{F})}_{crit,(3,1)} = 0.1$, then the bias of the estimator $\widehat{(1,0)\boldsymbol{\beta}}$ is $(0.0096, 0.0064)'$, what is admissible. However the value $\triangle^{(\boldsymbol{F})}_{crit,(3,1)} = 1$ leads to a non-admissible bias.

In situation B according Remark 3.4 formulas (9), (10) and from the Remark 3.5 formula (11) we will determine:

$$\triangle^{(\boldsymbol{F})}_{\boldsymbol{crit}} = \begin{pmatrix} -0.0298 & -0.0573 \\ -0.0894 & -0.1720 \\ 0.0894 & 0.1720 \\ 0.0298 & 0.0573 \end{pmatrix}, \quad \delta\mathbf{F}^* = \begin{pmatrix} -0.0106 & -0.0055 \\ -0.0035 & -0.0018 \\ 0.0035 & 0.0018 \\ 0.0106 & 0.0055 \end{pmatrix},$$

$${}_1\triangle^{(\boldsymbol{V})}_{\boldsymbol{crit}} = \begin{pmatrix} -2.2905 & -9.9767 \\ -2.8165 & -8.4173 \\ -3.3428 & -7.0840 \\ -3.7190 & -5.9767 \end{pmatrix}, \quad {}_2\triangle^{(\boldsymbol{V})}_{\boldsymbol{crit}} = \begin{pmatrix} 3.7190 & 5.9767 \\ 3.3428 & 7.0840 \\ 2.8165 & 8.4173 \\ 2.2905 & 9.9767 \end{pmatrix},$$

and from the Remark 4.2 formulas (12) and (13) we will determine

$$\triangle^{(\boldsymbol{\Sigma})}_{\boldsymbol{crit}} = \begin{pmatrix} 0.0086 & 0.0069 & 0.0053 & 0.0105 \\ 0.0069 & 0.0169 & 0.0105 & 0.0053 \\ 0.0053 & 0.0105 & 0.0169 & 0.0069 \\ 0.0105 & 0.0053 & 0.0069 & 0.0086 \end{pmatrix}.$$

Let for $\delta\mathbf{F} = \delta\mathbf{F}^*$ the value of the estimator (8) from Remark 3.3 be compared with $\mathbf{h}'\hat{\boldsymbol{\beta}} = \mathbf{h}'\begin{pmatrix} 1.00 \\ 0.52 \end{pmatrix}$; $\mathbf{h}'\tilde{\boldsymbol{\beta}} = \mathbf{h}'\hat{\boldsymbol{\beta}} - \mathrm{Tr}[(\delta\mathbf{F}^*)'\mathbf{l}_h\hat{\boldsymbol{\beta}}'] + \mathrm{Tr}[(\delta\mathbf{F}^*)'\mathbf{v}\mathbf{h}'\mathbf{C}^{-1}]$.

If $\mathbf{h} = (1,0)'$, then $\mathbf{h}'\tilde{\boldsymbol{\beta}} - \mathbf{h}'\hat{\boldsymbol{\beta}} = 0.9755 - 1.0000 = -0.0245$.

If $\mathbf{h} = (0,1)'$, then $\mathbf{h}'\tilde{\boldsymbol{\beta}} - \mathbf{h}'\hat{\boldsymbol{\beta}} = 0.5111 - 0.5200 = -0.0089$.

Example 5.2 Let the regression model be

$$y_i = \frac{\beta_1 x_i}{\beta_2 + x_i}, \quad i = 1,2,3,4,5 \tag{14}$$

and results of measurement of y at points $x_1, \ldots, x_5$ be

x	1	2	3	4	5
y	3.2	4.9	6.2	6.5	7.3

$$\boldsymbol{\Sigma} = \begin{pmatrix} 0.1^2, & 0, & 0, & 0, & 0 \\ 0, & 0.1^2, & 0, & 0, & 0 \\ 0, & 0, & 0.2^2, & 0, & 0 \\ 0, & 0, & 0, & 0.2^2, & 0 \\ 0, & 0, & 0, & 0, & 0.2^2 \end{pmatrix}.$$

Equations (14) enable us to obtain an approximate values $\boldsymbol{\beta}^{(0)}$.

For 1^{st} and 5^{nd} measurement two equations for unknown parameters lead to an approximate values $\boldsymbol{\beta}^{(0)} = (10,2)'$.

The linear version of the functions (14) obtained by the using the Taylor expansion at the approximate point $\boldsymbol{\beta}^{(0)}$ is in the form $\mathbf{Y} - \mathbf{g}(\boldsymbol{\beta}^{(0)}) = \mathbf{F}\delta\boldsymbol{\beta}$, where $\mathbf{F} = \frac{\partial g(\boldsymbol{\beta}^{(0)})}{\partial \boldsymbol{\beta}'}$ and $\mathbf{g}(\boldsymbol{\beta}^{(0)}) = (g_1(\boldsymbol{\beta}^{(0)}),\dots,g_5(\boldsymbol{\beta}^{(0)}))'$, $g_i(\boldsymbol{\beta}^{(0)}) = \frac{\beta_1^{(0)}x_i}{\beta_2^{(0)}+x_i}$, $i = 1,2,3,4,5$.

In our case we will determine

$$\mathbf{F} = \begin{pmatrix} 0.3333 & -1.1111 \\ 0.5000 & -1.2500 \\ 0.6000 & -1.2000 \\ 0.6667 & -1.1111 \\ 0.7143 & -1.0204 \end{pmatrix}, \quad \mathbf{y}_i^0 = \frac{\beta_1^0 x_i}{\beta_2^0 + x_i}, \; i = 1,2,3,4,5$$

$$\mathbf{y}^0 = (3.3333, 5.0000, 6.0000, 6.6667, 7.1429)',$$

$$\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}^{(0)} + \delta\widehat{\boldsymbol{\beta}} = \boldsymbol{\beta}^{(0)} + (\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F})^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\mathbf{y}^0) = \begin{pmatrix} 10.5230 \\ 2.2754 \end{pmatrix},$$

$$\mathbf{v} = \mathbf{y} - \mathbf{F}\widehat{\boldsymbol{\beta}} = (-0.0127, -0.0226, 0.2158, -0.2075, 0.0681)'.$$

Let $\mathbf{h} = (1,0)'$, $\sigma = 0.1$, $\varepsilon = 0.2$. Then in our linearized model we will determine numerically from the Remark 3.4 formula (9) and from the Remark 3.5 formula (11)

$$\triangle_{\boldsymbol{crit}}^{(\boldsymbol{F})} = \begin{pmatrix} -0.0681 & -0.1294 \\ -0.4915 & -0.9333 \\ 0.3349 & 0.6359 \\ 0.1672 & 0.3174 \\ 0.1184 & 0.2248 \end{pmatrix}, \quad \delta\mathbf{F}^* = \begin{pmatrix} -0.0342 & -0.0180 \\ -0.0047 & -0.0025 \\ 0.0070 & 0.0037 \\ 0.0140 & 0.0073 \\ 0.0197 & 0.0104 \end{pmatrix},$$

and from the Remark 3.4 formulas (9), (10) and from the Remark 3.5 formula (11)

$${}_1\triangle_{\boldsymbol{crit}}^{(\boldsymbol{V})} = \begin{pmatrix} -0.5132 & -1.2038 \\ -0.3135 & -0.7568 \\ -0.3516 & -0.8476 \\ -0.2763 & -0.6640 \\ -0.2308 & -0.5531 \end{pmatrix}, \quad {}_2\triangle_{\boldsymbol{crit}}^{(\boldsymbol{V})} = \begin{pmatrix} 0.1342 & 0.3216 \\ 0.2530 & 0.6107 \\ 0.5799 & 1.3966 \\ 0.7268 & 1.7326 \\ 0.8581 & 2.0160 \end{pmatrix},$$

and from the Remark 4.2 formulas (12) and (13)

$$\triangle_{\boldsymbol{crit}}^{(\boldsymbol{\Sigma})} = \begin{pmatrix} 0.0095 & 0.0083 & 0.0116 & 0.0126 & 0.0160 \\ 0.0083 & 0.0536 & 0.0310 & 0.0172 & 0.0125 \\ 0.0116 & 0.0310 & 0.0600 & 0.0303 & 0.0226 \\ 0.0126 & 0.0172 & 0.0303 & 0.0329 & 0.0296 \\ 0.0160 & 0.0125 & 0.0226 & 0.0296 & 0.0273 \end{pmatrix}.$$

Let $\mathbf{h} = (0,1)'$, $\varepsilon = 0.2$. Then

$$\triangle_{crit}^{(F)} = \begin{pmatrix} 0.0027 & -0.1200 \\ 0.0043 & -0.1912 \\ 0.0217 & -0.9609 \\ -0.0083 & 0.3674 \\ -0.0038 & 0.1699 \end{pmatrix}, \quad \delta\mathbf{F}^* = \begin{pmatrix} -0.0315 & -0.0166 \\ -0.0114 & -0.0060 \\ 0.0031 & 0.0016 \\ 0.0082 & 0.0043 \\ 0.0125 & 0.0066 \end{pmatrix},$$

$${}_1\triangle_{crit}^{(V)} = \begin{pmatrix} -0.5941 & -1.3733 \\ -0.5255 & -1.1809 \\ -0.6593 & -1.4756 \\ -0.5643 & -1.2680 \\ -0.4958 & -1.1190 \end{pmatrix}, \quad {}_2\triangle_{crit}^{(V)} = \begin{pmatrix} 0.2216 & 0.5018 \\ 0.3422 & 0.7678 \\ 0.8050 & 1.8022 \\ 0.9644 & 2.1739 \\ 1.1108 & 2.5353 \end{pmatrix},$$

$$\triangle_{crit}^{(\Sigma)} = \begin{pmatrix} 0.0076 & 0.0080 & 0.0097 & 0.0108 & 0.0143 \\ 0.0080 & 0.0166 & 0.0229 & 0.0165 & 0.0129 \\ 0.0097 & 0.0229 & 0.1013 & 0.0389 & 0.0262 \\ 0.0108 & 0.0165 & 0.0389 & 0.0415 & 0.0347 \\ 0.0143 & 0.0129 & 0.0262 & 0.0347 & 0.0321 \end{pmatrix}.$$

6 Concluding remarks

The aim in linear statistical models is to determine an estimator of the parameter $\boldsymbol{\beta}$ on the basis of the observation vector $\mathbf{Y}$.

In this article we concentrated on a fundamental questions – how uncertainty of the design and covariance matrices influence the bias and the variance of estimators.

The quantities $\triangle_{crit}^{(F)}$, $\delta\mathbf{F}^*$, $\triangle_{crit}^{(V)}$, $\triangle_{crit}^{(\Sigma)}$ enables to judge how precise the record of the design matrix and the covariance matrix must be.

In the last example it can be seen that in the situation B for $\varepsilon = 0.2$ the record of the design matrix must take into account the values 0.001 and that record of the covariance matrix must take into account the values 0.01.

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