On Structure Space of $\Gamma$-Semigroups

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Abstract

In this paper we introduce the notion of the structure space of $\Gamma$-semigroups formed by the class of uniformly strongly prime ideals. We also study separation axioms and compactness property in this structure space.

Key words: $\Gamma$-semigroup; uniformly strongly prime ideal; Noetherian $\Gamma$-semigroup, hull-kernel topology, structure space.

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1 Introduction

In [4], L. Gillman studied “Rings with Hausdorff structure space” and in [7], C. W. Kohls studied “The space of prime ideals of a ring”. In [1], M. R. Adhikari and M. K. Das studied ‘Structure spaces of semirings’.

In [9], M. K. Sen and N. K. Saha introduced the notion of $\Gamma$-Semigroup. Some works on $\Gamma$-Semigroups may be found in [10], [8], [5], [6], [2] and [3].

In this paper we introduce and study the structure space of $\Gamma$-Semigroups. For this we consider the collection $\mathcal{A}$ of all proper uniformly strongly prime ideals of a $\Gamma$-Semigroup $S$ and we give a topology $\tau_\mathcal{A}$ on $\mathcal{A}$ by means of closure operator defined in terms of intersection and inclusion relation among these ideals of the $\Gamma$-Semigroup $S$. We call the topological space $(\mathcal{A}, \tau_\mathcal{A})$—the structure space of the $\Gamma$-Semigroup $S$. We study separation axioms, compactness and connectedness in this structure space.
2 Preliminaries

Definition 2.1 Let $S = \{a, b, c, \ldots\}$ and $\Gamma = \{\alpha, \beta, \gamma, \ldots\}$ be two nonempty sets. $S$ is called a $\Gamma$-semigroup if

(i) $a a b \in S$, for all $a \in \Gamma$ and $a, b \in S$ and
(ii) $(a a b) \beta c = a a (b \beta c)$, for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

$S$ is said to be $\Gamma$-semigroup with zero if there exists an element $0 \in S$ such that $0 a a = a a 0 = 0$ for all $\alpha \in \Gamma$.

Example 2.2 Let $S$ be a set of all negative rational numbers. Obviously $S$ is not a semigroup under usual product of rational numbers. Let $\Gamma = \{-\frac{1}{p} : p \text{ is prime}\}$.

Definition 2.3 Let $S$ be a $\Gamma$-semigroup and $\alpha \in \Gamma$. Then $e \in S$ is said to be an $\alpha$-idempotent if $e a e = e$. The set of all $\alpha$-idempotents is denoted by $E_\alpha$ and we denote $\bigcup_{\alpha \in \Gamma} E_\alpha$ by $E(S)$. The elements of $E(S)$ are called idempotent element of $S$.

Definition 2.4 A nonempty subset $I$ of a $\Gamma$-semigroup $S$ is called an ideal if $\Pi S \subseteq I$ and $STI \subseteq I$ where for subsets $U, V$ of $S$ and $\Delta$ of $\Gamma$, $U \Delta V = \{u a v : u \in U, v \in V, \alpha \in \Delta\}$.

Definition 2.5 A nonempty subset $I$ of a $\Gamma$-semigroup $S$ is called an ideal if $\Pi S \subseteq I$ and $STI \subseteq I$ where for subsets $U, V$ of $S$ and $\Delta$ of $\Gamma$, $U \Delta V = \{u a v : u \in U, v \in V, \alpha \in \Delta\}$. An ideal $I$ of $S$ is called a proper ideal if $I \neq S$.

Definition 2.6 A proper ideal $P$ of a $\Gamma$-Semigroup $S$ is called a prime ideal of $S$ if $A \Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any two ideals $A, B$ of $S$.

Definition 2.7 An ideal $I$ of a $\Gamma$-semigroup $S$ is said to be full if $E(S) \subseteq I$.

An ideal $I$ of a $\Gamma$-semigroup $S$ is said to be a prime full ideal if it is both prime and full.

Theorem 2.8 Let $S$ be a $\Gamma$-semigroup. For an ideal $P$ of $S$, the following are equivalent.

(i) If $A$ and $B$ are ideals of $S$ such that $A \Gamma B \subseteq P$ then either $A \subseteq P$ or $B \subseteq P$.
(ii) If $a \Gamma ST b \subseteq P$ then either $a \in P$ or $b \in P(a, b \in S)$
(iii) If $I_1$ and $I_2$ are two right ideals of $S$ such that $I_1 \Gamma I_2 \subseteq P$ then either $I_1 \subseteq P$ or $I_2 \subseteq P$.
(iv) If $J_1$ and $J_2$ are two left ideals of $S$ such that $J_1 \Gamma J_2 \subseteq P$ then either $J_1 \subseteq P$ or $J_2 \subseteq P$. 

Proof (i) ⇒ (ii): Suppose \( a \Gamma S b \subseteq P \). Then \( <a> \Gamma <a> \Gamma <b> \Gamma <b> \subseteq P \). Since \( <a> \Gamma <a> \), \( <b> \Gamma <b> \) are ideals of \( S \), so by (i) we have either \( <a> \Gamma <a> \subseteq P \) or \( <b> \Gamma <b> \subseteq P \). By repeated uses of (i) we get \( a \in <a> \subseteq P \) or \( b \in <b> \subseteq P \).

(ii) ⇒ (iii): Let \( I_1 \Gamma I_2 \subseteq P \). Let \( I_1 \not\subseteq P \). Then there exists an element \( a_1 \in I_1 \) such that \( a_1 \notin P \). Then for every \( a_2 \in I_2 \) we have \( a_1 \Gamma S a_2 \subseteq I_1 \Gamma I_2 \subseteq P \). Hence from (ii) \( a_2 \notin P \). Thus \( I_2 \subseteq P \). Similarly (ii) implies (iv).

The proof is completed by observing that (i) is implied obviously either by (iii) or by (iv).

 Definition 2.9 An ideal \( P \) of a \( \Gamma \)-Semigroup \( S \) is called a uniformly strongly prime ideal (usp ideal) if \( S \) and \( \Gamma \) contain finite subsets \( F \) and \( \Delta \) respectively such that \( x \Delta F \Delta y \subseteq P \) implies that \( x \in P \) or \( y \in P \) for all \( x, y \in S \).

Theorem 2.10 Let \( S \) be a \( \Gamma \)-semigroup. Then every uniformly strongly prime ideal is a prime ideal.

Proof Let \( P \) be a uniformly strongly prime ideal of \( S \). Then \( S \) and \( \Gamma \) contain finite subsets \( F \) and \( \Delta \) respectively such that \( x \Delta F \Delta y \subseteq P \) implies that \( x \in P \) or \( y \in P \) for all \( x, y \in S \). Now let \( a \Gamma S b \subseteq P \). Thus we have \( a \Delta F \Delta b \subseteq a \Gamma S b \subseteq P \) and hence we have \( a \in P \) or \( b \in P \). Hence \( P \) is prime ideal by Theorem 2.8.

Throughout this paper \( S \) will always denote a \( \Gamma \)-Semigroup with zero and unless otherwise stated a \( \Gamma \)-Semigroup means a \( \Gamma \)-Semigroup with zero.

3 Structure space of \( \Gamma \)-semigroups

Suppose \( \mathcal{A} \) is the collection of all uniformly strongly prime ideals of a \( \Gamma \)-Semigroup \( S \). For any subset \( A \) of \( \mathcal{A} \), we define

\[ \overline{A} = \{ I \in \mathcal{A}: \bigcap_{I_\alpha \in A} I_\alpha \subseteq I \}. \]

It is easy to see that \( \overline{\emptyset} = \emptyset \).

Theorem 3.1 Let \( A, B \) be any two subsets of \( \mathcal{A} \). Then

(i) \( A \subseteq \overline{A} \)

(ii) \( \overline{A} = \overline{\overline{A}} \)

(iii) \( A \subseteq B \implies \overline{A} \subseteq \overline{B} \)

(iv) \( A \cup B = \overline{A \cup B} \)

Proof (i): Clearly, \( \bigcap_{I_\alpha \in A} I_\alpha \subseteq I_\alpha \) for each \( \alpha \) and hence \( A \subseteq \overline{A} \).

(ii): By (i), we have \( \overline{A} \subseteq \overline{\overline{A}} \). For converse part, let \( I_\beta \in \overline{A} \). Then \( \bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I_\beta \). Now \( I_\alpha \in \overline{A} \) implies that \( \bigcap_{I_\gamma \in \overline{A}} I_\gamma \subseteq I_\alpha \) for all \( \alpha \in \Lambda \). Thus

\[ \bigcap_{I_\gamma \in \Lambda} I_\gamma \subseteq \bigcap_{I_\alpha \in \overline{A}} I_\alpha \subseteq I_\beta \] i.e. \( \bigcap_{I_\alpha \in \Lambda} I_\gamma \subseteq I_\beta \).
So \( I_\beta \in \overline{\mathcal{A}} \) and hence \( \overline{\mathcal{A}} \subseteq \overline{\mathcal{A}} \). Consequently, \( \overline{\mathcal{A}} = \overline{\mathcal{A}} \).

(iii): Suppose that \( A \subseteq B \). Let \( I_\alpha \in \overline{\mathcal{A}} \). Then \( \bigcap_{\beta \in A} I_\beta \subseteq I_\alpha \). Since \( A \subseteq B \), it follows that

\[
\bigcap_{\beta \in B} I_\beta \subseteq \bigcap_{\beta \in A} I_\beta \subseteq I_\alpha.
\]

This implies that \( I_\alpha \in B \) and hence \( \overline{\mathcal{A}} \subseteq B \).

(iv): Clearly, \( \overline{A \cup B} \subseteq \overline{A} \cup \overline{B} \).

For the reverse part, let \( I_\alpha \in \overline{A \cup B} \). Then \( \bigcap_{I_\beta \in A \cup B} I_\beta \subseteq I_\alpha \).

It is easy to see that

\[
\bigcap_{I_\beta \in A \cup B} I_\beta = \left( \bigcap_{I_\beta \in A} I_\beta \right) \cap \left( \bigcap_{I_\beta \in B} I_\beta \right).
\]

Since \( \bigcap_{I_\beta \in A} I_\beta \) and \( \bigcap_{I_\beta \in B} I_\beta \) are ideals of \( S \), we have

\[
\left( \bigcap_{I_\beta \in A} I_\beta \right) \cap \left( \bigcap_{I_\beta \in B} I_\beta \right) \subseteq \left( \bigcap_{I_\beta \in A} I_\beta \right) \cap \left( \bigcap_{I_\beta \in B} I_\beta \right) = \bigcap_{I_\beta \in A \cup B} I_\beta \subseteq I_\alpha.
\]

Since every uniformly strongly prime ideal is prime, \( I_\alpha \) is a prime ideal of \( S \) and hence either \( \bigcap_{I_\beta \in A} I_\beta \subseteq I_\alpha \) or \( \bigcap_{I_\beta \in B} I_\beta \subseteq I_\alpha \) i.e. either \( I_\alpha \in \overline{A} \) or \( I_\alpha \in \overline{B} \) i.e. \( I_\alpha \in \overline{A \cup B} \). Consequently, \( \overline{A \cup B} \subseteq \overline{A} \cup \overline{B} \) and hence \( \overline{A \cup B} = \overline{A} \cup \overline{B} \). \( \square \)

**Definition 3.2** The closure operator \( A \rightarrow \overline{A} \) gives a topology \( \tau_A \) on \( A \). This topology \( \tau_A \) is called the hull-kernel topology and the topological space \((A, \tau_A)\) is called the structure space of the \( \Gamma \)-Semigroup \( S \).

Let \( I \) be a ideal of a \( \Gamma \)-Semigroup \( S \). We define

\[\Delta(I) = \{ I' \in A : I \subseteq I' \} \quad \text{and} \quad C\Delta(I) = A \setminus \Delta(I) = \{ I' \in A : I \nsubseteq I' \}.\]

Now we have the following result:

**Proposition 3.3** Any closed set in \( A \) is of the form \( \Delta(I) \), where \( I \) is a ideal of a \( \Gamma \)-Semigroup \( S \).

**Proof** Let \( \overline{A} \) be any closed set in \( A \), where \( A \subseteq A \). Let \( A = \{ I_\alpha : \alpha \in \Lambda \} \) and

\[
I = \bigcap_{I_\alpha \in A} I_\alpha.
\]

Then \( I \) is a ideal of \( S \). Let \( I' \in \overline{A} \). Then \( \bigcap_{I_\alpha \in A} I_\alpha \subseteq I' \). This implies that \( I \subseteq I' \). Consequently, \( I' \in \Delta(I) \). So \( \overline{A} \subseteq \Delta(I) \).

Conversely, let \( I' \in \Delta(I) \). Then \( I \subseteq I' \) i.e. \( \bigcap_{I_\alpha \in A} I_\alpha \subseteq I' \). Consequently, \( I' \in \overline{A} \) and hence \( \Delta(I) \subseteq \overline{A} \). Thus \( \overline{A} = \Delta(I) \). \( \square \)

**Corollary 3.4** Any open set in \( A \) is of the form \( C\Delta(I) \), where \( I \) is a ideal of \( S \).
Let $S$ be a $\Gamma$-Semigroup and $a \in S$. We define

$$\Delta(a) = \{I \in \mathcal{A}: a \in I\} \quad \text{and} \quad C\Delta(a) = \mathcal{A} \setminus \Delta(a) = \{I \in \mathcal{A}: a \notin I\}.$$ 

Then we have the following result:

**Proposition 3.5** \{C\Delta(a): a \in S\} forms an open base for the hull-kernel topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$.

**Proof** Let $U \in \tau_{\mathcal{A}}$. Then $U = C\Delta(I)$, where $I$ is an ideal of $S$. Let $J \in U = C\Delta(I)$. Then $I \not\subseteq J$. This implies that there exists $a \in I$ such that $a \notin J$. Thus $J \in C\Delta(a)$. Now it remains to show that $C\Delta(a) \subseteq U$. Let $K \in C\Delta(a)$. Then $a \notin K$. This implies that $I \not\subseteq K$. Consequently, $K \in U$ and hence $C\Delta(a) \subseteq U$. So we find that $J \in C\Delta(a) \subseteq U$. Thus \{C\Delta(a): a \in S\} is an open base for the hull-kernel topology $\tau_{\mathcal{A}}$ on $\mathcal{A}$.

**Theorem 3.6** The structure space $(\mathcal{A}, \tau_{\mathcal{A}})$ is a $T_0$-space.

**Proof** Let $I_1$ and $I_2$ be two distinct elements of $\mathcal{A}$. Then there is an element $a$ either in $I_1 \setminus I_2$ or in $I_2 \setminus I_1$. Suppose that $a \in I_1 \setminus I_2$. Then $C\Delta(a)$ is a neighbourhood of $I_2$ not containing $I_1$. Hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is a $T_0$-space.

**Theorem 3.7** $(\mathcal{A}, \tau_{\mathcal{A}})$ is a $T_1$-space if and only if no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.

**Proof** Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a $T_1$-space. Suppose that $I_1$ and $I_2$ be any two distinct elements of $\mathcal{A}$. Then each of $I_1$ and $I_2$ has a neighbourhood not containing the other. Since $I_1$ and $I_2$ are arbitrary elements of $\mathcal{A}$, it follows that no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$.

Conversely, suppose that no element of $\mathcal{A}$ is contained in any other element of $\mathcal{A}$. Let $I_1$ and $I_2$ be any two distinct elements of $\mathcal{A}$. Then by hypothesis, $I_1 \not\subseteq I_2$ and $I_2 \not\subseteq I_1$. This implies that there exist $a, b \in S$ such that $a \in I_1$ but $a \notin I_2$ and $b \in I_2$ but $b \notin I_1$. Consequently, we have $I_1 \in C\Delta(b)$ but $I_1 \notin C\Delta(a)$ and $I_2 \in C\Delta(a)$ but $I_2 \notin C\Delta(b)$ i.e. each of $I_1$ and $I_2$ has a neighbourhood not containing the other. Hence $(\mathcal{A}, \tau_{\mathcal{A}})$ is a $T_1$-space.

**Corollary 3.8** Let $\mathcal{M}$ be the set of all proper maximal ideals of a $\Gamma$-Semigroup $S$ with unities. Then $(\mathcal{M}, \tau_{\mathcal{M}})$ is a $T_1$-space, where $\tau_{\mathcal{M}}$ is the induced topology on $\mathcal{M}$ from $(\mathcal{A}, \tau_{\mathcal{A}})$.

**Theorem 3.9** $(\mathcal{A}, \tau_{\mathcal{A}})$ is a Hausdorff space if and only if for any two distinct pair of elements $I, J$ of $\mathcal{A}$, there exist $a, b \in S$ such that $a \notin I$, $b \notin J$ and there does not exist any element $K$ of $\mathcal{A}$ such that $a \notin K$ and $b \notin K$.

**Proof** Let $(\mathcal{A}, \tau_{\mathcal{A}})$ be a Hausdorff space. Then for any two distinct elements $I, J$ of $\mathcal{A}$, there exist basic open sets $C\Delta(a)$ and $C\Delta(b)$ such that $I \in C\Delta(a)$, $J \in C\Delta(b)$ and $C\Delta(a) \cap C\Delta(b) = \emptyset$. Now $I \in C\Delta(a)$ and $J \in C\Delta(b)$ imply
that \( a \notin I \) and \( b \notin J \). If possible, let \( K \) be any element of \( A \) such that \( a \notin K \) and \( b \notin K \). Then \( K \in C\Delta(a) \), \( K \in C\Delta(b) \) and hence \( K \in C\Delta(a) \cap C\Delta(b) \), a contradiction, since \( C\Delta(a) \cap C\Delta(b) = \emptyset \). Thus there does not exist any element \( K \) of \( A \) such that \( a \notin K \) and \( b \notin K \).

Conversely, suppose that the given condition holds and \( I, J \in A \) such that \( I \neq J \). Let \( a, b \in S \) be such that \( a \notin I \), \( b \notin J \) and there does not exist any \( K \) of \( A \) such that \( a \notin K \) and \( b \notin K \). Then \( I \in C\Delta(a) \), \( J \in C\Delta(b) \) and \( C\Delta(a) \cap C\Delta(b) = \emptyset \). This implies that \( (A, \tau_A) \) is a Hausdorff space.

**Corollary 3.10** If \((A, \tau_A)\) is a Hausdorff space, then no proper uniformly strongly prime ideal contains any other proper uniformly strongly prime ideal.

If \((A, \tau_A)\) contains more than one element, then there exist \( a, b \in S \) such that \( A = C\Delta(a) \cup C\Delta(b) \cup \Delta(I) \), where \( I \) is the ideal generated by \( a, b \).

**Proof** Suppose that \((A, \tau_A)\) is a Hausdorff space. Since every Hausdorff space is a \( T_1 \)-space, \((A, \tau_A)\) is a \( T_1 \)-space. Hence by Theorem 3.7, it follows that no proper uniformly strongly prime ideal contains any other proper uniformly strongly prime ideal. Now let \( J, K \in A \) be such that \( J \neq K \). Since \((A, \tau_A)\) is a Hausdorff space, there exist basic opens sets \( C\Delta(a) \) and \( C\Delta(b) \) such that \( J \in C\Delta(a) \), \( K \in C\Delta(b) \) and \( C\Delta(a) \cap C\Delta(b) = \emptyset \). Let \( I \) be the ideal generated by \( a, b \). Then \( I \) is the smallest ideal containing \( a \) and \( b \). Let \( K \in A \). Then either \( a \in K \), \( b \notin K \) or \( a \notin K \), \( b \in K \) or \( a, b \in K \). The case \( a \notin K \), \( b \notin K \) is not possible, since \( C\Delta(a) \cap C\Delta(b) = \emptyset \). Now in the first case, \( K \in C\Delta(b) \) and hence \( A \subseteq C\Delta(a) \cup C\Delta(b) \cup \Delta(I) \). In the second case, \( K \in C\Delta(a) \) and hence \( A \subseteq C\Delta(a) \cup C\Delta(b) \cup \Delta(I) \). In the third case, \( K \in \Delta(I) \) and hence \( A \subseteq C\Delta(a) \cup C\Delta(b) \cup \Delta(I) \). So we find that \( A \subseteq C\Delta(a) \cup C\Delta(b) \cup \Delta(I) \). Again, clearly \( C\Delta(a) \cup C\Delta(b) \cup \Delta(I) \subseteq A \). Hence \( A = C\Delta(a) \cup C\Delta(b) \cup \Delta(I) \).

**Theorem 3.11** \((A, \tau_A)\) is a regular space if and only if for any \( I \in A \) and \( a \notin I \), \( a \in S \), there exist an ideal \( J \) of \( S \) and \( b \in S \) such that \( I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\Delta(a) \).

**Proof** Let \((A, \tau_A)\) be a regular space. Let \( I \in A \) and \( a \notin I \). Then \( I \in C\Delta(a) \) and \( A \setminus C\Delta(a) \) is a closed set not containing \( I \). Since \((A, \tau_A)\) is a regular space, there exist disjoint open sets \( U \) and \( V \) such that \( I \in U \) and \( A \setminus C\Delta(a) \subseteq V \). This implies that \( A \setminus V \subseteq C\Delta(a) \). Since \( V \) is open, \( A \setminus V \) is closed and hence there exists an ideal \( J \) of \( S \) such that \( A \setminus V = \Delta(J) \), by Proposition 3.3. So we find that \( \Delta(J) \subseteq C\Delta(a) \). Again, since \( U \cap V = \emptyset \), we have \( V \subseteq A \setminus U \). Since \( U \) is open, \( A \setminus U \) is closed and hence there exists an ideal \( K \) of \( S \) such that \( A \setminus U = \Delta(K) \). Since \( I \in U \), \( I \notin A \setminus U = \Delta(K) \). This implies that \( K \notin I \). Thus there exists \( b \in K \) such that \( b \notin I \). So \( I \in C\Delta(b) \). Now we show that \( V \subseteq \Delta(b) \). Let \( M \in V \subseteq \Delta(K) \). Then \( K \subseteq M \). Since \( b \in K \), it follows that \( b \in M \) and hence \( M \in \Delta(b) \). Consequently, \( V \subseteq \Delta(b) \). This implies that \( A \setminus \Delta(b) \subseteq A \setminus V = \Delta(J) \implies C\Delta(b) \subseteq \Delta(J) \). Thus we find that \( I \in C\Delta(b) \subseteq \Delta(J) \subseteq C\Delta(a) \).
Conversely, suppose that the given condition holds. Let \( I \in \mathcal{A} \) and \( \Delta(K) \) be any closed set not containing \( I \). Since \( I \notin \Delta(K) \), we have \( K \nsubseteq I \). This implies that there exists \( a \in K \) such that \( a \notin I \). Now by the given condition, there exists an ideal \( J \) of \( S \) and \( b \in S \) such that \( I \subseteq C \Delta(b) \subseteq \Delta(J) \subseteq C \Delta(a) \). Since \( a \in K \), \( C \Delta(a) \cap \Delta(K) = \emptyset \). This implies that \( \Delta(K) \subseteq A \setminus C \Delta(a) \subseteq A \setminus \Delta(J) \). Since \( \Delta(J) \) is a closed set, \( A \setminus \Delta(J) \) is an open set containing the closed set \( \Delta(K) \). Clearly, \( C \Delta(b) \cap (A \setminus \Delta(J)) = \emptyset \). So we find that \( C \Delta(b) \) and \( A \setminus \Delta(J) \) are two disjoints open sets containing \( I \) and \( \Delta(K) \) respectively. Consequently, \((\mathcal{A}, \tau_{\mathcal{A}})\) is a regular space.

**Theorem 3.12** \((\mathcal{A}, \tau_{\mathcal{A}})\) is a compact space if and only if for any collection \( \{a_\alpha\}_{\alpha \in \Lambda} \subset S \) there exists a finite subcollection \( \{a_i: i = 1, 2, \ldots, n\} \) in \( S \) such that for any \( I \in \mathcal{A} \), there exists \( a_i \) such that \( a_i \notin I \).

**Proof** Let \((\mathcal{A}, \tau_{\mathcal{A}})\) be a compact space. Then the open cover \( \{C \Delta(a_\alpha): a_\alpha \in S\} \) of \((\mathcal{A}, \tau_{\mathcal{A}})\) has a finite subcover \( \{C \Delta(a_i): i = 1, 2, \ldots, n\} \). Let \( I \) be any element of \( \mathcal{A} \). Then \( I \subseteq C \Delta(a_i) \) for some \( a_i \in S \). This implies that \( a_i \notin I \). Hence \( \{a_i: i = 1, 2, \ldots, n\} \) is the required finite subcollection of elements of \( S \) such that for any \( I \in \mathcal{A} \), there exists \( a_i \) such that \( a_i \notin I \).

Conversely, suppose that the given condition holds. Let \( \{C \Delta(a_\alpha): a_\alpha \in S\} \) be an open cover of \( \mathcal{A} \). Suppose to the contrary that no finite subcollection of \( \{C \Delta(a_\alpha): a_\alpha \in S\} \) covers \( \mathcal{A} \). This means that for any finite set \( \{a_1, a_2, \ldots, a_n\} \) of elements of \( S \),

\[
C \Delta(a_1) \cup C \Delta(a_2) \cup \ldots \cup C \Delta(a_n) \neq \mathcal{A}
\]

\[
\Rightarrow \quad \Delta(a_1) \cap \Delta(a_2) \cap \ldots \cap \Delta(a_n) \neq \emptyset
\]

\[
\Rightarrow \quad \text{there exists } I \in \mathcal{A} \text{ such that } I \subseteq \Delta(a_1) \cap \Delta(a_2) \cap \ldots \cap \Delta(a_n)
\]

\[
\Rightarrow \quad a_1, a_2, \ldots, a_n \in I, \text{ which contradicts our hypothesis .}
\]

So the open cover \( \{C \Delta(a_\alpha): a_\alpha \in S\} \) has a finite subcover and hence \((\mathcal{A}, \tau_{\mathcal{A}})\) is compact.

**Corollary** If \( S \) is finitely generated, then \((\mathcal{A}, \tau_{\mathcal{A}})\) is a compact space.

**Proof** Let \( \{a_i: i = 1, 2, \ldots, n\} \) be a finite set of generators of \( S \). Then for any \( I \in \mathcal{A} \), there exists \( a_i \) such that \( a_i \notin I \), since \( I \) is a proper uniformly strongly prime ideal of \( S \). Hence by Theorem 3.12, \((\mathcal{A}, \tau_{\mathcal{A}})\) is a compact space.

**Definition 3.14** A \( \Gamma \)-Semigroup \( S \) is called a Noetherian \( \Gamma \)-Semigroup if it satisfies the ascending chain condition on ideals of \( S \) i.e. if \( I_1 \subseteq I_2 \subseteq \ldots \subseteq I_n \subseteq \ldots \) is an ascending chain of ideals of \( S \), then there exists a positive integer \( m \) such that \( I_n = I_m \) for all \( n \geq m \).

**Theorem 3.15** If \( S \) is a Noetherian \( \Gamma \)-Semigroup, then \((\mathcal{A}, \tau_{\mathcal{A}})\) is countably compact.

**Proof** Let \( \{\Delta(I_n)\}_{n=1}^\infty \) be a countable collection of closed sets in \( \mathcal{A} \) with finite intersection property (FIP). Let us consider the following ascending chain of prime ideals of \( S \):

\[
< I_1 > \subseteq < I_1 \cup I_2 > \subseteq < I_1 \cup I_2 \cup I_3 > \subseteq \ldots
\]
Since $S$ is a Noetherian $\Gamma$-Semigroup, there exists a positive integer $m$ such that $<I_1 \cup I_2 \cup \ldots \cup I_m> = <I_1 \cup I_2 \cup \ldots \cup I_{m+1}> = \ldots$

Thus it follows that $<I_1 \cup I_2 \cup \ldots \cup I_m> \in \bigcap_{n=1}^{\infty} \Delta(I_n)$. Consequently, $\bigcap_{n=1}^{\infty} \Delta(I_n) \neq \emptyset$ and hence $(A, \tau_A)$ is countably compact.

\begin{corollary}
If $S$ is a Noetherian $\Gamma$-Semigroup and $(A, \tau_A)$ is second countable, then $(A, \tau_A)$ is compact.
\end{corollary}

\begin{proof}
Proof follows from Theorem 3.15 and the fact that a second countable space is compact if it is countably compact.
\end{proof}

\begin{remark}
Let $\{I_\alpha\}$ be a collection of prime ideals of a $\Gamma$-semigroup $S$. Then $\bigcap I_\alpha$ is an ideal of $S$ but it may not be a prime ideal of $S$, in general.

However; in particular, we have the following result:

\begin{proposition}
Let $\{I_\alpha\}$ be a collection of prime ideals of a $\Gamma$-semigroup $S$ such that $\{I_\alpha\}$ forms a chain. Then $\bigcap I_\alpha$ is a prime ideal of $S$.
\end{proposition}

\begin{proof}
Clearly, $\bigcap I_\alpha$ is an ideal of $S$. Let $A \Gamma B \subsetneq \bigcap I_\alpha$ for any two ideals $A, B$ of $S$. If possible, let $A, B \not\subseteq \bigcap I_\alpha$. Then there exist $\alpha$ and $\beta$ such that $A \not\subseteq I_\alpha$ and $B \not\subseteq I_\beta$. Since $I_\alpha$ is a chain, let $I_\alpha \subseteq I_\beta$. This implies that $B \not\subseteq I_\alpha$. Since $A \Gamma B \subsetneq \bigcap I_\alpha$ and $I_\alpha$ is prime, we must have either $A \subseteq I_\alpha$ or $B \subseteq I_\alpha$, a contradiction. Therefore, either $A \subseteq \bigcap I_\alpha$ or $B \subseteq \bigcap I_\alpha$. Consequently, $\bigcap I_\alpha$ is a prime ideal of $S$.
\end{proof}

\begin{definition}
The structure space $(A, \tau_A)$ is called irreducible if for any decomposition $A = A_1 \cup A_2$, where $A_1$ and $A_2$ are closed subsets of $A$, we have either $A = A_1$ or $A = A_2$.

\begin{theorem}
Let $A$ be a closed subset of $A$. Then $A$ is irreducible if and only if $\bigcap_{I_\alpha \in A} I_\alpha$ is a prime ideal of $S$.
\end{theorem}

\begin{proof}
Let $A$ be irreducible. Let $P$ and $Q$ be two ideals of $S$ such that $P \Gamma Q \subsetneq \bigcap_{I_\alpha \in A} I_\alpha$. Then $P \Gamma Q \subsetneq I_\alpha$ for all $\alpha$. Since $I_\alpha$ is prime, either $P \not\subseteq I_\alpha$ or $Q \not\subseteq I_\alpha$ which implies for $I_\alpha \in A$ either $I_\alpha \in \{P\}$ or $I_\alpha \in \{Q\}$. Hence $A = (A \cap P) \cup (A \cap Q)$. Since $A$ is irreducible and $(A \cap P), (A \cap Q)$ are closed, it follows that $A = A \cap P$ or $A = A \cap Q$ and hence $A \subseteq P$ or $A \subseteq Q$. This implies that $P \subseteq \bigcap_{I_\alpha \in A} I_\alpha$ or $Q \subseteq \bigcap_{I_\alpha \in A} I_\alpha$. Consequently, $\bigcap_{I_\alpha \in A} I_\alpha$ is a prime ideal of $S$.

Conversely, suppose that $\bigcap_{I_\alpha \in A} I_\alpha$ is a prime ideal of $S$. Let $A = A_1 \cup A_2$, where $A_1$ and $A_2$ are closed subsets of $A$. Then $\bigcap_{I_\alpha \in A} I_\alpha \subseteq \bigcap_{I_\alpha \in A_1} I_\alpha$ and $\bigcap_{I_\alpha \in A} I_\alpha \subseteq \bigcap_{I_\alpha \in A_2} I_\alpha$. Also

$$\bigcap_{I_\alpha \in A} I_\alpha = \bigcap_{I_\alpha \in A_1 \cup A_2} I_\alpha = \left( \bigcap_{I_\alpha \in A_1} I_\alpha \right) \cap \left( \bigcap_{I_\alpha \in A_2} I_\alpha \right).$$
Now
\[
\left( \bigcap_{I_a \in A_1} I_a \right)^\Gamma \left( \bigcap_{I_a \in A_2} I_a \right) \subseteq \left( \bigcap_{I_a \in A_1} I_a \right) \quad \text{and} \quad \left( \bigcap_{I_a \in A_1} I_a \right)^\Gamma \left( \bigcap_{I_a \in A_2} I_a \right) \subseteq \left( \bigcap_{I_a \in A_2} I_a \right).
\]

Thus we have
\[
\left( \bigcap_{I_a \in A_1} I_a \right)^\Gamma \left( \bigcap_{I_a \in A_2} I_a \right) \subseteq \left( \bigcap_{I_a \in A_1} I_a \right) \cap \left( \bigcap_{I_a \in A_2} I_a \right).
\]

Since \( \bigcap_{I_a \in A} I_a \) is prime, it follows that either
\[
\bigcap_{I_a \in A_1} I_a \subseteq \bigcap_{I_a \in A} I_a \quad \text{or} \quad \bigcap_{I_a \in A_2} I_a \subseteq \bigcap_{I_a \in A} I_a.
\]

So we find that
\[
\bigcap_{I_a \in A} I_a = \bigcap_{I_a \in A_1} I_a \quad \text{or} \quad \bigcap_{I_a \in A} I_a = \bigcap_{I_a \in A_2} I_a.
\]

Let \( I_\beta \in A \). Then we have
\[
\bigcap_{I_a \in A_1} I_a \subseteq I_\beta \quad \text{or} \quad \bigcap_{I_a \in A_2} I_a \subseteq I_\beta.
\]

Since \( A_1, A_2 \subseteq A \), so either \( I_a \subseteq I_\beta \) for all \( I_a \in A_1 \) or \( I_a \subseteq I_\beta \) for all \( I_a \in A_2 \). Thus \( I_\beta \in A_1 \) or \( I_\beta \in A_2 \), since \( A_1 \) and \( A_2 \) are closed. i.e. \( A = A_1 \) or \( A = A_2 \).

Let \( C \) be the collection of all uniformly strongly prime full ideals of a \( \Gamma \)-semigroup \( S \). Then we see that \( C \) is a subset of \( A \) and hence \( (C, \tau_C) \) is a topological space, where \( \tau_C \) is the subspace topology.

In general, \((A, \tau_A)\) is not compact and connected. But in particular, for the topological space \((C, \tau_C)\), we have the following results:

**Theorem 3.21** \((C, \tau_C)\) is a compact space.

**Proof** Let \( \{\Delta(I_\alpha)\} : \alpha \in A \) be any collection of closed sets in \( C \) with finite intersection property. Let \( I \) be the uniformly strongly prime full ideal generated by \( E(S) \). Since any uniformly strongly prime full ideal \( J \) contains \( E(S) \), \( J \) contains \( I \). Hence \( I \in \bigcap_{\alpha \in A} \Delta(I_\alpha) \neq \emptyset \). Consequently, \((C, \tau_C)\) is a compact space. \( \square \)

**Theorem 3.22** \((C, \tau_C)\) is a connected space.

**Proof** Let \( I \) be the uniformly strongly prime ideal generated by \( E(S) \). Since any uniformly strongly prime full ideal \( J \) contains \( E(S) \), \( J \) contains \( I \). Hence \( I \) belongs to any closed set \( \Delta(I') \) of \( C \). Consequently, any two closed sets of \( C \) are not disjoint. Hence \((C, \tau_C)\) is a connected space. \( \square \)
References