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Varieties Satisfying the Triangular Scheme Need Not Be Congruence Distributive*

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Abstract

A diagrammatic scheme characterizing congruence distributivity of congruence permutable algebras was introduced by the first author in 2001. It is known under the name Triangular Scheme. It is known that every congruence distributive algebra satisfies this scheme and an algebra satisfying the Triangular Scheme which is not congruence distributive was found by E. K. Horváth, G. Czédli and the autor in 2003. On the other hand, it was an open problem if a variety of algebras satisfying the Triangular Scheme must be congruence distributive. We get a negative solution by presenting an example.

Key words: Congruence distributivity; Triangular Scheme, variety of algebras; Jónsson terms.

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Congruence distributive varieties were characterized by B. Jónsson [7] by means of the Maltsev condition. For the reader's convenience, we can repeat this result:

Proposition 1 *A variety $\mathcal{V}$ is congruence distributive if and only if there exist ternary terms $t_0, \dots, t_n$ such that $t_0(x,y,z) = x$, $t_n(x,y,z) = z$ and*

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(a) *for all* $i = 0, \dots, n$ *it holds* $t_i(x, y, x) = x$

(b) *for* i *even,* $t_i(x, x, y) = t_{i+1}(x, x, y)$

(c) *for* i *odd,* $t_i(x, y, y) = t_{i+1}(x, y, y)$.

These terms $t_0, \dots, t_n$ are referred to be *Jónsson terms.*

Unfortunately, a similar characterization of congruence distributivity for a single algebra is missing. It motivated us to introduce the following concept (see [1], [4]).

Let L be a sublattice of an equivalence lattice (known also as a partition lattice) on a non-void set A. We say that L *satisfies the Triangular Scheme* if for each $\alpha, \beta, \gamma \in L$ with $\alpha \cap \beta \subseteq \gamma$ and for $x, y, z \in A$ such that $\langle x, y\rangle \in \gamma$, $\langle x, z\rangle \in \alpha$, $\langle z, y\rangle \in \beta$ we have $\langle z, y\rangle \in \gamma$.

This can be visualized as follows

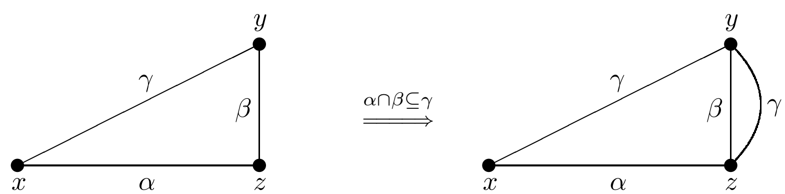

Fig. 1

We say that an algebra $\mathcal{A}$ satisfies the Triangular Scheme if the congruence lattice $\mathrm{Con}\,\mathcal{A}$ satisfies this condition. A variety $\mathcal{V}$ fulfils the Triangular Scheme if each $\mathcal{A} \in \mathcal{V}$ has this property.

The following was proved in [1], [4].

Proposition 2 *If an algebra is congruence distributive then it satisfies the Triangular Scheme. If an algebra is congruence permutable then it is congruence distributive if and only if it satisfies the Triangular Scheme.*

An example of algebra satisfying the Triangular Scheme but which is not congruence distributive was found in [5].

Let us note that similar schemes for congruence semidistributivity were involved in [3] and conclusions of the Triangular Scheme for n-permutable algebras were treated in [2], [3], [5]. For congruence modular algebras and varieties it was done in [5] where it is explicitely proved that for a variety, the assumption of congruence permutability of Proposition 2 can be replaced by a weaker one of congruence modularity. However, there still was an open question if a variety satisfying the Triangular Scheme is necessarily congruence distributive. To solve this question, we first characterize the Triangular Scheme for varieties by a Maltsev condition.

Theorem 1 *Let* $\mathcal{V}$ *be a variety of algebras. The following are equivalent:*

(1) *For each* $\mathcal{A} \in \mathcal{V}$, $\mathrm{Con}\,\mathcal{A}$ *satisfies the Triangular Scheme;*

(2) *there exist ternary terms* $t_0, \ldots, t_n$ *such that* $t_0(x,y,z) = x$, $t_n(x,y,z) = z$ *and*

   (a) *for* i *even,* $t_i(x,y,x) = t_{i+1}(x,y,x)$, $t_i(x,x,y) = t_{i+1}(x,x,y)$,

   (b) *for* i *odd,* $t_i(x,y,y) = t_{i+1}(x,y,y)$.

Proof Suppose $\mathcal{V}$ satisfies the Triangular Scheme, $\mathcal{F}_{\mathcal{V}}(x,y,z)$ is a free algebra of $\mathcal{V}$ with three free generators and $\alpha = \theta(x,y)$, $\beta = \theta(x,z)$ and $\gamma = (\alpha \cap \beta) \vee \theta(y,z)$. Then $\alpha \cap \beta \subseteq \gamma$ and, by Triangular Scheme, $\langle x, z \rangle \in \gamma$. Hence, there exists an integer $n \geq 0$ and ternary terms $t_0, \ldots, t_n$ such that

$$x = t_0(\alpha \cap \beta)t_1\theta(y,z)t_2(\alpha \cap \beta)t_3 \ldots t_n = z.$$

Applying the standard procedure, we easily derive that $t_0(x,y,z) = x$, $t_n(x,y,z) = z$ and $t_i(x,y,x) = t_{i+1}(x,y,x)$ and $t_i(x,x,y) = t_{i+1}(x,x,y)$ for i even, and $t_i(x,y,y) = t_{i+1}(x,y,y)$ for i odd.

Prove the converse. Let $\mathcal{A} = (A,F) \in \mathcal{V}$, $a,b,c \in A$, $\alpha, \beta, \gamma \in \operatorname{Con} \mathcal{A}$ and $\alpha \cap \beta \subseteq \gamma$. Suppose $\langle c, b \rangle \in \gamma$, $\langle a, b \rangle \in \beta$ and $\langle a, c \rangle \in \alpha$. Then

$$t_i(a,b,c)\alpha t_i(a,b,a) = t_{i+1}(a,b,a)\alpha t_{i+1}(a,b,c)$$

$$t_i(a,b,c)\beta t_i(a,a,c) = t_{i+1}(a,a,c)\beta t_{i+1}(a,b,c)$$

for i even and

$$t_i(a,b,c)\gamma t_i(a,b,b) = t_{i+1}(a,b,b)\gamma t_{i+1}(a,b,c)$$

for i odd. Altogether, we conclude

$$a = t_0(a,b,c)(\alpha \cap \beta)t_1(a,b,c)\gamma t_2(a,b,c)(\alpha \cap \beta) \ldots t_n(a,b,c) = c$$

thus

$$\langle a, c \rangle \in (\alpha \cap \beta) \circ \gamma \circ (\alpha \cap \beta) \circ \gamma \circ \ldots \subseteq \gamma \circ \gamma \circ \ldots \circ \gamma = \gamma.$$

This together with $\langle b, c \rangle \in \gamma$ yields $\langle a, b \rangle \in \gamma$. Hence, $\mathcal{A}$ and also $\mathcal{V}$ satisfies the Triangular Scheme. □

Remark 1 When comparing our terms of Theorem 2 with Jónsson terms, the difference is that we do not ask $t_i(x,y,x) = t_{i+1}(x,y,x)$ for i odd. It motivates us to suppose that this variety need not be necessarily congruence distributive. However, if $n \leq 3$ then $t_0(x,y,x) = x$ and $t_3(x,y,x) = x$ yield that also $t_1(x,y,x) = x$ and $t_2(x,y,x) = x$. To find an example of a variety which is not congruence distributive but still satisfying the Triangular Scheme, we must suppose that $n \geq 4$. We are ready to construct such an example:

Example 1 Consider a variety $\mathcal{V}$ of type $(2,1,1)$ whose operations are denoted by $\wedge$ and f, g and satisfying the identities

$$x \wedge x = x, \quad x \wedge y = y \wedge x, \quad x \wedge (y \wedge z) = (x \wedge y) \wedge z$$

(i.e. the $\wedge$-reducts of its members are semilattices) and

$$f(f(x)) = x$$

$$x \wedge g(g(x) \wedge g(y)) = x$$

$$x \wedge g(g(y)) = x \wedge f(f(x) \wedge f(y)).$$

Hence it follows also

$$x \wedge g(g(x)) = x.$$

We can take $n = 6$ and establish the following terms:

$$t_0(x,y,z) = x$$

$$t_1(x,y,z) = x \wedge g(g(y) \wedge g(z))$$
$$t_2(x,y,z) = x \wedge g(g(y)) \wedge f(f(x) \wedge f(z))$$
$$t_3(x,y,z) = x \wedge g(g(y)) \wedge f(f(y) \wedge f(z))$$
$$t_4(x,y,z) = x \wedge y \wedge z$$
$$t_5(x,y,z) = y \wedge z$$
$$t_6(x,y,z) = z.$$

Then for i even we have
$i = 0$:

$$t_0(x,x,y) = x = x \wedge g(g(x) \wedge g(y)) = t_1(x,x,y)$$
$$t_0(x,y,x) = x = x \wedge g(g(y) \wedge g(x)) = t_1(x,y,x)$$

$i = 2$:

$$t_2(x,x,y) = x \wedge g(g(x)) \wedge f(f(x) \wedge f(y)) = t_3(x,x,y)$$

$$t_2(x,y,x) = x \wedge g(g(y)) \wedge f(f(x)) = x \wedge g(g(y)) =$$
$$= x \wedge g(g(y)) \wedge f(f(y) \wedge f(x)) = t_3(x,y,x)$$

$i = 4$:

$$t_4(x,x,y) = x \wedge y = t_5(x,x,y)$$
$$t_4(x,y,x) = x \wedge y = t_5(x,y,x).$$

For i odd we have
$i = 1$:

$$t_1(x,y,y) = x \wedge g(g(y)) = x \wedge g(g(y)) \wedge f(f(x) \wedge f(y)) = t_2(x,y,y)$$

$i = 3$:

$$t_3(x,y,y) = x \wedge g(g(y)) \wedge f(f(y)) = x \wedge y \wedge g(g(y)) = x \wedge y = t_4(x,y,y)$$

$i = 5$:

$$t_5(x,y,y) = y \wedge y = y = t_6(x,y,y).$$

We have shown that our variety $\mathcal{V}$ satisfies the Triangular Scheme. Consider now a four element $\wedge$-semilattice as drawn in Fig. 2 where f and g

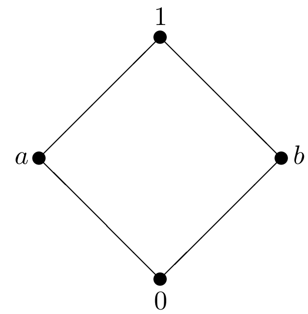

Fig. 2

are determined by the table

x	$f(x)$	$g(x)$
0	a	1
a	0	1
b	1	a
1	b	a

It is an easy exercise to check that $\mathcal{A} = (\{0, a, b, 1\}; \wedge, f, g) \in \mathcal{V}$. Consider the partitions:

$\alpha = \{0, a\}$

$\beta = \{0, a\}, \{b, 1\}$

$\gamma = \{0, b\}, \{a, 1\}$.

Then apparently $\operatorname{Con} \mathcal{A} = \{\omega, \alpha, \beta, \gamma, A \times A\}$ as shown in Fig. 3 thus $\mathcal{A}$ is not congruence distributive.

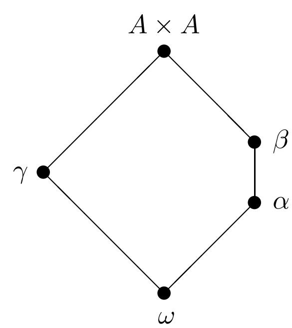

Fig. 3

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