Volterra-Prabhakar function of distributed order and Applications

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Overview

Definitions

Volterra's function is defined as follows [1,2]

$$\mu(t,\beta,\alpha) = \frac{1}{\Gamma(1+\beta)} \int_0^\infty \frac{t^{u+\alpha}\, u^\beta}{\Gamma(u+\alpha+1)} du, \quad \beta > -1 \quad \text{and} \quad t > 0, \tag{1}$$

whose particular cases are

$$\begin{aligned} \alpha = \beta = 0: &\qquad \nu(t) = \mu(t,0,0), \\ \alpha \neq 0,\ \beta = 0: &\qquad \nu(t,\alpha) = \mu(t,0,\alpha), \\ \alpha = 0,\ \beta \neq 0: &\qquad \mu(t,\beta) = \mu(t,\beta,0). \end{aligned}$$

The Laplace transform of the Volterra's function $\mu(t,\beta,\alpha)$ is given by

$$L[\mu(t,\beta,\alpha); s] = \frac{1}{s^{\alpha+1} \log^{\beta+1} s}. \tag{2}$$

Volterra integral equation with logarithmic kernel

V.Volterra (1916) [3], Garrapa-Mainardi (2016) [4]

$$\int_0^t u(\tau)\log(t-\tau)d\tau = f(t),\ f(0)=0 \tag{3}$$

$$f(t)=t \implies u(t)=-\nu(te^{-\gamma}) \tag{4}$$

Ramanujan identity, Hardy (1940), Twelve lectures on subjects suggested by his life and work,Cambridge University Press, Cambridge, England, New York.

$$\nu(t)=e^t-\int_0^\infty \frac{e^{-rt}}{r(\log^2 r+\pi^2)}dr \tag{5}$$

$$\int_0^\infty \frac{1}{r(\log^2 r+\pi^2)}dr=1 \tag{6}$$

Definitions

$$e^{\gamma}_{\alpha,\beta}(\lambda; t) = t^{\beta-1} E^{\gamma}_{\alpha,\beta}(-\lambda t^{\alpha}), \tag{7}$$

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{r=0}^{\infty} \frac{(\gamma)_r z^r}{r!\Gamma(\beta + \alpha r)} \tag{8}$$

$$\frac{d^n}{dt^n} e^{\gamma}_{\alpha,\beta}(\lambda; t) = e^{\gamma}_{\alpha,\beta-n}(\lambda; t) \tag{9}$$

$$\epsilon^{\gamma}_{\alpha,\, p}(\lambda; t) = \int_0^{\infty} e^{\gamma}_{\alpha,u+p+1}(\lambda; t) du \tag{10}$$

Volterra-Prabhakar function.

- $\gamma = 0$ and/or $\lambda = 0$ reduces to $\nu(t, p)$.

$$\frac{d^n}{dt^n}[\epsilon^{\gamma}_{\alpha,\, kn}(\lambda; t)] = \epsilon^{\gamma}_{\alpha,\, (k-1)n}(\lambda; t), \quad k \in \mathbb{N}.$$

Properties

For $t > 0$, $\alpha > 0$, and $p \in \mathbb{R}$ we have

$$\int_0^t \xi^{\alpha-1}\nu(t-\xi,p)d\xi = \Gamma(\alpha)\,\nu(t,\alpha+p) \qquad (11)$$

$$\int_0^t \xi^{\alpha-1}\nu(t-\xi,-\alpha)d\xi = \Gamma(\alpha)\,\nu(t).$$

For $t > 0$, $\alpha \in (0,1)$, $\gamma \in \mathbb{R}$, and $p \in \mathbb{R}$ we get

$$\int_0^t \nu(t-\xi,p)\,e^{\gamma}_{\alpha,0}(\lambda;\xi)d\xi = \sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}(\gamma)_n\,\nu(t,\alpha n+p), \qquad (12)$$

$$\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{n!}(\gamma)_n\,\nu(t,\alpha n+p) = \int_0^{\infty} e^{\gamma}_{\alpha,u+p+1}(t;\lambda)du \qquad (13)$$

Convolution properties

$$L\left[\epsilon^{\gamma}_{\alpha,\,p}(\lambda;t);s\right]=\frac{s^{\alpha\gamma-p-1}}{(s^{\alpha}+\lambda)^{\gamma}\log s}=L\left[e^{\gamma}_{\alpha,\,p}(\lambda;t)\star\nu(t);s\right]. \qquad (14)$$

$$L\left[\epsilon^{\gamma}_{\alpha,\,p}(\lambda;t);s\right]L\left[\mu(t,\beta-1,\alpha-1);s\right]=L\left[e^{\gamma}_{\alpha,\,p}(\lambda;t)\right];s\right]L\left[\mu(t,\beta,\alpha);s\right],$$

$$\epsilon^{\gamma}_{\alpha,\,p}(\lambda;t)\star\mu(t,\beta-1,\alpha-1)=e^{\gamma}_{\alpha,\,p}(\lambda;t)\star\mu(t,\beta,\alpha).$$

$$L\left[\epsilon^{\gamma}_{\alpha,\,p}(\lambda;t);s\right]L\left[\epsilon^{\gamma'}_{\alpha,\,p'}(\lambda;t);s\right]=L\left[\epsilon^{\gamma+\gamma'}_{\alpha,\,p+p'}(\lambda;t);s\right]L\left[\nu(t);s\right],$$

convolution semigroup property

$$\epsilon^{\gamma}_{\alpha,\,p}(\lambda;t)\star\epsilon^{\gamma'}_{\alpha,\,p'}(\lambda;t)=\epsilon^{\gamma+\gamma'}_{\alpha,\,p+p'}(\lambda;t)\star\nu(t).$$

$$\epsilon^{\gamma}_{\alpha,\,p}(\lambda;t)\star\epsilon^{-\gamma}_{\alpha,\,-p}(\lambda;t)=\mu(t,1,1).$$

Integral representations

$$\epsilon^{\gamma}_{\alpha,\,p}(1;t) \equiv \epsilon^{\gamma}_{\alpha,\,p}(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{s^{\alpha\gamma-p}}{(s^{\alpha}+1)^{\gamma} s\log s}ds = \frac{e^{t}}{2^{\gamma}} + f^{\gamma}_{\alpha,p}(t),$$

$$f^{\gamma}_{\alpha,p}(t) = \frac{1}{2\pi i}\int_{H} e^{st}\frac{s^{\alpha\gamma-p}}{(s^{\alpha}+1)^{\gamma} s\log s}ds = L\left[K^{\gamma}_{\alpha,\,p}(r);t\right].$$

$$\begin{aligned} K^{\gamma}_{\alpha,p}(r) &= -\frac{1}{\pi}\Im\left\{\frac{r^{\alpha\gamma-p}e^{i\pi(\alpha\gamma-p)}}{(r^{\alpha}e^{i\pi\alpha}+1)^{\gamma}(re^{i\pi})\log(re^{i\pi})}\right\} \\ &= -\frac{r^{\alpha\gamma-p-1}}{\pi}\Im\left\{\frac{e^{i\pi(\alpha\gamma-p-1)}}{(r^{\alpha}e^{i\pi\alpha}+1)^{\gamma}(i\pi+\log r)}\right\} \end{aligned} \tag{15}$$

$$= \frac{r^{\alpha\gamma-p-1}}{\pi}\frac{(\ln r)\sin[\pi(\alpha\gamma-p)-\gamma\theta_{\alpha}(r)] - \pi\cos[\pi(\alpha\gamma-p)-\gamma\theta_{\alpha}(r)]}{[r^{2\alpha}+2r^{\alpha}\cos(\pi\alpha)+1]^{\gamma/2}(\pi^{2}+\log^{2} r)}.$$

Integral representations

$$\epsilon_{\alpha,\,p}^{\gamma}(t) = \frac{e^t}{2^\gamma} - \int_0^\infty e^{-rt} \tilde{K}_{\alpha,p}^{\gamma}(r)dr, \tag{16}$$

$$\widetilde{K}_{\alpha,p}^{\gamma}(r) = -K_{\alpha,p}^{\gamma}(r). \tag{17}$$

$$\int_0^\infty \widetilde{K}_{\alpha,p}^{\gamma}(r)dr = \frac{1}{2^\gamma}.$$

$$\nu(t,p) = \epsilon_{\alpha,p}^{0}(t) = e^t - \frac{1}{\pi}\int_0^\infty \frac{e^{-rt}}{r^{p+1}} \frac{(\log r)\sin(\pi p) + \pi\cos(\pi p)}{\pi^2 + \log^2 r} dr.$$

$$\int_0^\infty \frac{1}{\pi} r^{p+1} \frac{(\ln r)\sin(\pi p) + \pi\cos(\pi p)}{\pi^2 + \log^2 r} dr = 1.$$

$$\nu(t) = \epsilon_{\alpha,0}^{0}(t) = e^t - \int_0^\infty \frac{e^{-rt}}{r} \frac{1}{\pi^2 + \log^2 r} dr.$$

$$\int_0^\infty \frac{1}{r} \frac{1}{\pi^2 + \log^2 r} dr = 1.$$

Complete monotonicity

Theorem

Let $\gamma > 0$. The following assertions hold true:

(a) $\epsilon^{\gamma}_{\alpha,p}(t) - \frac{e^t}{2^\gamma}$ is CM on $(0,\infty)$ for $\alpha \in (0,1/2]$ and $\alpha\gamma - p = 2k$, $k \in \mathbb{Z}$;

(b) $\epsilon^{\gamma}_{\alpha,\alpha\gamma}(t) - \frac{e^t}{2^\gamma}$ is CM on $(0,\infty)$ for $\alpha \in (0,1/2]$;

(c) $\epsilon^{\gamma}_{\alpha,0}(t) - \frac{e^t}{2^\gamma}$ is CM on $(0,1)$ for $\alpha \in (1/2,1]$.

Theorem

The following assertions hold true:

(a) The function $e^t - \nu(t,p)$ is CM and log-convex on $(0,\infty)$, for $p \in [0,1/2]$;

(b) The function $e^t - \nu(t)$ is CM and log-convex on $(0,\infty)$.

Further generalizations

$$\epsilon^{\gamma}_{\alpha,\,\beta,\,p}(\lambda; t) = \int_0^{\infty} u^{\beta} e^{\gamma}_{\alpha, u+p+1}(\lambda; t)du,$$

In particular, $\epsilon^{\gamma}_{\alpha,\,0,\,p}(\lambda; t) = \epsilon^{\gamma}_{\alpha,\,p}(\lambda; t)$.

Theorem

For $t > 0$, $\alpha, \beta > 0$, $\lambda, \gamma \in \mathbb{R}$ we have

(a)

$$\int_0^t \mu(t-\xi, \beta, \alpha)\, e^{\gamma}_{\alpha,\beta}(\lambda; \xi)d\xi = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}(\gamma)_n\, \mu(t, \beta, \alpha n + \alpha + \beta),$$

(b)

$$\epsilon^{\gamma}_{\alpha,\,\beta,\,\alpha+\beta}(\lambda; t) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!}(\gamma)_n\, \mu(t, \beta, \alpha n + \alpha + \beta).$$

Theorem

The Laplace transform of $\epsilon^{\gamma}_{\alpha,\,\beta,\,p}(\lambda; t)$ for $\beta > 0$ reads

$$L[\epsilon^{\gamma}_{\alpha,\,\beta,\,p}(\lambda; t); s] = \Gamma(1+\beta)\frac{s^{\alpha\gamma-p-1}}{(s^{\alpha}+\lambda)^{\gamma}\,(\log s)^{1+\beta}}.$$

Furthermore, the following convolution relation holds true,

$$\epsilon^{\gamma}_{\alpha,\,\beta,\,p}(\lambda; t) = e^{\gamma}_{\alpha,p}(\lambda; t) \star \mu(t, \beta, 0).$$

Theorem

For $\alpha > 0,\ 0 < \beta < 1,\ 0 < s < \gamma \leq 1-\beta,\ p \in \mathbb{R}$ the Mellin transform of $\epsilon^{\gamma}_{\alpha,\,\beta,\,p}(t)$ is given by

$$M[\epsilon^{\gamma}_{\alpha,\,\beta,\,p}(t); s]$$
$$= \frac{-\alpha^{\beta}\pi}{[\sin(\pi\beta)]\,\Gamma(1-s)}\sum_{k=0}^{\infty}\binom{-\gamma}{k}[(\frac{p+s}{\alpha}+k)^{\beta} - (-\gamma+\frac{p+s}{\alpha}-k)^{\beta}].$$

Langeven equation

$$\ddot{x}(t) + \int_0^t \gamma(t-t')\dot{x}(t')\mathrm{d}t' = \xi(t), \quad \dot{x}(t) = v(t). \tag{18}$$

$$\gamma(t) = (k_B T)^{-1} \int_0^1 \frac{t^{-\lambda}}{\Gamma(1-\lambda)} d\lambda, \tag{19}$$

Here the friction memory kernel is assumed to satisfy

$$\lim_{t\to\infty} \gamma(t) = \lim_{s\to 0} s\hat{\gamma}(s) = 0, \tag{20}$$

where $\hat{\gamma}(s) = L\{\gamma(t); s\}$.

$$\hat{\gamma}(s) = (k_B T)^{-1} \frac{s-1}{s\log s} \tag{21}$$

$$\lim_{s\to 0} s\hat{\gamma}(s) = (k_B T)^{-1} \lim_{s\to 0} \frac{s-1}{\log s} = 0 \tag{22}$$

Tauberian Theorems

If the Laplace transform pair $\hat{r}(s)$ of the function $r(t)$ behaves like

$$\hat{r}(s) \simeq s^{-\rho} L(s^{-1}), \quad s \to 0, \quad \rho > 0, \tag{23}$$

where $L(t)$ is a slowly varying function at infinity, then $r(t)$ has the following asymptotic behavior

$$r(t) \simeq \frac{1}{\Gamma(\rho)} t^{\rho-1} L(t), \quad t \to \infty. \tag{24}$$

A slowly varying function at infinity means that

$$\lim_{t\to\infty} \frac{L(at)}{L(t)} = 1, \quad a > 0. \tag{25}$$

The Tauberian theorem works also for the opposite asymptotic, i.e., for $t \to 0$.

Langeven equation

$$x(t) = \langle x(t) \rangle + \int_0^t G(t-t')\xi(t')dt' \tag{26}$$

$$\hat{G}(s) = \frac{1}{s^2 + s\hat{\gamma}(s)} = \frac{1}{s^2 + (k_B T)^{-1} \frac{s-1}{\log s}}$$
$$= \sum_{n=0}^{\infty} (-\frac{1}{k_B T})^n \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{s^{n+k+2} \log^n s}$$

$$G(t) = t + \sum_{n=1}^{\infty} (-\frac{1}{k_B T})^n \sum_{k=1}^{n} \binom{n}{k} (-1)^k \mu(t, n-1, n+k+1) \tag{27}$$

$$\langle x^2(t) \rangle = 2K_B T \int_0^t G(\xi)d\xi \tag{28}$$

Langeven equation

Anomalos diffusion:

$$\langle x^2(t)\rangle \simeq t^\alpha \ (0 < \alpha < 1) \tag{29}$$

Ultraslow diffusion:

$$\langle x^2(t)\rangle \simeq \log^\nu t \tag{30}$$

$\nu = 4$: Sinai diffusion model (1982).

$$\begin{aligned}\langle x^2(t)\rangle =&2k_B T\left[\frac{t^2}{2} + \sum_{n=1}^{\infty}\Big(-\frac{1}{k_B T}\Big)^n \sum_{k=1}^{n}\binom{n}{k}(-1)^k\right.\\ &\left.\times\, \mu(t, n-1, n+k+2)\right],\end{aligned} \tag{31}$$

which in the long time limit becomes

$$\begin{aligned}\langle x^2(t)\rangle \simeq&2\,(k_B T)^2[\gamma + \log t + e^t E_1(t)]\\ \simeq&2\,(k_B T)^2(\gamma + \log t).\end{aligned} \tag{32}$$

Langeven equation

$$\hat{I}(s) = s^{-1}\hat{G}(s) = \frac{s^{-1}}{s^2 + s\hat{\gamma}(s)}. \tag{33}$$

Mainardi et al. [6]

$$I(t) = L^{-1}\left\{\frac{s^{-1}}{s^2 + (k_B T)^{-1}\frac{s-1}{\log s}}; t\right\} = L^{-1}\left\{\frac{1}{s^3}\frac{1}{1 + (k_B T)^{-1}\frac{s-1}{s^2 \log s}}; t\right\}$$
$$\sim L^{-1}\left\{\frac{1}{s^3}(1 - (k_B T)^{-1}\frac{s-1}{s^2 \log s}); t\right\}$$
$$= \frac{t^2}{2} - (k_B T)^{-1}\mu(t,0,3) + (k_B T)^{-1}\mu(t,0,4), t \to \infty.$$

Generalized Langeven equation

$$\ddot{x}(t) + \int_0^t \gamma(t-t')\dot{x}(t')\mathrm{d}t' + \frac{\mathrm{d}V(x)}{\mathrm{d}x} = \xi(t), \quad \dot{x}(t) = v(t). \tag{34}$$

The second fluctuation-dissipation theorem (FDT) is then valid in a thermal bath of temperature T, where fluctuations and dissipation come from the same source. The FDT relates the friction memory kernel $\gamma(t)$ with the correlation function $\xi(t)$ of the random force. The FDT allows one to write

$$\langle \xi(t+t')\xi(t') \rangle = C(t) = k_B T \gamma(t). \tag{35}$$

LE with power-logarithmic distributed order noises

$$\gamma(t) = (k_B T)^{-1} \int_0^1 \Gamma(3/2-\lambda) \frac{\log^{\lambda-1} t}{\sqrt{t}} \mathrm{d}\lambda \quad \text{with} \quad \left(\hat{\gamma}(s) = \frac{\pi}{k_B T} \frac{s-1}{\sqrt{s} \log s} \right), \tag{36}$$

which satisfies the condition (20). The relaxation function $I(t)$ defined by

$$I(t) = L^{-1} \left\{ \frac{s^{-1}}{s^2 + \frac{\pi}{k_B T} \frac{s(s-1)}{\sqrt{s} \log s} + \omega^2}; t \right\}. \tag{37}$$

For $s \ll 1$,

$$\hat{I}(s) \simeq \frac{s^{-1}}{\omega^2} \frac{1}{1 - \frac{\pi}{k_B T \omega^2} \frac{s(1-s)}{\sqrt{s} \log s}}$$

$$\simeq \frac{1}{\omega^2} \left[1 + \frac{\pi}{k_B T \omega^2} \frac{1}{\sqrt{s} \log s} - \frac{\pi}{k_B T \omega^2} \frac{\sqrt{s}}{\log s} \right],$$

LE with power-logarithmic distributed order noises

For $s \to 0$,

$$\hat{I}(s) \simeq \frac{1}{\omega^2}\left[1 + \frac{\pi}{k_B T \omega^2}\frac{1}{\sqrt{s}\log s}\right]. \tag{38}$$

$$I(t) \simeq \frac{1}{\omega^2}\left[1 + \frac{\pi}{k_B T \omega^2}\nu(t, -1/2)\right]. \tag{39}$$

For $t \to \infty$,

$$\langle x^2(t)\rangle \simeq \frac{2k_B T}{\omega^2} + \frac{2\pi}{\omega^4}\nu(t, -1/2).$$

Note that for $\omega = 0$ and $t \to \infty$ we get

$$\langle x^2(t)\rangle = 2k_B T L^{-1}\left\{\frac{s^{-1}}{s^2 + \frac{\pi}{k_B T}\frac{s(s-1)}{\sqrt{s}\log s}}; t\right\} \simeq \frac{4(k_B T)^2}{\pi^{3/2}}\sqrt{t}\log t.$$

LE with distributed-order Mittag-Leffler function

$$\gamma(t) = \frac{1}{k_B T} \int_0^1 E_\lambda\left(-t^\lambda\right) \mathrm{d}\lambda \quad \left(\hat{\gamma}(s) = \frac{1}{k_B T} \frac{\log \frac{s+1}{2}}{s \log s}\right), \tag{40}$$

which satisfies the condition (20).

$$I(t) = L^{-1}\left\{\frac{s^{-1}}{s^2 + \frac{1}{k_B T}\frac{\log \frac{s+1}{2}}{\log s} + \omega^2}; t\right\} \sim \frac{1}{\omega^2} L^{-1}\left\{\frac{1}{s} - \frac{1}{k_B T \omega^2}\frac{\log 2}{s \log s}; t\right\}$$
$$= \frac{1}{\omega^2}\left[1 - \frac{\log 2}{k_B T \omega^2}\nu(t)\right]. \tag{41}$$

In the force-free case with $\omega = 0$ we observe ultraslow diffusion, since the relaxation function $I(t)$ has a logarithmic time dependence,

$$I(t) \sim \frac{k_B T}{\log 2} L^{-1}\left\{\frac{\log \frac{1}{s}}{s}; t\right\} = \frac{k_B T}{\log 2} \log t = k_B T \log_2 t. \tag{42}$$

LE with distributed-order Volterra function

$$\gamma_1(t) = \frac{1}{k_B T}\int_0^1 \nu(t,-\alpha)\mathrm{d}\alpha \quad \text{and} \quad \hat{\gamma}_1(s) = \frac{1}{k_B T}\frac{s-1}{s\log^2 s}, \tag{43}$$

For the GLE for the stochastic harmonic oscillator ($\omega \neq 0$) the long time limit ($t \to \infty$) of the relaxation function $I_1(t)$ yields in the form

$$I_1(t) = L^{-1}\left\{\frac{s^{-1}}{s^2 + \frac{1}{k_B T}\frac{s-1}{\log^2 s} + \omega^2}; t\right\} \simeq \frac{1}{\omega^2}\left[1 + \frac{\mu(t,1)}{k_B T \omega^2}\right]. \tag{44}$$

In the force-free case ($\omega = 0$) we obtain ([5], Eq.45)

$$\begin{aligned} I_1(t) &= L^{-1}\left\{\frac{s^{-1}}{s^2 + \frac{1}{k_B T}\frac{s-1}{\log^2 s}}; t\right\} \simeq k_B T L^{-1}\left\{\frac{\log^2 s}{s(s-1)}; t\right\} \\ &= k_B T\Big[-(C^2 + \pi^2/6)e^t + 2te^t\Phi_{1;1}^{\star,(1,1)}(-t,3,1) - 2(C + \log t)e^t E_1(t) \\ &- (2C + \log t)(e^t + 1)(\log t) - C^2 + \pi^2/6\Big]. \end{aligned} \tag{45}$$

LE with distributed-order Volterra function

$$\gamma_2(t) = \frac{1}{k_B T} \int_0^1 \mu(t, -\beta) d\beta \quad \text{and} \quad \hat{\gamma}_2(s) = \frac{1}{k_B T} \frac{(\log s) - 1}{s(\log s)(\log \log s)}. \tag{46}$$

Here we consider only the force-free case, i.e., $\omega = 0$. In that case we find the asumptotic of $I_2(t)$ at long times. Using that $\log \log s$ is a slowly varying function, by Tauberian theorems we find that

$$\begin{aligned} I_2(t) &\simeq k_B T L^{-1} \left\{ \frac{(\log s)(\log \log s)}{s(\log s - 1)}; t \right\} = k_B T L^{-1} \left\{ \frac{\log \log s}{s(1 - \frac{1}{\log s})}; t \right\} \\ &\simeq k_B T L^{-1} \left\{ \frac{1}{s} \log \log \frac{1}{s^{-1}}; t \right\} \simeq k_B T \log \log \frac{1}{t}. \end{aligned} \tag{47}$$

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