The Fuzzy Weighted Average Operation in Decision Making Models

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Abstract

The paper deals with the operation of fuzzy weighted average of fuzzy numbers. The operation can be applied to the aggregation of partial fuzzy evaluations in the fuzzy models of multiple-criteria decision making and to the computation of the expected fuzzy evaluations of alternatives in the discrete fuzzy-stochastic models of decision making under risk. Normalized fuzzy weights figuring in the operation have to form a special structure of fuzzy numbers; its properties will be studied. The practical procedures for setting the normalized fuzzy weights will be shown. The operation of a fuzzy weighted average of fuzzy numbers will be defined and an effective algorithm of its calculation will be described.

Keywords: Multiple-criteria decision making, decision making under risk, normalized fuzzy weights, fuzzy weighted average, fuzzy weights of criteria, fuzzy probabilities

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1 Introduction

The standard operation of a weighted average is used for aggregating the partial evaluations of alternatives in the most of multiple-criteria decision making models and for computing the expected evaluations of alternatives in the discrete models of decision making under risk. The applied normalized weights, i.e. non-negative real numbers whose sum is equal to one, express different importance of considered criteria in the first case and probabilities of states of the world in the second case.

Whereas the probabilities of states of the world have a unique mathematical meaning, weights of criteria are understand differently in different models. The most general definition says that the weights of criteria are non-negative real numbers whose ordering expresses the importance of criteria. According to the definition, the weights mean the measurements of criteria importance that are defined on an ordinal scale. But in most of the multiple-criteria decision making models, the weights of criteria represent some kind of cardinal information concerning the importance of criteria. In the model of multiple-criteria decision making that is described in [7] the weights of criteria represent shares of the corresponding partial objectives of evaluation in the overall one. This concept of criteria weights will be considered in this paper.

The weights of criteria as well as the probabilities of states of the world are usually set subjectively, i.e. they are more or less uncertain. In this paper, it will be shown how this kind of uncertain information can be expressed by means of the tools of the fuzzy sets theory. A special structure of fuzzy numbers, so called normalized fuzzy weights, will be introduced for that purpose. The operation of
a fuzzy weighted average of fuzzy numbers, where the normalized fuzzy weights are used, will be defined. An effective computing algorithm of the fuzzy weighted average will be described.

2 Applied Notions of the Fuzzy Sets Theory

A fuzzy set \(A\) on a universal set \(X\), \(X \neq \emptyset\), is given by its membership function \(A : X \rightarrow [0, 1]\). For any element \(x \in X\), \(A(x)\) is called the membership degree of the element \(x\) to the fuzzy set \(A\). A set \(\text{Supp} \ A = \{x \in X \mid A(x) > 0\}\) is called a support of \(A\). Sets \(A_\alpha = \{x \in X \mid A(x) \geq \alpha\}\), \(\alpha \in (0, 1]\), are \(\alpha\)-cuts of \(A\). A set \(\text{Ker} \ A = \{x \in X \mid A(x) = 1\}\) is called a kernel of \(A\). A fuzzy set \(A\) is called normal if \(\text{Ker} \ A \neq \emptyset\).

A fuzzy number is defined as a fuzzy set \(C\) on the set of all real numbers \(\mathbb{R}\) that fulfills the following conditions: a) \(C\) is a normal fuzzy set, b) \(\text{Supp} \ C\) is a bounded set. It can be proved (see [4]) that the membership function \(C(x)\) is upper semicontinuous and that there exist real numbers \(c^1 \leq c^2 \leq c^3 \leq c^4\) (so called significant values of \(C\)) such that \([c^1, c^4]\) = \(\text{Cl}(\text{Supp} \ C)\), where \(\text{Cl}(\text{Supp} \ C)\) means the closure of \(\text{Supp} \ C\), \([c^2, c^3]\) = \(\text{Ker} \ C\), \(C(x)\) is non-decreasing for \(x \in [c^1, c^2]\) and non-increasing for \(x \in [c^3, c^4]\). A fuzzy number \(C\) is said to be defined on \([a, b]\), if \(\text{Supp} \ C \subseteq [a, b]\).

For a fuzzy number \(C\), let us denote \(C_\alpha = [\xi(\alpha), \tau(\alpha)]\) for any \(\alpha \in (0, 1]\), and \(\text{Cl}(\text{Supp} \ C) = [\xi(0), \tau(0)]\). Then it can be seen that each fuzzy number \(C\) is determined not only by its membership function \(C(x)\), \(x \in \mathbb{R}\), but also by the couple of real functions \((\xi(\alpha), \tau(\alpha))\), \(\alpha \in (0, 1]\). The functions \(\xi\) and \(\tau\) are the pseudo-inverse functions to the functions \(C_L\) and \(C_R\) that represent restrictions of the membership function \(C\) to the closed intervals \([c^1, c^2]\) and \([c^3, c^4]\), respectively.

If \(\xi(\alpha) = c = \tau(\alpha)\) for all \(\alpha \in [0, 1]\), then \(C\) represents a real number; \(C = c\), \(c \in \mathbb{R}\). A fuzzy number \(C\) is called symmetric, if \(\xi(\alpha) + \tau(\alpha)\) is a constant function of \(\alpha\), \(\alpha \in [0, 1]\). A fuzzy number \(C\) is called linear, if \(\xi(\alpha)\), \(\tau(\alpha)\), \(\alpha \in [0, 1]\), are linear functions. Any linear fuzzy number \(C\) is fully determined by its significant values \(c^1 \leq c^2 \leq c^3 \leq c^4\); its functions \(\xi\), \(\tau\) are given as follows: \(\xi(\alpha) = c^1 + \alpha(c^2 - c^1)\) and \(\tau(\alpha) = c^4 - \alpha(c^4 - c^3)\) for any \(\alpha \in [0, 1]\). For \(c^2 \neq c^3\) the linear fuzzy number \(C\) is often called trapezoidal, for \(c^2 = c^3\) triangular.

For any fuzzy number \(C\), the non-increasing, non-negative function \(\sigma_C(\alpha) = \tau(\alpha) - \xi(\alpha), \alpha \in [0, 1]\), is called a span function of \(C\). It describes in details, for any membership degree \(\alpha\), the uncertainty of \(C\). The real numbers \(\sigma_C(1), \sigma_C(\alpha)\) and \(\sigma_C(0)\), are called the span of the kernel, the span of the \(\alpha\)-cut and the span of the support, respectively. Obviously, \(\sigma_C(\alpha) = 0\) for all \(\alpha \in [0, 1]\) if and only if \(C\) is a real number. It holds \(\int_0^1 \sigma_C(\alpha) \, d\alpha = \int_\mathbb{R} C(x) \, dx\); the second member in the equality represents the usual aggregated measure of imprecision of the fuzzy number \(C\).

Calculations with fuzzy numbers are based on a so-called extension principle. Let \(y = f(x_1, x_2, \ldots, x_n)\) be a continuous function on \(\mathbb{R}^n\). For any fuzzy numbers \(X_1, X_2, \ldots, X_n\), \(f(X_1, X_2, \ldots, X_n)\) is defined as a fuzzy number \(Y\) whose membership function is given for each \(y \in \mathbb{R}\) as follows

\[
Y(y) = \begin{cases} 
\min \{ \min_{i=1,...,n} |X_i(x_i)| \mid y = f(x_1, \ldots, x_n) \} & \text{if such } x_1, \ldots, x_n \text{ exist,} \\
0 & \text{otherwise.}
\end{cases}
\]

(1)

In the fuzzy models of decision making, special structures of fuzzy numbers are used. The first of them is a fuzzy scale that enables a finite representation of an infinite interval by fuzzy numbers; the second, normalized fuzzy weights, will be a topic of the next section. A fuzzy scale on \([a, b]\) (see [7]) is a finite set of fuzzy numbers \(A_1, A_2, \ldots, A_n\) that are defined on \([a, b]\), form a fuzzy partition on \([a, b]\),
i.e. $\sum_{i=1}^{m} A_i(x) = 1$ holds for any $x \in [a, b]$, and are numbered according to their linear ordering. If a fuzzy scale represents a mathematical meaning of a natural linguistic scale, then it is called a linguistic fuzzy scale.

3 Normalized Fuzzy Weights

3.1 Definitions and Basic Properties

In decision making models, weights of criteria as well as probabilities of states of the world are usually set subjectively, i.e. they are uncertain. Therefore the models are more realistic if the weights of criteria and the probabilities of states of the world are expressed by means of the tools of the fuzzy sets theory. In this section, a special structure of fuzzy numbers, so called normalized fuzzy weights, will be introduced and its general properties will be studied. From the general point of view, the structure makes it possible to model mathematically an uncertain division of a unit into fractions.

Normalized fuzzy weights represent a fuzzification of the structure of normalized weights; $m$-tuple of real numbers $v_1, v_2, \ldots, v_m$ forms normalized weights if $v_i \geq 0$, $i = 1, 2, \ldots, m$, and $\sum_{i=1}^{m} v_i = 1$. Normalized weights can be used for describing a division of a unit into $m$ uniquely specified fractions.

Fuzzy numbers $V_i$, $i = 1, 2, \ldots, m$, defined on $[0, 1]$ are called normalized fuzzy weights (see [6]) if for all $\alpha \in (0, 1]$ and for all $i \in \{1, 2, \ldots, m\}$ the following holds: for any $v_i \in V_{i\alpha}$ there exist $v_j \in V_{j\alpha}$, $j = 1, 2, \ldots, m$, $j \neq i$, such that

$$ v_i + \sum_{j=1, j\neq i}^{m} v_j = 1. \tag{2} $$

The purpose of this definition is a generalization of the condition of normalization from the case of real numbers into the case of fuzzy numbers. Obviously, normalized weights $v_1, v_2, \ldots, v_m$ are a special case of normalized fuzzy weights. The membership degree $V_i(v_i)$ can be interpreted as the possibility of the fact that the weight of $i$-th criterion or the probability of $i$-th state of the world equals $v_i$; from the general point of view $V_i(v_i)$ means the possibility that the share of $i$-th subject in the whole is $v_i \cdot 100\%$.

The structure of normalized fuzzy weights was introduced in [6] in order to fuzzify the weights of criteria in the method of weighted average of partial fuzzy evaluations that was described in [7]. Later on, the normalized fuzzy weights were applied to modeling uncertain probabilities in fuzzy probability spaces with a finite set of elementary events (see [8], [9]). The same structure of fuzzy numbers was applied to modelling uncertain probabilities in fuzzy probability spaces with a finite number of elementary events (see [8], [9]). The same structure of fuzzy numbers was introduced for modelling fuzzy probabilities also in [5] (it was called "a tuple of fuzzy probabilities") as a generalization of the structure of interval probabilities developed in [1].

The fact whether fuzzy numbers $V_i$, $i = 1, 2, \ldots, m$, defined on $[0, 1]$ form normalized fuzzy weights or not can be verified in the following way (see [5] or [6]): For $i = 1, 2, \ldots, m$ let us denote $V_{i\alpha} = [\omega_{i\alpha}, \varpi_{i\alpha}])$, $\sigma_{V_i}(\alpha) = \varpi_{i\alpha} - \omega_{i\alpha}$ for any $\alpha \in [0, 1]$. Furthermore, let us denote $\omega_{V_1, \ldots, V_m}(\alpha) = 1 - \sum_{i=1}^{m} \omega_{i\alpha} \alpha \in [0, 1]$. Fuzzy numbers $V_i$, $i = 1, 2, \ldots, m$, defined on $[0, 1]$ represent normalized fuzzy weights if and only if for any $\alpha \in [0, 1]$ and for $i^*(\alpha) \in \{1, 2, \ldots, m\}$, such that $\sigma_{V_{i^*(\alpha)}}(\alpha) = \max_{i=1, 2, \ldots, m} \{\sigma_{V_i}(\alpha)\}$, the following holds

$$ \sigma_{V_{i^*(\alpha)}}(\alpha) \leq \omega_{V_1, \ldots, V_m}(\alpha) \leq \sum_{j=1, j\neq i^*(\alpha)}^{m} \sigma_{V_j}(\alpha). \tag{3} $$

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Obviously, for \( m = 2 \) from (3) follows that \( V_2 = 1 - V_1 \). In the case of linear fuzzy numbers \( V_i = (v_i^1, v_i^2, v_i^3, v_i^4), i = 1, 2, \ldots, m \), it is sufficient to verify only the following two conditions

\[
\sigma_{V_i^* (1)} (1) \leq \omega_{V_1, \ldots, V_m (1)} \leq \sum_{j=1, j \neq i}^{m} \sigma_{V_j} (1) \quad (4)
\]

\[
\sigma_{V_i^* (0)} (0) \leq \omega_{V_1, \ldots, V_m (0)} \leq \sum_{j=1, j \neq i}^{m} \sigma_{V_j} (0). \quad (5)
\]

The uncertainty of the structure of normalized fuzzy weights can be characterized by two functions defined on \([0, 1]\). The first of them, so called expansibility function \( \omega_{V_1, \ldots, V_m (\alpha)} \), \( \alpha \in [0, 1] \), describes for each membership degree \( \alpha \) the total value which is available for expansion of \( V_i (\alpha), i = 1, 2, \ldots, m \). The second, so called overall span function \( \sigma_{V_1, \ldots, V_m (\alpha)} = \sum_{i=1}^{m} \sigma_{V_i (\alpha)}, \alpha \in [0, 1] \), reflects for any \( \alpha \) the overall variability of the possible vectors of normalized weights. In [6], the following relation between these functions was proved

\[
\frac{m}{m-1} \cdot \omega_{V_1, \ldots, V_m (\alpha)} \leq \sigma_{V_1, \ldots, V_m (\alpha)} \leq m \cdot \omega_{V_1, \ldots, V_m (\alpha)} \quad \text{for all } \alpha \in [0, 1], \quad (6)
\]

where the first equality holds if and only if \( \sigma_{V_i (\alpha)} = \frac{\omega_{V_1, \ldots, V_m (\alpha)}}{m-1} \) for each \( i \in \{1, 2, \ldots, m\} \), and the second equality holds if and only if \( \sigma_{V_i (\alpha)} = \omega_{V_1, \ldots, V_m (\alpha)} \) for each \( i \in \{1, 2, \ldots, m\} \).

### 3.2 Procedures of Setting Normalized Fuzzy Weights

In the practical applications, it is useful to set normalized fuzzy weights as linear fuzzy numbers \( V_i = (v_i^1, v_i^2, v_i^3, v_i^4), i = 1, 2, \ldots, m \), where \( [v_i^2, v_i^3] \) represents the interval of fully possible values of \( i \)-th weight, while outside the interval \([v_i^1, v_i^2]\) the \( i \)-th weight cannot lie.

The easiest way of setting normalized fuzzy weights is following: First, an expert sets crisp normalized weights \( v_1, v_2, \ldots, v_m, v_i \geq 0, i = 1, 2, \ldots, m, \sum_{i=1}^{m} v_i = 1 \). Then he/she fuzzifies them into triangular fuzzy numbers \( V_i = (v_i - s, v_i, v_i + s), i = 1, 2, \ldots, m \), where \( s \geq 0 \) satisfies \( v_i - s \geq 0 \) and \( v_i + s \leq 1 \) for all \( i \in \{1, 2, \ldots, m\} \).

A more general method of setting normalized fuzzy weights results from the conditions (4) and (5). Let \( v_1, v_2, \ldots, v_m, v_i \geq 0, i = 1, 2, \ldots, m, \sum_{i=1}^{m} v_i = 1 \), represent again a crisp estimation of normalized weights. Let real numbers \( k_i, s_i, 0 \leq k_i \leq s_i, v_i - s_i \geq 0, v_i + s_i \leq 1, i = 1, 2, \ldots, m \), characterize the width of kernels and supports of the particular fuzzy weights. Then linear fuzzy numbers \( V_i = (v_i - s_i, v_i - k_i, v_i + k_i, v_i + s_i), i = 1, 2, \ldots, m \), represent normalized fuzzy weights, if and only if the numbers \( k_i \) and \( s_i, i = 1, 2, \ldots, m \), satisfy the following conditions

\[
\sum_{i=1, i \neq j}^{m} k_i \geq k_j \quad \text{and} \quad \sum_{j=1, j \neq i}^{m} s_j \geq s_i, \quad (7)
\]

where \( k_i = \max_{i=1, \ldots, m} \{k_i\} \) and \( s_j = \max_{j=1, \ldots, m} \{s_j\} \).

The normalized fuzzy weights, which express either the importance of criteria or the probabilities of states of the world in the decision making models, can be set also linguistically. For the purpose, a special linguistic fuzzy scale has to be used. It can consists e.g. of the following linguistic terms: "negligible minority", "minority", "approximately half", "majority", "absolute majority". The mathematical meanings of these terms have to form a symmetrical fuzzy scale on \([0, 1]\), i.e. a fuzzy scale \( T_1, T_2, \ldots, T_3 \) satisfying \( T_j (x) = T_{6-j} (1-x) \) for \( x \in [0, 1] \) and \( j = 1, 2, 3 \).

For example it can be set: \( T_1 = (0, 0, 0, \frac{1}{6}), T_2 = (0, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}), T_3 = (\frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{2}), \)
$T_4 = (\frac{1}{2}, \frac{2}{5}, \frac{5}{6}, 1), T_5 = (\frac{2}{5}, 1, 1, 1)$. Each couple of fuzzy numbers $T_j, T_{6-j}$, $j = 1, 2, 3$, represents a couple of normalized fuzzy weights, i.e. it describes an uncertain division of a unit into two fractions; the situation also corresponds with the linguistic descriptions. Let us consider now the problem of setting criteria weights by means of such linguistic terms. Let us suppose that the criteria are linearly ordered according to their importance, i.e. $V_1 > V_2 > \ldots > V_m$. We express the share of the partial evaluation according to the most important criterion $K_1$ in the total evaluation, i.e. evaluation according to $K_1, K_2, \ldots, K_m$, by an adequate linguistic term $T_{j_1}$ (e.g. ”majority”). Then the share of the evaluation according to $K_2, K_3, \ldots, K_m$ in the total evaluation is $T_{6-j_1}$ (”minority”). We repeat this procedure for each $i = 2, \ldots, m - 1$, denoting by $T_i$ the share of the evaluation according to $K_i$ in the total evaluation according to $K_1, K_{i+1}, \ldots, K_m$. Then the normalized fuzzy weights of the criteria $K_1, K_2, \ldots, K_m$ are given in the following way

$$
V_1 = T_{j_1}, V_2 = T_{6-j_1} \cdot T_{j_2}, V_3 = T_{6-j_1} \cdot T_{6-j_2} \cdot T_{j_3}, \\
\ldots \\
V_m = T_{6-j_1} \cdot T_{6-j_2} \cdot T_{6-j_3} \cdots T_{6-j_{m-1}}.
$$

(8)

Normalized fuzzy weights defined in this way are not linear, but they can be approximated by linear fuzzy numbers.

Normalized fuzzy weights of criteria can be set by means of the linguistic terms analogously; e.g. we can say that the state $S$ is divided into “approximately equal” fractions; for example by means of triangular fuzzy numbers $(\frac{1}{k-1}, \frac{1}{k}, \frac{k+2}{k(k+1)})$, $k = 2, 3, \ldots$

For setting uncertain probabilities of states of the world we can use the linguistic terms analogously; e.g. we can say that the state $S_1$ occurs in the ”approximately half” of all cases, otherwise in the ”majority” of cases a state $S_2$ occurs, and if not, then a state $S_3$ comes.

4 The Fuzzy Weighted Average of Fuzzy Numbers

A fuzzy weighted average $U$ of fuzzy numbers $U_1, U_2, \ldots, U_m$ with non-negative fuzzy weights $W_1, W_2, \ldots, W_m$ is defined in [2] as a fuzzification according to the extension principle (1) of the operation $\sum_{i=1}^{m} w_i = 1, w_i \geq 0, i = 1, 2, \ldots, m, \sum_{i=1}^{m} w_i \neq 0$. The membership function $U(u)$ is given for each $u \in \mathbb{R}$ in the following way

$$
U(u) = \max \\{ \min \{ U_1(u_1), U_2(u_2), \ldots, U_m(u_m), W_1(w_1), W_2(w_2), \ldots, W_m(w_m) \} | \\
u = \sum_{i=1}^{m} w_i u_i, \sum_{i=1}^{m} w_i \neq 0 \}.
$$

(9)

The algorithm for computing the fuzzy weighted average given by (9) is studied in [3].

The fuzzy weighted average given by (9) cannot be used for aggregating partial fuzzy evaluations in multiple-criteria decision making models where uncertain weights of criteria express the shares of partial objectives of evaluation in the overall one (see [7], [8]). The same holds also for computing expected fuzzy evaluations in models of decision making under risk with uncertain probabilities of states of the world (see [8]). The problem is illustrated by the following example:

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Let $S_1$, $S_2$ and $S_3$ be states of the world, their uncertain probabilities be, for simplicity, described by intervals $P_1 = [0.1, 0.2]$, $P_2 = [0.4, 0.6]$ and $P_3 = [0.3, 0.4]$ ($P_1$, $P_2$ and $P_3$ form normalized fuzzy weights) and the evaluations of an alternative $x$ under states of the world $S_1$, $S_2$, $S_3$ be given by $U_1 = [0.1, 0.2]$, $U_2 = [0.4, 0.5]$ and $U_3 = [0.8, 0.9]$. Then, according to (9) where we replace $W_i$ by $P_i$ for $i = 1, 2, 3$, the fuzzy expected value of $x$ is $U = [0.45, 0.64]$. Let us notice that for computing the minimum value of $U$, i.e. 0.45, the third weight $p_3 = 0.3$ has to be replaced by $0.27$ and $0.27 \notin P_3$. So this value lies outside the expertly set range of possible probabilities of $S_3$.

In our case, a fuzzy weighted average of fuzzy numbers has to be defined as a fuzzification of an operation $\sum_{i=1}^{m} v_i \cdot u_i$, $\sum_{i=1}^{m} v_i = 1$, $v_i \geq 0$, $i = 1, 2, \ldots, m$. The fuzzy weighted average (see [6]) of fuzzy numbers $U_i$, $i = 1, 2, \ldots, m$, defined on $[a, b]$ with normalized fuzzy weights $V_i$, $i = 1, 2, \ldots, m$, is a fuzzy number $U$ on $[a, b]$ whose membership function is given for each $u \in [a, b]$ by the following formula:

$$U(u) = \max\left\{\min\{V_1(v_1), \ldots, V_m(v_m), U_1(u_1), \ldots, U_m(u_m)\}\right\}$$

$$\sum_{i=1}^{m} v_i \cdot u_i = u, \quad \sum_{i=1}^{m} v_i = 1, \quad i = 1, 2, \ldots, m.$$  \hfill (10)

The following denotation will be used for the fuzzy weighted average given by (10)

$$U = (\mathcal{F}) \sum_{i=1}^{m} V_i \cdot U_i.$$  \hfill (11)

In the case of normalized weights the fuzzy weighted averages given by (9) and (10) coincide. But for normalized fuzzy weights it does not hold. A fuzzy number representing the result of the fuzzy weighted average operation given by (10) is a subset of that one given by (9). For example, in the above mentioned example, if $U = (\mathcal{F}) \sum_{i=1}^{m} P_i \cdot U_i$, then $U = [0.46, 0.63]$. In the following text the fuzzy weighted average given by (10) will be considered.

The following algorithm of computing the fuzzy weighted average of fuzzy numbers is based on an algorithm that was originally developed for computing the expected value of a discrete random variable with interval probabilities (see [1]).

For each $\alpha \in [0, 1]$ let $\{i_k\}_{k=1}^{m}$ be such a permutation on an index set $\{1, 2, \ldots, m\}$ that it holds $\underline{v}_{i_1}(\alpha) \leq \underline{v}_{i_2}(\alpha) \leq \ldots \leq \underline{v}_{i_m}(\alpha)$. For $k \in \{1, 2, \ldots, m\}$ let us denote $v_{i_k}(\alpha) = 1 - \sum_{j=1}^{k-1} \underline{v}_{i_j}(\alpha) - \sum_{j=k+1}^{m} \overline{v}_{i_j}(\alpha)$. Let $k^* \in \{1, 2, \ldots, m\}$ be such an index that it holds $\underline{v}_{i_{k^*}}(\alpha) \leq v_{i_{k^*}}(\alpha) \leq \overline{v}_{i_{k^*}}(\alpha)$. Then

$$\underline{v}(\alpha) = \sum_{j=1}^{k^*-1} \underline{v}_{i_j}(\alpha) \cdot v_{i_j}(\alpha) + v_{i_{k^*}}(\alpha) \cdot \underline{v}_{i_{k^*}}(\alpha) + \sum_{j=k^*+1}^{m} \overline{v}_{i_j}(\alpha) \cdot v_{i_j}(\alpha).$$  \hfill (12)

Let $\{i_h\}_{h=1}^{m}$ be such a permutation on an index set $\{1, 2, \ldots, m\}$ that it holds $\overline{v}_{i_1}(\alpha) \geq \overline{v}_{i_2}(\alpha) \geq \ldots \geq \overline{v}_{i_m}(\alpha)$. For $h \in \{1, 2, \ldots, m\}$ let us denote $v_{i_h}(\alpha) = 1 - \sum_{j=1}^{h-1} \underline{v}_{i_j}(\alpha) - \sum_{j=h+1}^{m} \overline{v}_{i_j}(\alpha)$. Let $h^* \in \{1, 2, \ldots, m\}$ be such an index that it holds $\overline{v}_{i_{h^*}}(\alpha) \leq v_{i_{h^*}}(\alpha) \leq \overline{v}_{i_{h^*}}(\alpha)$. Then

$$\overline{v}(\alpha) = \sum_{j=1}^{h^*-1} \underline{v}_{i_j}(\alpha) \cdot \overline{v}_{i_j}(\alpha) + v_{i_{h^*}}(\alpha) \cdot \overline{v}_{i_{h^*}}(\alpha) + \sum_{j=h^*+1}^{m} \overline{v}_{i_j}(\alpha) \cdot v_{i_j}(\alpha).$$  \hfill (13)

The span function of the fuzzy weighted average $\sigma_U(\alpha)$, $\alpha \in [0, 1]$, depends on the span functions $\sigma_{U_1}, \sigma_{U_2}, \ldots, \sigma_{U_m}$ and $\sigma_{V_1}, \sigma_{V_2}, \ldots, \sigma_{V_m}$ on the other hand and on the variability of the fuzzy numbers $U_1, U_2, \ldots, U_m$ on the other hand. It is illustrated by the following example:

Let $K_1$ and $K_2$ be a couple of criteria, corresponding normalized fuzzy weights $V_1$ and $V_2$ be set as triangular fuzzy numbers $V_1 = (0.4, 0.6, 0.8)$ and $V_2 = (0.2, 0.4, 0.6)$,
and the partial evaluations of alternatives \( x_1, x_2 \) and \( x_3 \) according to \( K_1, K_2 \) be, for simplicity, given as real numbers \( u_{11} = 0.9, u_{12} = 0.1; u_{21} = 0.1, u_{22} = 0.9; u_{31} = 0.5, u_{32} = 0.5 \). Then the fuzzy weighted averages \( U_i = (F) \sum_{j=1}^{r} V_j \cdot u_{ij}, i = 1, 2, 3, \) expressing overall evaluations of the alternatives, are \( U_1 = (0.42, 0.58, 0.74), U_2 = (0.26, 0.42, 0.58) \) and \( U_3 = 0.5 \).

We can see practically in the above example that the overall fuzzy evaluations of alternatives that are evaluated very differently according to the particular criteria are much more uncertain than the overall fuzzy evaluations of alternatives whose partial evaluations are almost uniform in multiple-criteria decision making models. Similarly, in decision making under risk, the expected fuzzy evaluations of alternatives that depend strongly on states of the world are more uncertain than expected fuzzy evaluations of alternatives which are relatively stable.

5 Applications of the Fuzzy Weighted Average in the Decision Making Models

The operation of a fuzzy weighted average can be applied to multiple-criteria decision making and decision making under risk in the following way:

First, let us consider the problem of multiple-criteria decision making, where the best of alternatives \( x_1, x_2, \ldots, x_n \) is to be chosen. The alternatives will be evaluated with respect to a given objective that is partitioned into \( m \) partial objectives associated with criteria \( K_1, K_2, \ldots, K_m \). Let the uncertain information concerning the shares of the partial objectives in the overall one be given by normalized fuzzy weights \( V_1, V_2, \ldots, V_m \). Let \( U_{i,j}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m \), be uncertain partial fuzzy evaluations of alternatives \( x_i, i = 1, 2, \ldots, n \), with respect to the criteria \( K_j, j = 1, 2, \ldots, m \). \( U_{i,j}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, m \), are supposed to be the fuzzy degrees of fulfillment of the corresponding partial objectives and to be expressed by fuzzy numbers defined on \([0, 1]\). Then the overall fuzzy evaluation \( U_i \) of the alternative \( x_i \) will be calculated for \( i = 1, 2, \ldots, n \) as the fuzzy weighted average of the partial fuzzy evaluations \( U_{i,j} \) with the normalized fuzzy weights \( V_j, j = 1, 2, \ldots, m \), i.e.

\[
U_i = (F) \sum_{j=1}^{m} V_j \cdot U_{i,j} .
\]

The overall fuzzy evaluations \( U_i, i = 1, 2, \ldots, n \), express again the fuzzy degrees of fulfillment of the overall objective of evaluation. The best alternative is either the first alternative in an ordering of the fuzzy numbers \( U_i, i = 1, 2, \ldots, n \), or the closest to the ideal alternative whose evaluation is equal to 1. The overall fuzzy evaluations of alternatives can also be approximated linguistically by linearly ordered elements of a proper linguistic evaluation scale defined on \([0, 1]\). For more details about metrics, ordering of fuzzy numbers and linguistic approximation see [7].

Second, let us consider a problem of decision making under risk, where the best of risk alternatives \( x_1, x_2, \ldots, x_n \) is being chosen according to their degrees of fulfillment of an objective associated with a criterion \( K \). The degrees of the objective fulfillment depend not only on the alternatives themselves but also on the fact which of the states of the world \( S_1, S_2, \ldots, S_r \) occurs. Uncertain probabilities of the states of the world are expertly set by normalized fuzzy weights \( P_1, P_2, \ldots, P_r \). The fuzzy evaluations of the alternatives \( x_i, i = 1, 2, \ldots, n \), under the states of the world \( S_k, k = 1, 2, \ldots, r \), are described as fuzzy numbers \( U_{i,k}, i = 1, 2, \ldots, n, k = 1, 2, \ldots, r \), defined on \([0, 1]\), that express to which extend the alternatives fulfill the given objective under the given states of the world. Expected fuzzy evaluations \( FEU_i \) of the alternatives \( x_i, i = 1, 2, \ldots, n \), will be calculated according to the formula
\[ FEU_i = (\mathcal{F}) \sum_{k=1}^{r} P_k \cdot U_{i,k}. \] (15)

The best alternative will be chosen in an analogical way as in the previous case.

Examples of practical applications of such decision making models, namely in finance and banking, are described in [8] and [9].

References


