Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **55**, 2 (2016) 41–55

Some Classes of Lorentzian α-Sasakian Manifolds Admitting a Quarter-symmetric Metric Connection

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(Received August 5, 2016)

Abstract

The object of the present paper is to study a quarter-symmetric metric connection in an Lorentzian α-Sasakian manifold. We study some curvature properties of an Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection. We study locally ϕ-symmetric, ϕ-symmetric, locally projective ϕ-symmetric, ξ-projectively flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection.

Key words: Quarter-symmetric metric connection, Lorentzian α-Sasakian manifold, locally ϕ-symmetric manifold, locally projective ϕ-symmetric manifold, ξ-projectively flat Lorentzian α-Sasakian manifold.

2010 Mathematics Subject Classification: 53C25, 53C15

1 Introduction

The idea of semi-symmetric linear connection on a differentiable manifold was introduced by Friedmann and Schouten ([4]). Further, Hayden ([6]), introduced

*The first author is supported by DST/INSPIRE Fellowship/2013/1041, Govt. of India.

the idea of metric connection with torsion on a Riemannian manifold. In ([22]), Yano studied some curvature conditions for semi-symmetric connections in Riemannian manifolds. In 1975, Golab ([5]) defined and studied quarter-symmetric connection in a differentiable manifold.

A linear connection $\tilde{\nabla}$ on an n-dimensional Riemannian manifold (M^n, g) is said to be a quarter-symmetric connection ([5]) if its torsion tensor $\tilde{T}$ defined by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y], \tag{1.1}$$

is of the form

$$\tilde{T}(X,Y) = \eta(Y)\phi X - \eta(X)\phi Y, \tag{1.2}$$

where η is 1-form and ϕ is a tensor field of type $(1,1)$. In addition, if a quarter-symmetric linear connection $\tilde{\nabla}$ satisfies the condition

$$(\tilde{\nabla}_X g)(Y,Z) = 0, \tag{1.3}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the set of all differentiable vector fields on M, then $\tilde{\nabla}$ is said to be a quarter-symmetric metric connection. In particular, if $\phi X = X$ and $\phi Y = Y$ for all $X, Y \in \chi(M)$, then the quarter-symmetric connection reduces to a semi-symmetric connection [4].

In 1980, R. S. Mishra and S. N. Pandey ([15]) studied quarter-symmetric metric connection and in particular, Ricci quarter-symmetric symmetric metric connection on Riemannian, Sasakian and Kaehlerian manifolds. Note that a quarter-symmetric metric connection is a Hayden connection with the torsion tensor of the form (1.2). Studies of various types of quarter-symmetric metric connection and their properties by various authors in ([1, 17, 18, 23]) among others.

The notion of locally symmetry of Riemannian manifolds have been weakened by many authors in several ways to a different extent. As a weaker version of local symmetry, T. Takahashi ([20]) introduced the notion of locally ϕ-symmetry on Sasakian manifolds. In the context of contact geometry the notion of ϕ-symmetry is introduced and studied by E. Boeckx, P. Buecken and L. Vanhecke ([2]) with several examples.

In monograph [8] are presented many properties of symmetric, recurrent, semi-symmetric, Einstein, Sasakian and other manifolds, see also [3, 10, 7, 9, 11, 12, 14, 13, 19].

In 2005, Yildiz and Murathan ([24]) studied Lorentzian α-Sasakian manifolds and proved that conformally flat and quasi conformally flat Lorentzian α-Sasakian manifolds are locally isometric with a sphere. In 2012, Yadav and Suthar ([25]) studied Lorentzian α-Sasakian manifolds.

Definition 1.1. An Lorentzian α-Sasakian manifold M^n is said to be locally ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{1.4}$$

for all vector fields X, Y, Z, W orthogonal to ξ. This notion was introduced by Takahashi for Sasakian manifolds ([20]).

Definition 1.2. An Lorentzian α-Sasakian manifold M^n is said to be ϕ-symmetric if

$$\phi^2((\nabla_W R)(X,Y)Z) = 0, \tag{1.5}$$

for arbitrary vector fields X, Y, Z, W.

The Projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a n-dimensional Riemannian manifold. If there exists a one-to-one correspondence between each co-ordinate neighbourhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the projective curvature tensor vanishes. Here the projective curvature tensor P is defined by

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y], \tag{1.6}$$

for $X, Y, Z \in \chi(M)$, where S is the Ricci tensor of the manifold. In fact M is projectively flat if and only if it is of constant curvature. Thus the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature.

Definition 1.3. An Lorentzian α-Sasakian manifold M^n is said to be locally projective ϕ-symmetric if

$$\phi^2((\nabla_W P)(X,Y)Z) = 0, \tag{1.7}$$

for all vector fields X, Y, Z, W orthogonal to ξ, where P is the projective curvature tensor defined in (1.6).

Definition 1.4. A Lorentzian α-Sasakian manifold M^n is said to be ξ projective flat if

$$P(X,Y)\xi = 0, \tag{1.8}$$

for all vector fields $X, Y \in \chi(M)$, This notion was first defined by Tripathi and Dwivedi ([21]). If equation (1.8) holds for X, Y orthogonal to ξ, we called such a manifold a horizontal ξ-projectively flat manifold.

In the present paper, we study Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection. Motivated by the above studies in this paper we give the relation between the Levi-Civita connection and the quarter-symmetric metric connection on a Lorentzian α-Sasakian manifold. We characterize locally ϕ-symmetric Lorentzian α-Sasakian with respect to quarter-symmetric metric connection. Then we study ϕ-symmetric Lorentzian α-Sasakian manifolds with respect to quarter-symmetric metric connection. We also study locally projective ϕ-symmetric Lorentzian α-Sasakian with respect to quarter-symmetric metric connection. Next we cultivate ξ-projectively flat Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection. Finally we give an example of 3-dimensional Lorentzian α-Sasakian manifolds with respect to quarter-symmetric metric connection.

2 Preliminaries

A $n(=2m+1)$-dimensional differentiable manifold M is said to be a Lorentzian α-Sasakian manifold if it admits a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and Lorentzian metric g which satisfy the following conditions

$$\phi^2 X = X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = -1, \phi\xi = 0, \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$g(X,\xi) = \eta(X), \tag{2.4}$$

$$(\nabla_X \phi)(Y) = \alpha g(X,Y)\xi + \eta(Y)X \tag{2.5}$$

$\forall X, Y \in \chi(M)$ and for smooth functions α on M, ∇ denotes the covariant differentiation with respect to Lorentzian metric g ([16, 26]).

For a Lorentzian α-Sasakian manifold, it can be shown that ([16], [26]):

$$\nabla_X \xi = \alpha\phi X, \tag{2.6}$$

$$(\nabla_X \eta)(Y) = \alpha g(\phi X, Y), \tag{2.7}$$

for all $X, Y \in TM$. Further on a Lorentzian α-Sasakian manifold, the following relations hold ([16])

$$g(R(X,Y)Z,\xi) = \eta(R(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{2.8}$$

$$R(\xi,X)Y = \alpha^2[g(X,Y)\xi - \eta(Y)X], \tag{2.9}$$

$$R(X,Y)\xi = \alpha^2[\eta(Y)X - \eta(X)Y], \tag{2.10}$$

$$R(\xi,X)\xi = \alpha^2[X + \eta(X)\xi], \tag{2.11}$$

$$S(X,\xi) = S(\xi,X) = (n-1)\alpha^2\eta(X), \tag{2.12}$$

$$S(\xi,\xi) = -(n-1)\alpha^2, \tag{2.13}$$

$$Q\xi = (n-1)\alpha^2\xi, \tag{2.14}$$

where Q is the Ricci operator, i.e.

$$g(QX,Y) = S(X,Y). \tag{2.15}$$

If ∇ is the Levi-Civita connection manifold M, then quarter-symmetric metric connection $\tilde{\nabla}$ in M is denoted by

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y)\phi(X). \tag{2.16}$$

3 Curvature tensor and Ricci tensor of Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

Let $\tilde{R}(X,Y)Z$ and $R(X,Y)Z$ be the curvature tensors with respect to the quarter-symmetric metric connection $\tilde{\nabla}$ and with respect to the Riemannian connection ∇ respectively on a Lorentzian α-Sasakian manifold M. A relation between the curvature tensors $\tilde{R}(X,Y)Z$ and $R(X,Y)Z$ on M is given by

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + \alpha[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X] \\ &+ \alpha\eta(Z)[\eta(Y)X - \eta(X)Y].\end{aligned} \tag{3.1}$$

Also from (3.1), we obtain

$$\tilde{S}(X,Y) = S(X,Y) + \alpha[g(X,Y) + n\eta(X)\eta(Y)], \tag{3.2}$$

where $\tilde{S}$ and S are the Ricci tensor with respect to $\tilde{\nabla}$ and ∇ respectively. Contracting (3.2), we obtain,

$$\tilde{r} = r, \tag{3.3}$$

where $\tilde{r}$ and r are the scalar curvature tensor with respect to $\tilde{\nabla}$ and ∇ respectively.

Also we have

$$\tilde{R}(\xi,X)Y = -\tilde{R}(X,\xi)Y = \alpha^2[g(X,Y))\xi - \eta(Y)X] + \alpha\eta(Y)[X + \eta(X)\xi], \tag{3.4}$$

$$\eta(\tilde{R}(X,Y)Z) = \alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{3.5}$$

$$\tilde{R}(X,Y)\xi = (\alpha^2 - \alpha)[\eta(Y)X - \eta(X)Y], \tag{3.6}$$

$$\tilde{S}(X,\xi) = \tilde{S}(\xi,X) = (n-1)(\alpha^2 - \alpha)\eta(X), \tag{3.7}$$

$$\tilde{S}(\xi,\xi) = -(n-1)(\alpha^2 - \alpha), \tag{3.8}$$

$$\tilde{Q}X = QX - \alpha(n-1)X, \tag{3.9}$$

$$\tilde{Q}\xi = (n-1)(\alpha^2 - \alpha)\xi \tag{3.10}$$

$$\tilde{R}(\xi,X)\xi = (\alpha^2 - \alpha)[X + \eta(X)\xi], \tag{3.11}$$

4 Locally ϕ-symmetric Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection

A Lorentzian α-Sasakian manifold M^n is said to be locally ϕ-symmetric with respect to the quarter-symmetric metric connection if

$$\phi^2((\tilde{\nabla}_W\tilde{R})(X,Y)Z) = 0, \tag{4.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ.

From the equation (2.16) and (3.1), we have

$$(\tilde{\nabla}_W\tilde{R})(X,Y)Z = (\nabla_W\tilde{R})(X,Y)Z + \eta(\tilde{R}(X,Y)Z)\phi W. \tag{4.2}$$

Now differentiating equation (3.1) covariantly with respect to W, we get

$$\begin{aligned}(\nabla_W\tilde{R})(X,Y)Z &= (\nabla_W R)(X,Y)Z + \alpha[g((\nabla_W\phi)(X),Z)\phi Y \\ &+ g(\phi X,Z)(\nabla_W\phi)(Y) - g((\nabla_W\phi)(Y),Z)\phi X \\ &- g(\phi Y,Z)(\nabla_W\phi)(X)] + \alpha(\nabla_W\eta)(Z)[\eta(Y)X \\ &- \eta(X)Y] + \alpha\eta(Z)[(\nabla_W\eta)(Y)X - (\nabla_W\eta)(X)Y].\end{aligned} \tag{4.3}$$

In view of the equation (2.5) and (2.7), the above equation becomes

$$
\begin{aligned}
(\nabla_W \tilde{R})(X,Y)Z = (\nabla_W R)(X,Y)Z &+ \alpha^2 g(W,X)\eta(Z)\phi Y \\
&+ \alpha^2 g(W,Z)\eta(X)\phi Y + \alpha^2 g(\phi X, Z)[g(W,Y)\xi \\
&+ \eta(Y)W] - \alpha^2 g(W,Y)\eta(Z)\phi X \\
&- \alpha^2 g(W,Z)\eta(Y)\phi X - \alpha^2 g(\phi Y, Z)[g(W,X)\xi \\
&+ \eta(X)W] + \alpha^2 g(\phi W, Z)[\eta(Y)X - \eta(X)Y] \\
&+ \alpha^2 \eta(Z).
\end{aligned} \tag{4.4}
$$

Now using the equation (3.5), (2.2) and (4.4) in (4.2), we have

$$
\begin{aligned}
\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z) &+ \alpha^2 g(W,X)\eta(Z)\phi^2(\phi Y) \\
&+ \alpha^2 g(W,Z)\eta(X)\phi^2(\phi Y) + \alpha^2 g(\phi X, Z)\eta(Y)\phi^2 W \\
&- \alpha^2 g(W,X)\eta(Z)\phi^2(\phi X) - \alpha^2 g(W,Z)\eta(Y)\phi^2(\phi X) \\
&- \alpha^2 g(\phi Y, Z)\eta(X)\phi^2 W + \alpha^2 g(\phi W, Z)\eta(Y)\phi^2 X \\
&- \alpha^2 g(\phi W, Z)\eta(X)\phi^2 Y + \alpha^2 \eta(Z)[g(\phi W, Y)\phi^2 X \\
&- g(\phi W, X)\phi^2 Y] + \alpha^2[g(Y,Z)\eta(X) \\
&- g(X,Z)\eta(Y)]\phi^2(\phi W).
\end{aligned} \tag{4.5}
$$

Consider X, Y, Z and W are orthogonal to ξ, then equation (4.5) yields

$$
\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \tag{4.6}
$$

Hence we can state the following

Theorem 4.1. *In a Lorentzian α-Sasakian manifold, the quarter-symmetric metric connection $\tilde{\nabla}$ is locally ϕ-symmetric iff the Levi-Civita connection ∇ is also locally ϕ-symmetric.*

5 ϕ-symmetric Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection

A Lorentzian α-Sasakian manifold M^n is said to be ϕ-symmetric with respect to the quarter-symmetric metric connection if

$$
\phi^2((\tilde{\nabla}_W \tilde{R})(X,Y)Z) = 0, \tag{5.1}
$$

for arbitrary vector fields X, Y, Z, W.

Let us consider a ϕ-symmetric Lorentzian α-Sasakian manifolds with respect to quarter-symmetric metric connection. Then by virtue of (2.1) and (5.1) we have

$$
((\tilde{\nabla}_W \tilde{R})(X,Y)Z) + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)\xi = 0 \tag{5.2}
$$

from which it follows that

$$
g((\tilde{\nabla}_W \tilde{R})(X,Y)Z, U) + \eta((\tilde{\nabla}_W \tilde{R})(X,Y)Z)g(\xi, U) = 0 \tag{5.3}
$$

Let e_i, $i = 1, 2, \ldots, n$ be an orthonormal basis of the tangent space at any pointof the manifold. Then putting $X = U = e_i$ in (5.3) and taking summation over i, $1 \leq i \leq n$, we have

$$(\tilde{\nabla}_W \tilde{S})(Y, Z) + \Sigma_{i=1}^{n} \eta((\tilde{\nabla}_W \tilde{R})(e_i, Y)Z)\eta(e_i) = 0 \tag{5.4}$$

The second term of (5.4) by putting $Z = \xi$ takes the form

$$\eta((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi)\eta(e_i) = g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi)g(e_i, \xi), \tag{5.5}$$

By using (2.16) and (4.2), we can write

$$g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = g((\nabla_W \tilde{R})(e_i, Y)\xi, \xi) + \eta(\tilde{R}(e_i, Y)\xi)\phi W \tag{5.6}$$

After some calculations, from (5.6) we have

$$g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = g((\nabla_W \tilde{R})(e_i, Y)\xi, \xi). \tag{5.7}$$

In Lorentzian α-Sasakian manifold, we have

$$g((\nabla_W \tilde{R})(e_i, Y)\xi, \xi) = 0.$$

So from (5.7) we get

$$g((\tilde{\nabla}_W \tilde{R})(e_i, Y)\xi, \xi) = 0 \tag{5.8}$$

By replacing $Z = \xi$ in (5.4) and using (5.8), we get

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = 0 \tag{5.9}$$

we know that

$$(\tilde{\nabla}_W \tilde{S})(Y, \xi) = \tilde{\nabla}_W \tilde{S}(Y, \xi) - \tilde{S}(\tilde{\nabla}_W Y, \xi) - \tilde{S}(Y, \tilde{\nabla}_W \xi). \tag{5.10}$$

Now using (2.6), (2.12), (2.16) and (3.7), we obtain

$$\begin{aligned}(\tilde{\nabla}_W \tilde{S})(Y, \xi) &= (n-1)(\alpha^2 - \alpha)\alpha g(Y, \phi W) \\ &- (\alpha - 1)[S(Y, \phi W) + \alpha g(Y, \phi W)]\end{aligned} \tag{5.11}$$

Applying (5.11) in (5.9), we obtain

$$S(Y, \phi W) = g(Y, \phi W)[(n-1)\alpha^2 - \alpha] \tag{5.12}$$

Replacing W by ϕW we get

$$S(Y, W) = g(Y, W)[(n-1)\alpha^2 - \alpha] - \alpha\eta(Y)\eta(W), \tag{5.13}$$

Contracting (5.13), we get

$$r = (n-1)\alpha[n\alpha - 1] \tag{5.14}$$

This leads to the following theorem

Theorem 5.1. *Let M be a ϕ-symmetric Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection $\tilde{\nabla}$. Then the manifold has a scalar curvature $r = (n-1)\alpha[n\alpha - 1]$ with respect to Levi-Civita connection ∇ of M.*

6 Locally Projective ϕ-symmetric Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection

A Lorentzian α-Sasakian manifold M^n is said to be locally projective ϕ-symmetric with respect to the quarter-symmetric metric connection if

$$\phi^2((\tilde{\nabla}_W \tilde{P})(X,Y)Z) = 0, \tag{6.1}$$

for all vector fields X, Y, Z, W orthogonal to ξ, where $\tilde{P}$ is the projective curvature tensor defined as follows:

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y], \tag{6.2}$$

where $\tilde{R}$ and $\tilde{S}$ are the Riemannian curvature tensor and Ricci tensor with respect to quarter-symmetric metric connection $\tilde{\nabla}$.

Using equation (2.16), we can write

$$(\tilde{\nabla}_W \tilde{P})(X,Y)Z = (\nabla_W \tilde{P})(X,Y)Z + \eta(\tilde{P}(X,Y)Z)\phi W, \tag{6.3}$$

Now differentiating equation (6.2) with respect to W, we get

$$\begin{aligned}(\nabla_W \tilde{P})(X,Y)Z &= (\nabla_W \tilde{R})(X,Y)Z \\ &- \frac{1}{n-1}[(\nabla_W \tilde{S})(Y,Z)X - (\nabla_W \tilde{S})(X,Z)Y].\end{aligned} \tag{6.4}$$

In view of equations (4.4) and (3.2) above equation reduces to

$$\begin{aligned}(\nabla_W \tilde{P})(X,Y)Z = {} & (\nabla_W R)(X,Y)Z + \alpha^2 g(W,X)\eta(Z)\phi Y \\ &+ \alpha^2 g(W,Z)\eta(X)\phi Y + \alpha^2 g(\phi X, Z)[g(W,Y)\xi \\ &+ \eta(Y)W] - \alpha^2 g(W,Y)\eta(Z)\phi X \\ &- \alpha^2 g(W,Z)\eta(Y)\phi X - \alpha^2 g(\phi Y, Z)[g(W,X)\xi \\ &+ \eta(X)W] + \alpha^2 g(\phi W, Z)[\eta(Y)X - \eta(X)Y] \\ &+ \alpha^2 \eta(Z) - \frac{1}{n-1}[(\nabla_W S)(Y,Z)X - (\nabla_W S)(X,Z)Y \\ &+ \alpha^2 n\{g(\phi W, Y)\eta(Z)X + g(\phi W, Z)\eta(Y)X\} \\ &- \alpha^2 n\{g(\phi W, X)\eta(Z)Y + \phi W, Z)\eta(X)Y\}],\end{aligned} \tag{6.5}$$

which on using equation (6.2) reduces to

$$\begin{aligned}(\nabla_W \tilde{P})(X,Y)Z = &(\nabla_W P)(X,Y)Z + \alpha^2 g(W,X)\eta(Z)\phi Y \\ &+ \alpha^2 g(W,Z)\eta(X)\phi Y + \alpha^2 g(\phi X, Z)[g(W,Y)\xi \\ &+ \eta(Y)W] - \alpha^2 g(W,Y)\eta(Z)\phi X \\ &- \alpha^2 g(W,Z)\eta(Y)\phi X - \alpha^2 g(\phi Y, Z)[g(W,X)\xi \\ &+ \eta(X)W] + \alpha^2 g(\phi W, Z)[\eta(Y)X - \eta(X)Y] \\ &+ \alpha^2\eta(Z) - \frac{\alpha^2 n}{n-1}[\{g(\phi W, Y)\eta(Z)X \\ &+ g(\phi W, Z)\eta(Y)X\} - \{g(\phi W, X)\eta(Z)Y + \phi W, Z)\eta(X)Y\}]. \end{aligned} \tag{6.6}$$

Now using (3.5) on (6.2), we have

$$\begin{aligned}\eta(\tilde{P}(X,Y)Z) = \;&\alpha^2[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)] - \frac{1}{n-1}[\tilde{S}(Y,Z)\eta(X) \\ &- \tilde{S}(X,Z)\eta(Y)]. \end{aligned} \tag{6.7}$$

Applying the equations (2.2), (6.6) and (6.7) in (6.3), we get

$$\begin{aligned}\phi^2((\tilde{\nabla}_W \tilde{P})(X,Y)Z) = &\phi^2((\nabla_W P)(X,Y)Z) + \alpha^2 g(W,X)\eta(Z)\phi^2(\phi Y) \\ &+ \alpha^2 g(W,Z)\eta(X)\phi^2(\phi Y) + \alpha^2 g(\phi X, Z)\eta(Y)\phi^2 W \\ &- \alpha^2 g(W,X)\eta(Z)\phi^2(\phi X) - \alpha^2 g(W,Z)\eta(Y)\phi^2(\phi X) \\ &- \alpha^2 g(\phi Y, Z)\eta(X)\phi^2 W + \alpha^2 g(\phi W, Z)\eta(Y)\phi^2 X \\ &- \alpha^2 g(\phi W, Z)\eta(X)\phi^2 Y + \alpha^2\eta(Z)[g(\phi W, Y)\phi^2 X \\ &- g(\phi W, X)\phi^2 Y] + \alpha^2[g(Y,Z)\eta(X) \\ &- g(X,Z)\eta(Y)]\phi^2(\phi W) - \frac{1}{n-1}[\tilde{S}(Y,Z)\eta(X) \\ &- \tilde{S}(X,Z)\eta(Y)]\phi^2(\phi W) - \frac{\alpha^2 n}{n-1}[\{g(\phi W, Y)\eta(Z)\phi^2 X \\ &+ g(\phi W, Z)\eta(Y)\phi^2 X\} - \{g(\phi W, X)\eta(Z)\phi^2 Y \\ &+ \phi W, Z)\eta(X)\phi^2 Y\}]. \end{aligned} \tag{6.8}$$

By assuming X, Y, Z, W orthogonal to ξ, above equation reduces to

$$\phi^2((\tilde{\nabla}_W \tilde{P})(X,Y)Z) = \phi^2((\nabla_W P)(X,Y)Z). \tag{6.9}$$

Hence we can state as follows:

Theorem 6.1. *A n-dimensional Lorentzian α-Sasakian manifold is locally projective ϕ-symmetric with respect to quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is locally projective ϕ-symmetric with respect to the Levi-Civita connection ∇.*

Again using the equations (2.2), (6.5) and (6.7) in (6.3), we get

$$\begin{aligned}\phi^2((\tilde{\nabla}_W\tilde{P})(X,Y)Z) &= \phi^2((\nabla_W R)(X,Y)Z) + \alpha^2 g(W,X)\eta(Z)\phi^2(\phi Y)\\
&+ \alpha^2 g(W,Z)\eta(X)\phi^2(\phi Y) + \alpha^2 g(\phi X,Z)\eta(Y)\phi^2 W\\
&- \alpha^2 g(W,X)\eta(Z)\phi^2(\phi X) - \alpha^2 g(W,Z)\eta(Y)\phi^2(\phi X)\\
&- \alpha^2 g(\phi Y,Z)\eta(X)\phi^2 W + \alpha^2 g(\phi W,Z)\eta(Y)\phi^2 X\\
&- \alpha^2 g(\phi W,Z)\eta(X)\phi^2 Y + \alpha^2\eta(Z)[g(\phi W,Y)\phi^2 X\\
&- g(\phi W,X)\phi^2 Y] + \alpha^2[g(Y,Z)\eta(X)\\
&- g(X,Z)\eta(Y)]\phi^2(\phi W) - \frac{1}{n-1}[\tilde{S}(Y,Z)\eta(X)\\
&- \tilde{S}(X,Z)\eta(Y)]\phi^2(\phi W) - \frac{\alpha^2 n}{n-1}[\{g(\phi W,Y)\eta(Z)\phi^2 X\\
&+ g(\phi W,Z)\eta(Y)\phi^2 X\} - \{g(\phi W,X)\eta(Z)\phi^2 Y\\
&+ \phi W,Z)\eta(X)\phi^2 Y\}] - \frac{1}{n-1}[(\nabla_W S)(Y,Z)\phi^2 X\\
&- (\nabla_W S)(X,Z)\phi^2 Y]. \qquad (6.10)\end{aligned}$$

Taking X, Y, Z, W orthogonal to ξ in equation (6.10), we obtain by some calculation

$$\phi^2((\tilde{\nabla}_W\tilde{P})(X,Y)Z) = \phi^2((\nabla_W R)(X,Y)Z). \qquad (6.11)$$

Hence we can state as follows:

Theorem 6.2. *An n-dimensional Lorentzian α-Sasakian manifold is locally projective ϕ-symmetric with respect to quarter-symmetric metric connection $\tilde{\nabla}$ if and only if it is locally ϕ-symmetric with respect to the Levi-Civita connection ∇.*

7 ξ-projectively flat Lorentzian α-Sasakian manifold with respect to the quarter-symmetric metric connection

A Lorentzian α-Sasakian manifold M^n with respect to the quarter-symmetric metric connection is said to be ξ projective flat if

$$\tilde{P}(X,Y)\xi = 0, \qquad (7.1)$$

for all vector fields $X, Y \in \chi(M)$. This notion was first defined by Tripathi and Dwivedi ([21]). If equation (7.1) holds for X, Y orthogonal to ξ, we called such a manifold a horizontal ξ-projectively flat manifold.

Using (3.1) in (6.2), we get

$$\begin{aligned}\tilde{P}(X,Y)Z &= R(X,Y)Z + \alpha[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X]\\
&+ \alpha\eta(Z)[\eta(Y)X - \eta(X)Y] - \frac{1}{n-1}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y]. \qquad (7.2)\end{aligned}$$

Putting $Z = \xi$ and using (2.2), (2.10) and (3.7) in (7.2), we get

$$\tilde{P}(X,Y)\xi = 0. \qquad (7.3)$$

Hence we state the following theorem:

Theorem 7.1. *A n-dimensional Lorentzian α-Sasakian manifold is ξ-projectively flat with respect to the quarter-symmetric metric connection.*

Now using (3.2) in (7.2), we have

$$\begin{aligned}\tilde{P}(X,Y)Z &= R(X,Y)Z \\ &\quad + \alpha[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X] + \alpha\eta(Z)[\eta(Y)X - \eta(X)Y] \\ &\quad - \frac{1}{n-1}[S(Y,Z)X + \alpha X\{g(Y,Z) + n\eta(Y)\eta(Z)\} \\ &\quad - S(X,Z)Y - \alpha Y\{g(X,Z) + n\eta(X)\eta(Z)\}] \qquad (7.4)\end{aligned}$$

In view of (1.6), the above equation becomes

$$\begin{aligned}\tilde{P}(X,Y)Z &= P(X,Y)Z + \alpha[g(\phi X,Z)\phi Y - g(\phi Y,Z)\phi X] \\ &\quad + \alpha\eta(Z)[\eta(Y)X - \eta(X)Y] - \frac{1}{n-1}[\alpha X\{g(Y,Z) \\ &\quad + n\eta(Y)\eta(Z)\} - \alpha Y\{g(X,Z) + n\eta(X)\eta(Z)\}], \qquad (7.5)\end{aligned}$$

where P be the projective curvature tensor with respect to the Levi-Civita connection.

Putting $Z = \xi$ in (7.5) and using (2.2), it follows that

$$\begin{aligned}\tilde{P}(X,Y)\xi &= P(X,Y)\xi - \alpha[\eta(Y)X - \eta(X)Y] \\ &\quad - \frac{1}{n-1}[\alpha X\eta(Y) - n\alpha X\eta(Y) - \alpha Y\eta(X) + n\alpha Y\eta(X)]. \qquad (7.6)\end{aligned}$$

It implies that

$$\tilde{P}(X,Y)\xi = P(X,Y)\xi; \qquad (7.7)$$

$\forall X, Y$ orthogonal to ξ.

In view of above discussions we can state the following theorem:

Theorem 7.2. *A n-dimensional Lorentzian ξ-Sasakian manifold is horizontal ξ-projectively flat with respect to the semi-symmetric metric connection if and only if the manifold is ξ-projectively flat with respect to the Levi-Civita connection.*

8 Example of 3-dimensional Lorentzian α-Sasakian manifold with respect to quarter-symmetric metric connection

We consider a 3-dimensional manifold $M = \{(x,y,u) \in R^3\}$, where (x,y,u) are the standard coordinates of R^3. Let e_1, e_2, e_3 be the vector fields on M^3 given by

$$e_1 = e^u\frac{\partial}{\partial x}, \quad e_2 = e^u\frac{\partial}{\partial y}, \quad e_3 = e^u\frac{\partial}{\partial u}.$$

Clearly, $\{e_1, e_2, e_3\}$ is a set of linearly independent vectors for each point of M and hence a basis of $\chi(M)$. The Lorentzian metric g is defined by

$$g(e_1, e_2) = g(e_2, e_3) = g(e_1, e_3) = 0,$$
$$g(e_1, e_1) = 1, \quad g(e_2, e_2) = 1, \quad g(e_3, e_3) = -1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$ and the $(1, 1)$ tensor field ϕ is defined by

$$\phi e_1 = -e_1, \quad \phi e_2 = -e_2, \quad \phi e_3 = 0.$$

From the linearity of ϕ and g, we have

$$\eta(e_3) = -1, \qquad \phi^2 X = X + \eta(X)e_3$$

and

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for any $X \in \chi(M)$. Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian paracontact structure on M.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then we have

$$[e_1, e_2] = 0, \quad [e_1, e_3] = e_1 e^{-u}, \quad [e_2, e_3] = e_2 e^{-u}.$$

Koszul's formula is defined by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

Then from above formula we can calculate the followings:

$$\nabla_{e_1} e_1 = -e_3 e^u, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = -e_1 e^u,$$
$$\nabla_{e_2} e_1 = 0, \quad \nabla_{e_2} e_2 = -e_3 e^u, \quad \nabla_{e_2} e_3 = -e_2 e^u,$$
$$\nabla_{e_3} e_1 = 0, \quad \nabla_{e_3} e_2 = 0, \quad \nabla_{e_3} e_3 = 0.$$

From the above calculations, we see that the manifold under consideration satisfies $\eta(\xi) = -1$ and $\nabla_X \xi = \alpha \phi X$ for $\alpha = e^u$.

Hence the structure (ϕ, ξ, η, g) is a Lorentzian α-Sasakian manifold.

Using (2.16), we find $\tilde{\nabla}$, the quarter-symmetric metric connection on M following:

$$\tilde{\nabla}_{e_1} e_1 = -e_3 e^u, \quad \tilde{\nabla}_{e_1} e_2 = 0, \quad \tilde{\nabla}_{e_1} e_3 = e_1(1 - e^u),$$
$$\tilde{\nabla}_{e_2} e_1 = 0, \quad \tilde{\nabla}_{e_2} e_2 = -e_3 e^u, \quad \tilde{\nabla}_{e_2} e_3 = e_2(1 - e^u),$$
$$\tilde{\nabla}_{e_3} e_1 = 0, \quad \tilde{\nabla}_{e_3} e_2 = 0, \quad \tilde{\nabla}_{e_3} e_3 = 0.$$

Using (1.2), the torson tensor T, with respect to quarter-symmetric metric connection $\tilde{\nabla}$ as follows:

$$\tilde{T}(e_i, e_i) = 0, \quad \forall i = 1, 2, 3,$$
$$\tilde{T}(e_1, e_2) = 0, \quad \tilde{T}(e_1, e_3) = e_1, \quad \tilde{T}(e_2, e_3) = e_2.$$

Also,

$$(\tilde{\nabla}_{e_1} g)(e_2, e_3) = 0, \quad (\tilde{\nabla}_{e_2} g)(e_3, e_1) = 0, \quad (\tilde{\nabla}_{e_3} g)(e_1, e_2) = 0.$$

Thus M is Lorentzian α-Sasakian manifold with quarter-symmetric metric connection $\tilde{\nabla}$.

By using the above results, we can easily obtain the components of the curvature tensor as follows:

$$R(e_1, e_3)e_3 = -e_1\alpha^2, \quad R(e_2, e_1)e_1 = e_2\alpha^2, \quad R(e_2, e_3)e_3 = -e_2\alpha^2,$$
$$R(e_3, e_1)e_1 = e_3\alpha^2, \quad R(e_3, e_2)e_2 = e_3\alpha^2, \quad R(e_1, e_2)e_3 = 0,$$
$$R(e_2, e_3)e_2 = -e_3\alpha^2, \quad R(e_1, e_2)e_2 = e_1\alpha^2$$

and

$$\tilde{R}(e_1, e_3)e_3 = e_1(\alpha - \alpha^2), \quad \tilde{R}(e_2, e_1)e_1 = e_2(\alpha^2 - \alpha),$$
$$\tilde{R}(e_2, e_3)e_3 = e_2(\alpha - \alpha^2), \quad \tilde{R}(e_3, e_1)e_1 = e_3\alpha^2,$$
$$\tilde{R}(e_3, e_2)e_2 = e_3\alpha^2, \quad \tilde{R}(e_1, e_2)e_3 = 0, \quad \tilde{R}(e_2, e_3)e_2 = -e_3\alpha^2,$$
$$\tilde{R}(e_1, e_2)e_2 = e_1(\alpha^2 - \alpha).$$

Using the expressions of the curvature tensors, we find the values of the Ricci tensors as follows:

$$S(e_1, e_1) = 0, \quad S(e_2, e_2) = 0, \quad S(e_3, e_3) = -2\alpha^2,$$
$$S(e_1, e_2) = 0, \quad S(e_2, e_3) = 0, \quad S(e_1, e_3) = 0$$

and

$$\tilde{S}(e_1, e_1) = \alpha, \quad \tilde{S}(e_2, e_2) = \alpha, \quad \tilde{S}(e_3, e_3) = 2(\alpha - \alpha^2),$$
$$\tilde{S}(e_1, e_2) = 0, \quad \tilde{S}(e_2, e_3) = 0, \quad \tilde{S}(e_1, e_3) = 0.$$

By the above expressions and using the definition of Lorentzian α-Sasakian manifold, one can easily see that Theorems 4.1, 6.1 and 6.2 are verified below:

$$\phi^2((\tilde{\nabla}_W \tilde{R})(X, Y)Z) = \phi^2((\nabla_W R)(X, Y)Z),$$
$$\phi^2((\tilde{\nabla}_W \tilde{P})(X, Y)Z) = \phi^2((\nabla_W P)(X, Y)Z),$$
$$\phi^2((\tilde{\nabla}_W \tilde{P})(X, Y)Z) = \phi^2((\nabla_W R)(X, Y)Z).$$

Let X and Y are any two vector fields given by $X = a_1e_1 + a_2e_2 + a_3e_3$ and $Y = b_1e_1 + b_2e_2 + b_3e_3$.

Using (6.2) and above relations, we get

$$\tilde{P}(X,Y)\xi = 0,$$

which verifies the Theorem 7.1.

Acknowledgement. The authors wish to express their sincere thanks and gratitude to the referee for valuable suggestions towards the improvement of the paper.

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