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Projective Curvature Tensor in 3-dimensional Connected Trans-Sasakian Manifolds

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Abstract

The object of the present paper is to study ξ-projectively flat and ϕ-projectively flat 3-dimensional connected trans-Sasakian manifolds. Also we study the geometric properties of connected trans-Sasakian manifolds when it is projectively semi-symmetric. Finally, we give some examples of a 3-dimensional trans-Sasakian manifold which verifies our result.

Key words: Trans-Sasakian manifold, ξ-projectively flat, ϕ-projectively flat, Einstein manifold.

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1 Introduction

Trans-Sasakian manifolds arose in a natural way from the classification of almost contact metric structures by Chinea and Gonzales [6] and they appear as a natural generalization of both Sasakian and Kenmotsu manifolds. Again in the Gray–Hervella classification of almost Hermite manifolds [11], there appears a class W_4 of Hermitian manifolds which are closely related to locally conformally Kähler manifolds. An almost contact metric structure on a manifold M is called a trans-Sasakian structure [21] if the product manifold $M \times \mathbb{R}$ belongs to the class W_4. The class $C_6 \oplus C_5$ ([15, 16]) coincides with the class of trans-Sasakian structures of type (α, β). In [16], the local nature of the two subclasses C_5 and C_6

of trans-Sasakian structures is characterized completely. In [7], some curvature identities and sectional curvatures for C_5, C_6 and trans-Sasakian manifolds are obtained. It is known that [12] trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$, and $(\alpha, 0)$ are cosymplectic, β-Kenmotsu and α-Sasakian respectively where $\alpha, \beta \in \mathbb{R}$.

The local structure of trans-Sasakian manifolds of dimension $n \geq 5$ has been completely characterized by Marrero [15]. He proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu manifold. Hence proper trans-Sasakian manifold exists only for three dimension. In this context we can mention that some authors have studied $(2n+1)$-dimensional trans-Sasakian manifolds, such as ([1, 13]) and many others. But these results are not true for proper trans-Sasakian manifolds. Three-dimensional trans-Sasakian manifolds have been studied by De and Tripathi [10], De and Sarkar [9], De and De [8], Shukla and Singh [23] and many others. Sasakian spaces were studied by [17, 19, 18].

The projective curvature tensor is an important tensor from the differential geometric point of view. Let M be a n-dimensional Riemannian manifold. If there exist an one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidian space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 3$, M is locally projectively flat if and only if the well known projective curvature tensor P vanishes. Here P is defined by [20]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}, \tag{1.1}$$

for $X, Y, Z \in T(M)$, where R is the curvature tensor and S is the Ricci tensor. In fact, M is projectively flat (that is, $P = 0$) if and only if the manifold is of constant curvature [26, pp. 84–85]. Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature. A Riemannian or a semi-Riemannian manifold is said to be *semi-symmetric* ([14, 18, 24, 25]) if $R(X,Y).R = 0$, where R is the Riemannian curvature tensor and $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y. If a Riemannian manifold satisfies $R(X,Y).P = 0$, then the manifold is said to be projectively semi-symmetric manifold. In [18, p. 286, p. 329] there is proved that projectively semi-symmetric spaces are semi-symmetric.

The paper is organized as follows. In section 2, some preliminary results are recalled. After preliminaries in section 3, we prove that a 3-dimensional compact connected trans-Sasakian manifold is ξ-projectively flat if and only if the manifold is α-Sasakian. In the next section, we prove that a 3-dimensional connected trans-Sasakian manifold is ϕ-projectively flat if and only if it is an Einstein manifold provided $\alpha, \beta =$ constant. In section 5, we prove that a 3-dimensional connected trans-Sasakian manifold is projectively semisymmetric if and only if the manifold is projectively flat, provided $\phi(\operatorname{grad}\alpha) = \operatorname{grad}\beta$.

Finally, we construct some examples of a 3-dimensional trans-Sasakian manifold with constant function α, β on M.

2 Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an (1,1) tensor field, ξ is a vector field, η is a 1-form and g is a compatible Riemannian metric such that

$$\phi^2(X) = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \ \phi\xi = 0, \ \eta\phi = 0 \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X, \xi) = \eta(X) \tag{2.3}$$

for all X and Y tangent to M ([2, 3]).

The fundamental 2-form of the manifold is defined by

$$\Phi(X, Y) = g(X, \phi Y) \tag{2.4}$$

for all X and Y tangent to M.

An almost contact metric structure (ϕ, ξ, η, g) on a connected manifold M is called a trans-Sasakian structure [21] if $(M \times \mathbb{R}, \mathrm{J}, \mathrm{G})$ belongs to the class W_4 [11], where J is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(X, f\tfrac{d}{df}) = (\phi X - \mathrm{f}\xi, \eta(X)\tfrac{d}{dt}),$$

for any vector fields X on M, f is a smooth function on $M \times \mathbb{R}$ and G is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [4]

$$(\nabla_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.5}$$

for smooth functions α and β on M. Hence we say that the trans-Sasakian structure is of type (α, β). From (2.5) it follows that

$$\nabla_X \phi = -\alpha(\phi X) + \beta(X - \eta(X)\xi), \tag{2.6}$$

$$(\nabla_X \phi)Y = -\alpha g(\phi X, Y) + \beta g(\phi X, \phi Y). \tag{2.7}$$

An explicit example of a 3-dimensional proper trans-Sasakian manifold is constructed in [15]. In [10], Ricci tensor and curvature tensor for 3-dimensional trans-Sasakian manifolds are studied and their explicit formulae are given. From [10] we know that for a 3-dimensional trans-Sasakian manifold

$$2\alpha\beta + \xi\alpha = 0, \tag{2.8}$$

$$S(X, \xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(X) - X\beta - (\phi X)\alpha, \tag{2.9}$$

$$\begin{aligned} S(X, Y) = {} & \left(\tfrac{r}{2} + \xi\beta - (\alpha^2 - \beta^2)\right) g(X, Y) \\ & - \left(\tfrac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2)\right) \eta(X)\eta(Y) \\ & - (Y\beta + (\phi Y)\alpha)\eta(X) - (X\beta + (\phi X)\alpha)\eta(Y), \end{aligned} \tag{2.10}$$

$$
\begin{aligned}
R(X,Y)\xi = (\alpha^2-\beta^2)(\eta(Y)X-\eta(X)Y)\\
-\eta(Y)(X\beta)\xi+\phi(X)\alpha\xi+\eta(X)(Y\beta)\xi+\phi(Y)\alpha\xi\\
-(Y\beta)X+(X\beta)Y-(\phi(Y)\alpha)X+(\phi(X)\alpha)Y, \quad (2.11)
\end{aligned}
$$

and

$$
\begin{aligned}
R(X,Y)Z = \left(\tfrac{r}{2}+2\xi\beta-2(\alpha^2-\beta^2)\right)(g(Y,Z)X-g(X,Z)Y\\
-g(Y,Z)\big[\left(\tfrac{r}{2}+\xi\beta-3(\alpha^2-\beta^2)\right)\eta(X)\xi\\
-\eta(X)(\phi\,\mathrm{grad}\,\alpha-\mathrm{grad}\,\beta)+(X\beta+(\phi X)\alpha)\xi\big]\\
+g(X,Z)\big[\left(\tfrac{r}{2}+\xi\beta-3(\alpha^2-\beta^2)\right)\eta(Y)\xi\\
-\eta(Y)(\phi\,\mathrm{grad}\,\alpha-\mathrm{grad}\,\beta)+(Y\beta+(\phi Y)\alpha)\xi\big]\\
-\big[(Z\beta+(\phi Z)\alpha)\eta(Y)+(Y\beta+(\phi Y)\alpha)\eta(Z)\\
+\tfrac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(Y)\eta(Z)\big]X\\
+\big[(Z\beta+(\phi Z)\alpha)\eta(X)+(X\beta+(\phi X)\alpha)\eta(Z)\\
+\tfrac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(X)\eta(Z)\big]Y, \quad (2.12)
\end{aligned}
$$

where S is the Ricci tensor of type (0,2) and R is the curvature tensor of type (1,3) and r is the scalar curvature of the manifold M.

3 3-dimensional ξ-projectively flat trans-Sasakian manifolds

ξ-conformally flat K-contact manifolds have been studied by Zhen, Cabrerizo and Fernandez [28]. In this section we study ξ-projectively flat connected transSasakian manifolds. Analogous to the definition of ξ-conformally flat K-contact manifold we define ξ-projectively flat connected trans-Sasakian manifolds.

Definition 3.1. A connected trans-Sasakian manifold M is called *ξ-projectively flat* if the condition $P(X,Y)\xi=0$ holds on M, where projective curvature tensor P is defined by (1.1).

Putting $Z=\xi$ in (1.1) and using (2.9) and (2.11), we get

$$
\begin{aligned}
P(X,Y)\xi = -\tfrac{1}{2}\{(Y\beta)X-(X\beta)Y\}+\{(Y\beta)\eta(X)-(X\beta)\eta(Y)\}\xi\\
+(Y\alpha)\phi X-(X\alpha)\phi Y+2\alpha\beta\{\eta(Y)\phi X-\eta(X)\phi Y\}\\
+\tfrac{1}{2}[(\phi Y)\alpha X-(\phi X)\alpha Y+(\xi\beta)\{\eta(Y)X-\eta(X)Y\}]. \quad (3.1)
\end{aligned}
$$

Now assume that M is a 3-dimensional compact connected ξ-projectively flat trans-Sasakian manifold. Then from (3.1) we can write

$$-\tfrac{1}{2}\{(Y\beta)X-(X\beta)Y\}+\{(Y\beta)\eta(X)-(X\beta)\eta(Y)\}\xi$$
$$+(Y\alpha)\phi X-(X\alpha)\phi Y+2\alpha\beta\{\eta(Y)\phi X-\eta(X)\phi Y\}$$
$$+\tfrac{1}{2}[(\phi Y)\alpha X-(\phi X)\alpha Y+(\xi\beta)(\eta(Y)X-\eta(X)Y)\}]=0. \quad (3.2)$$

Putting $Y=\xi$ in the above equation and using (2.8), we obtain

$$(X\beta)\xi+(\phi X)\alpha\xi-(\xi\beta)\eta(X)\xi=0$$

which implies

$$(X\beta)+(\phi X)\alpha-(\xi\beta)\eta(X)=0. \quad (3.3)$$

The gradient of the function β is related to the exterior derivative $d\beta$ by the formula

$$d\beta(X)=g(\operatorname{grad}\beta,X). \quad (3.4)$$

Using (3.4) in (3.3) we obtain

$$d\beta(X)+g(\operatorname{grad}\alpha,\phi X)-d\beta(\xi)\eta(X)=0. \quad (3.5)$$

Differentiating (3.5) covariantly along Y, we get

$$(\nabla_Y d\beta)(X)+g(\nabla_Y\operatorname{grad}\alpha,\phi X)+g(\operatorname{grad}\alpha,(\nabla_Y\phi)X)$$
$$-(\nabla_Y d\beta)\xi\eta(X)-(\xi\beta)(\nabla_Y\eta)(X)=0. \quad (3.6)$$

Hence, by antisymmetrization with respect to X and Y, we have

$$g(\nabla_Y\operatorname{grad}\alpha,\phi X)-g(\nabla_X\operatorname{grad}\alpha,\phi Y)$$
$$+((\nabla_Y\phi)X-(\nabla_X\phi)Y)\alpha-(\nabla_Y d\beta)\xi\eta(X)+(\nabla_X d\beta)\xi\eta(Y)$$
$$-(\xi\beta)\{(\nabla_Y\eta)(X)-(\nabla_X\eta)(Y)\}=0. \quad (3.7)$$

From (2.4) and (2.7) we get

$$(\nabla_X\eta)Y-(\nabla_Y\eta)X=\alpha\Phi((X,Y)-\Phi(Y,X))=2\alpha\Phi(X,Y). \quad (3.8)$$

Using (3.8) in (3.7) we have

$$g(\nabla_Y\operatorname{grad}\alpha,\phi X)-g(\nabla_X\operatorname{grad}\alpha,\phi Y)+\{(\nabla_Y\phi)X\alpha-(\nabla_X\phi)Y\alpha\}$$
$$-(\nabla_Y d\beta)\xi\eta(X)+(\nabla_X d\beta)\xi\eta(Y)+2\alpha(\xi\beta)\Phi(X,Y)=0. \quad (3.9)$$

Let $\{e_1,e_2,\xi\}$ be an orthonormal ϕ-basis where $\phi e_1=-e_2$ and $\phi e_2=e_1$. Taking $X=e_1$ and $Y=e_2$ in (3.7), we find that

$$g(\nabla_{e_1}\operatorname{grad}\alpha,e_1)+g(\nabla_{e_2}\operatorname{grad}\alpha,e_2)=2\beta(\xi\alpha)+2\alpha(\xi\beta). \quad (3.10)$$

On the other hand (2.8) yields $g(\operatorname{grad}\alpha,\xi) = -2\alpha\beta$, whence by covariant differentiation we get, on account of (2.1)

$$g(\nabla_\xi \operatorname{grad}\alpha,\xi) = 2\alpha(\xi\beta) - 2\beta(\xi\alpha). \tag{3.11}$$

From (3.10) and (3.11) we get $\Delta\alpha = 0$, where Δ is the Laplacian defined by

$$\Delta\alpha = \sum_{i=0}^{2} g(\nabla_{e_i} \operatorname{grad}\alpha, e_i).$$

Since M is compact, we get α is constant.

Now if $\alpha \neq 0$, (2.8) implies $\beta = 0$. This implies M is a α-Sasakian manifold.

Conversely, if M is a α-Sasakian manifold, then from (3.1) it is easy to see that $P(X,Y)\xi = 0$. Hence we can state the following:

Theorem 3.1. *A 3-dimensional compact connected trans-Sasakian manifold is ξ-projectively flat if and only if it is a α-Sasakian manifold.*

4 3-dimensional ϕ-projectively flat trans-Sasakian manifolds

Analogous to the definition of ϕ-conformally flat contact metric manifold [5], we define ϕ-projectively flat trans-Sasakian manifold. In this connection we can mention the work of Ozgur [22] who has studied ϕ-projectively flat Lorentzian Para-Sasakian manifolds.

Definition 4.1. A 3-dimensional trans-Sasakian manifold satisfying the condition

$$\phi^2 P(\phi X, \phi Y)\phi Z = 0 \tag{4.1}$$

is called *ϕ-projectively flat.*

Let us assume that M is a 3-dimensional connected ϕ-projectively flat trans-Sasakian manifold. It can be easily seen that $\phi^2 P(\phi X, \phi Y)\phi Z = 0$ holds if and only if

$$g(P(\phi X, \phi Y)\phi Z, \phi W) = 0,$$

for X, Y, Z, $W \in T(M)$.

Using (1.1) and (2.1), ϕ-projectively flat means

$$g(R(\phi X,\phi Y)\phi Z,\phi W) = \tfrac{1}{2}\{S(\phi Y,\phi Z)g(\phi X,\phi W) - S(\phi X,\phi Z)g(\phi Y,\phi W)\}. \tag{4.2}$$

Let $\{e_1, e_2, \xi\}$ be a local orthonormal basis of the vector fields in M. Using the fact that $\{\phi e_1, \phi e_2, \xi\}$ is also a local orthonormal basis, if we put $X = W = e_i$ in (4.2) and summing up with respect to i, then we have

$$\sum_{i=1}^{2} g(R(\phi e_i,\phi Y)\phi Z,\phi e_i) = \frac{1}{2}\sum_{i=1}^{2}\{S(\phi Y,\phi Z)g(\phi e_i,\phi e_i) - S(\phi e_i,\phi Z)g(\phi Y,\phi e_i)\}. \tag{4.3}$$

It can be easily verified that

$$\sum_{i=1}^{2} g(R(\phi e_i, \phi Y)\phi Z, \phi e_i) = S(\phi Y, \phi Z) + (\xi\beta - \alpha^2 + \beta^2)g(\phi Y, \phi Z), \tag{4.4}$$

$$\sum_{i=1}^{2} g(\phi e_i, \phi e_i) = 2, \tag{4.5}$$

$$\sum_{i=1}^{2} S(\phi e_i, \phi Z)g(\phi Y, \phi e_i) = S(\phi Y, \phi Z). \tag{4.6}$$

So using (2.2), the equation (4.3) becomes

$$\left(\tfrac{r}{2} + 3(\xi\beta - \alpha^2 + \beta^2)\right)\{g(Y, Z) - \eta(Y)\eta(Z)\} = 0$$

which gives $r = -6(\xi\beta - \alpha^2 + \beta^2)$. So we state the following:

Proposition 4.1. *The scalar curvature r of a 3-dimensional connected ϕ-projectively flat trans-Sasakian manifold is $r = -6(\xi\beta - \alpha^2 + \beta^2)$.*

Also if $r = -6(\xi\beta - \alpha^2 + \beta^2)$, it follows from (2.10) that the manifold is an Einstein manifold provided $\alpha, \beta =$ constant. Hence we can state the following:

Proposition 4.2. *A 3-dimensional connected ϕ-projectively flat trans-Sasakian manifold is an Einstein manifold, provided $\alpha, \beta =$ constant.*

It is known [27] that a 3-dimensional Einstein manifold is a manifold of constant curvature. Also M is projectively flat if and only if it is of constant curvature [26]. Now trivially, projectively flatness implies ϕ-projectively flat. Hence using Proposition 4.2 we can state the following:

Theorem 4.1. *A 3-dimensional connected trans-Sasakian manifold is ϕ-projectively flat if and only if it is an Einstein manifold, provided $\alpha, \beta =$ constant.*

5 3-dimensional trans-Sasakian manifold satisfying $R(X, Y).P = 0$

Using (2.3), (2.12) in (1.1), we get

$$\eta(P(X, Y)Z) = (\alpha^2 - \beta^2)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] - \tfrac{1}{2}[S(Y, Z)\eta(X) - S(X, Z)\eta(Y), \tag{5.1}$$

provided $\phi(\operatorname{grad}\alpha) = \operatorname{grad}\beta$. Putting $Z = \xi$ in (5.1), we get

$$\eta(P(X, Y)\xi) = 0. \tag{5.2}$$

Again taking $X = \xi$ in (5.1), we have

$$\eta(P(\xi, Y)Z) = (\alpha^2 + \beta^2)g(Y, Z) - \frac{1}{2}S(Y, Z), \tag{5.3}$$

where (2.1) and (2.9) are used.

Now,

$$(R(X,Y)P)(U,V)Z = R(X,Y).P(U,V)Z - P(R(X,Y)U,V)Z - P(U,R(X,Y)V)Z - P(U,V)R(X,Y)Z.$$

As it has been considered $R(X,Y).P = 0$, so we have

$$R(X,Y).P(U,V)Z - P(R(X,Y)U,V)Z - P(U,R(X,Y)V)Z - P(U,V)R(X,Y)Z = 0. \quad (5.4)$$

Therefore,

$$g(R(\xi,Y).P(U,V)Z,\xi) - g(P(R(\xi,Y)U,V)Z,) - g(P(U,R(\xi,Y)V)Z,\xi) - g(P(U,V)R(\xi,Y)Z,\xi) = 0. \quad (5.5)$$

From this it follows that,

$$-\tilde{P}(U,V,Z,Y) + \eta(Y)\eta(P(U,V)Z) - \eta(U)\eta(P(Y,V)Z) + g(Y,U)\eta(P(\xi,V)Z) - \eta(V)\eta(P(U,Y)Z) + g(Y,V)\eta(P(U,\xi)Z) - \eta(Z)\eta(P(U,V)Y) = 0, \quad (5.6)$$

where $-\tilde{P}(U,V,Z,Y) = g(P(U,V)Z,Y)$.

Putting $Y = U$ in (5.6), we get

$$-\tilde{P}(U,V,Z,Y) + g(U,U)\eta(P(\xi,V)Z) - \eta(V)\eta(P(U,U)Z) + g(U,V)\eta(P(U,\xi)Z) - \eta(Z)\eta(P(U,V)U) = 0. \quad (5.7)$$

Let $\{e_1, e_2, \xi\}$ be a local orthonormal basis of the vector fields in M. If we put $U = e_i$ in (5.7) and summing up with respect to i, then we have

$$S(V,Z) = 2(\alpha^2 - \beta^2)g(V,Z) - \left[\tfrac{1}{2} - 3(\alpha^2 - \beta^2)\right]\eta(V)\eta(Z), \quad (5.8)$$

where (5.1) and (5.3) are used.

Taking $Z = \xi$ in (5.8) and using (2.9) we obtain

$$r = 6(\alpha^2 - \beta^2). \quad (5.9)$$

Now using (5.1), (5.2), (5.8) and (5.9) in (5.6) we get

$$\tilde{P}(U,V,Z,U) = 0. \quad (5.10)$$

From (5.10) it follows that

$$P(U,V)Z = 0. \quad (5.11)$$

Therefore, the trans-Sasakian manifold under consideration is projectively flat. Conversely, if the manifold is projectively flat, then obviously $R(X,Y).P = 0$ holds. Hence we can state the next theorem:

Theorem 5.1. *A 3-dimensional connected trans-Sasakian manifold is projectively semisymmetric if and only if the manifold is projectively flat, provided* $\phi(\operatorname{grad}\alpha) = \operatorname{grad}\beta$.

6 Example of a 3-dimensional trans-Sasakian manifold

Example 6.1. [8] We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, z \neq 0\}$, where (x, y, z) are standard co-ordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = z\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}\right), \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Let ϕ be the (1,1) tensor field defined by

$$\phi(e_1) = e_2, \quad \phi(e_2) = -e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have $\eta(e_3) = 1$,

$$\phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$, the set of all smooth vector fields on M.

Then for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to metric g and R be the curvature tensor of g. Then we have

$$[e_1, e_2] = ye_2 - z^2 e_3, \quad [e_1, e_3] = -\frac{1}{z}e_1, \quad [e_2, e_3] = -\frac{1}{z}e_2.$$

Taking $e_3 = \xi$ and using Koszul formula for the Riemannian metric g, we can easily calculate

$$\nabla_{e_1} e_3 = -\frac{1}{z}e_1 + \frac{1}{z^2}e_2, \quad \nabla_{e_1} e_2 = -\frac{1}{2}z^2 e_3, \quad \nabla_{e_1} e_1 = \frac{1}{z}e_3,$$
$$\nabla_{e_2} e_3 = -\frac{1}{z}e_2 - \frac{1}{2}z^2 e_1, \quad \nabla_{e_2} e_2 = ye_1 + \frac{1}{z}e_3, \quad \nabla_{e_2} e_1 = \frac{1}{2}z^2 e_2 - \frac{1}{2}z^2 e_3 - ye_2,$$
$$\nabla_{e_3} e_3 = 0, \quad \nabla_{e_3} e_2 = -\frac{1}{2}z^2 e_1, \quad \nabla_{e_3} e_1 = \frac{1}{2}z^2 e_2.$$

From the above it can be easily seen that (ϕ, ξ, η, g) is a trans-Sasakian structure on M. Consequently $M^3(\phi, \xi, \eta, g)$ is a trans-Sasakian manifold with $\alpha = -\frac{1}{2}z^2 \neq 0$ and $\beta = -\frac{1}{z} \neq 0$.

Example 6.2. We consider the 3-dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3, (x, y, z) \neq 0\}$, where (x, y, z) are standard co-ordinate of $\mathbb{R}^3$.

The vector fields

$$e_1 = \frac{\partial}{\partial z} - y\frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = 2\frac{\partial}{\partial x}$$

are linearly independent at each point of M.

Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_1, e_2) = g(e_2, e_3) = 0,$$
$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$.

Let ϕ be the (1,1) tensor field defined by

$$\phi(e_1) = -e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0.$$

Then using the linearity of ϕ and g, we have $\eta(e_3) = 1$,

$$\phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$.

Thus for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on M.

Let ∇ be the Levi-Civita connection with respect to metric g. Then we have

$$[e_1, e_2] = e_1e_2 - e_2e_1 = \left(\frac{\partial}{\partial z} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} - y\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x} = \frac{1}{2}e_3.$$

Similarly $[e_1, e_3] = 0$ and $[e_2, e_3] = 0$.

Taking $e_3 = \xi$ and using Koszul formula for the Riemannian metric g, we can easily calculate

$$\nabla_{e_1}e_3 = \frac{1}{4}e_2, \quad \nabla_{e_1}e_2 = -\frac{1}{4}e_3, \quad \nabla_{e_1}e_1 = 0,$$
$$\nabla_{e_2}e_3 = -\frac{1}{4}e_1, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_1 = \frac{1}{4}e_2,$$
$$\nabla_{e_3}e_3 = 0, \quad \nabla_{e_3}e_2 = -\frac{1}{4}e_1, \quad \nabla_{e_3}e_1 = \frac{1}{4}e_2.$$

We see that the structure (ϕ, ξ, η, g) satisfies the formula (2.6) for $\alpha = \frac{1}{4}$ and $\beta = 0$. Hence the manifold is a trans-Sasakian manifold of type $(\frac{1}{4}, 0)$.

Example 6.3. In [9] the authors cited an example of a 3-dimensional trans-Sasakian manifold of type $(0, -1)$. This is the classical example of the hyperbolic 3-space which is obviously of constant sectional curvature. Hence the manifold is Einstein manifold and projectively flat. Hence the manifold is ϕ-projectively flat. Thus Theorem 4.1 is verified.

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